SAMPLE 7.1 Springs in series versus springs in parallel: Two springs with spring constants $k_1 = 100\,\text{N/m}$ and $k_2 = 150\,\text{N/m}$ are attached together in their relaxed state (with no applied force) as shown in Fig. 7.11. In case (a), a vertical force $F = 10\,\text{N}$ is applied at point A, and in case (b), the same force is applied at the end point B. Find the force in each spring for static equilibrium. Also, find the equivalent stiffness for (a) and (b).

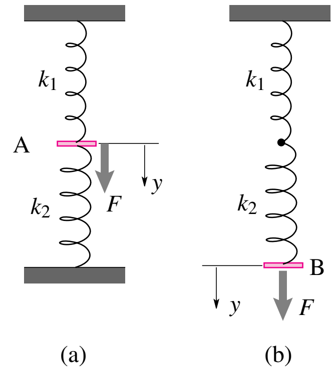

Figure 7.11

Solution In static equilibrium, let Δy be the displacement of the point of application of the force in each case. We can figure out the forces in the springs by writing force-balance equations in each case.

- **Case (a):** The free-body diagram of point A is shown in Fig. 7.12. As point A is displaced downwards by Δy, spring 1 gets stretched by Δy whereas spring 2 gets compressed by Δy. Therefore, the forces applied by the two springs, $k_1\Delta y$ and $\Delta k_2 y$, are in the same direction. Then, the force balance in the vertical direction, $\hat{\boldsymbol{\jmath}} \cdot (\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}})$, gives:

$$\begin{aligned} F &= F_1 + F_2 = (k_1 + k_2)\Delta y \\ \Rightarrow \quad \Delta y &= \frac{F}{k_1 + k_2} = \frac{10\,\text{N}}{(100+150)\,\text{N/m}} = 0.04\,\text{m} \\ \Rightarrow \quad F_1 &= k_1\Delta y = 100\,\text{N/m} \cdot 0.04\,\text{m} = 4\,\text{N} \\ \Rightarrow \quad F_2 &= k_2\Delta y = 150\,\text{N/m} \cdot 0.04\,\text{m} = 6\,\text{N}. \end{aligned}$$

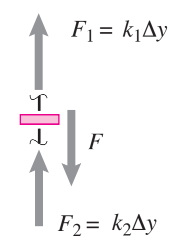

Figure 7.12: Free-body diagram of joint A.

The equivalent stiffness of the system is the stiffness of a single spring that will undergo the same displacement Δy under F. From the equilibrium equation above, it is easy to see that,

$$k_e \equiv \frac{F}{\Delta y} = k_1 + k_2 = 250\,\text{N/m}.$$

$$\boxed{F_1 = 4\,\text{N}, \quad F_2 = 6\,\text{N}, \quad k_e = 250\,\text{N/m}}$$

- **Case (b):** The free-body diagrams of the two springs are shown in Fig. 7.13 along with that of point B. In this case both springs stretch as point B is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = \Delta y.$$

Now, using the free-body diagram of point B and writing the force balance equation in the vertical direction, we get $F = k_2 y_2$, and from the free-body diagram of spring 2, we get $k_2 y_2 = k_1 y_1$. Thus the force in each spring is the same and equals the applied force, *i.e.*,

$$F_1 = k_1 y_1 = F = 10\,\text{N} \qquad \text{and} \qquad F_2 = k_2 y_2 = F = 10\,\text{N}.$$

The springs in this case are in series. Therefore, their equivalent stiffness, k_e, is

$$k_e = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1} = \left(\frac{1}{100\,\text{N/m}} + \frac{1}{150\,\text{N/m}}\right)^{-1} = 60\,\text{N/m}.$$

Note that the displacements y_1 and y_2 are different in this case. They can be easily found from $y_1 = F/k_1$ and $y_2 = F/k_2$.

$$\boxed{F_1 = F_2 = 10\,\text{N}, \quad k_e = 60\,\text{N/m}}$$

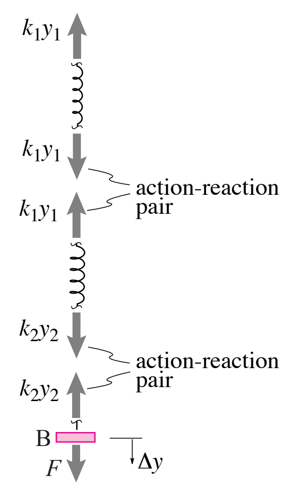

Figure 7.13: Free-body diagrams

Comments: Although the springs attached to point A do not visually seem to be in parallel, from a mechanics point of view they are parallel. Springs in *parallel* have the *same displacement* but *different forces*. Springs in *series* have *different displacements* but the *same force*.

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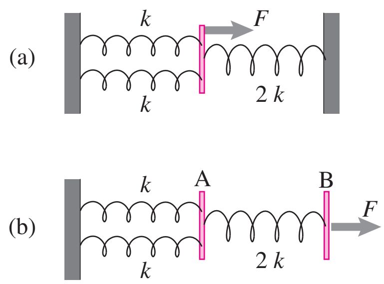

Figure 7.14

SAMPLE 7.2 Stiffness of three springs: For the spring networks shown in Fig. 7.14(a) and (b), find the equivalent stiffness of the springs in each case, given that each spring has a stiffness of $k = 20\,\mathrm{N/m}$.

Solution

1. In Fig. 7.14(a), all springs are in parallel since all of them undergo the same displacement Δx in order to balance the applied force F. Each of the two springs on the left stretches by Δx and the spring on the right compresses by Δx. Therefore, the equivalent stiffness of the three springs is

$$k_p = k + k + 2k = 4k = 80\,\mathrm{kN/m}.$$

Pictorially,

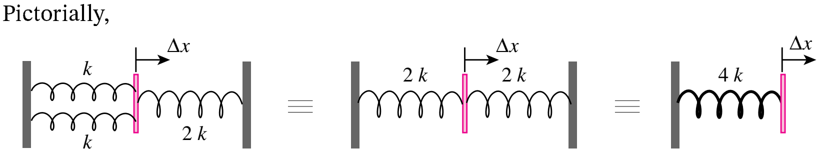

Figure 7.15:

$$k_{\text{equiv}} = 80\,\mathrm{kN/m}$$

2. In Fig. 7.14(b), the first two springs (on the left) are in parallel but the third spring is in series with the first two. To see this, imagine that, in equilibrium, point A moves to the right by Δx_A and point B moves to the right by Δx_B. Then each of the first two springs has the same stretch Δx_A while the third spring has a net stretch $= \Delta x_B - \Delta x_A$. Therefore, to find the equivalent stiffness, we can first replace the two parallel springs by a single spring of equivalent stiffness $k_p = k + k = 2k$. Then the springs with stiffnesses k_p and $2k$ are in series and therefore their equivalent stiffness k_s is found as follows:

$$\begin{aligned} \frac{1}{k_s} &= \frac{1}{k_p} + \frac{1}{2k} = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \\ \Rightarrow \quad k_s &= k = 20\,\mathrm{kN/m}. \end{aligned}$$

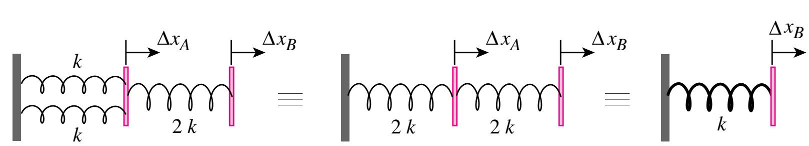

Figure 7.16:

$$k_{\text{equiv}} = 20\,\mathrm{kN/m}$$

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SAMPLE 7.3 Stiffness vs strength: Which of the two structures (network of springs) shown in the figure is stiffer and which one has more strength if each spring has stiffness $k = 10\,\mathrm{kN/m}$ and strength $T_0 = 10\,\mathrm{kN}$.

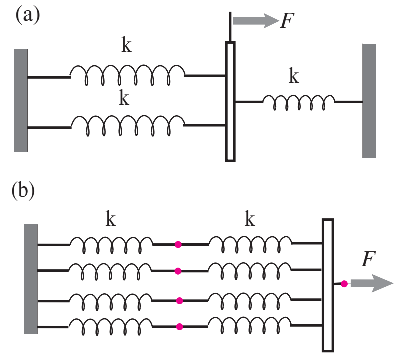

Figure 7.17

Solution In structure (a), all the three springs are in parallel. Therefore, the equivalent stiffness of the three springs is

$$k_a = k + k + k = 3k = 30\,\mathrm{kN/\,m}.$$

For figuring out the strength of the structure, we need to find the force in each spring. From the free-body diagram in Fig. 7.18 we see that,

$$\begin{aligned} k\Delta x + k\Delta x + k\Delta x &= F \\ \Rightarrow \quad \Delta x &= \frac{F}{3k}. \end{aligned}$$

Therefore, the force in each spring is

$$F_s = k\Delta x = \frac{F}{3}.$$

But the maximum force that a spring can take is $(F_s)_{\max} = T_0 = 10\,\mathrm{kN}$. Therefore, the maximum force that the structure can take (*i.e.,* the strength of the structure), is

$$F_{\max} = 3T_0 = 30\,\mathrm{kN}.$$

Stiffness = 30 kN/m, Strength = 30 kN

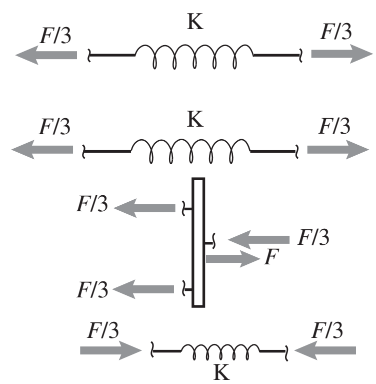

Figure 7.18

Now we carry out a similar analysis for structure (b). There are four parallel chains in this structure, each chain containing two springs in series. The stiffness of each chain, k_c, is found from

$$\begin{aligned} \frac{1}{k_c} &= \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \\ \Rightarrow \quad k_c &= \frac{k}{2} = 5\,\mathrm{kN/\,m}. \end{aligned}$$

So, the stiffness of the entire structure is

$$k_b = k_c + k_c + k_c + k_c = 4k_c = 20\,\mathrm{kN/\,m}.$$

From the free-body diagram shown in fig. 7.19, we find the force in each spring to be F/4 . Therefore, the maximum force that the structure can take is

$$F_{\max} = 4T_0 = 40\,\mathrm{kN}.$$

Stiffness = 20 kN/m, Strength = 40 kN

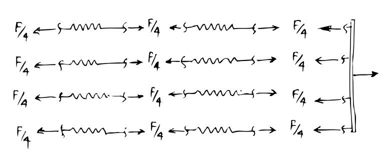
Figure 7.19

Thus, structure (a) is stiffer but structure (b) is stronger (higher strength).

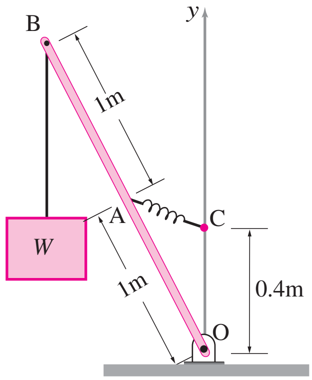

Figure 7.20

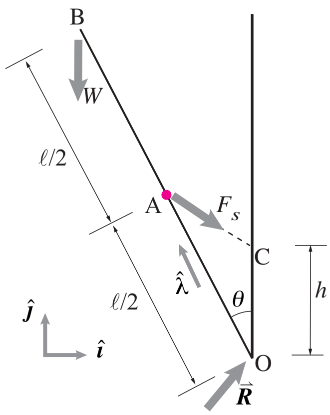

Figure 7.21

SAMPLE 7.4 Zero length springs are special. A rigid and massless rod OAB of length 2 m supports a weight $W = 100$ kg hung from point B. The rod is pinned at O and supported by a zero length (in relaxed state) spring attached at mid-point A and point C on the vertical wall. Find the equilibrium angle θ and the force in the spring.

Solution The free-body diagram of the rod is shown in Fig. 7.21 in an assumed equilibrium state. Let $\hat{\boldsymbol{\lambda}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$ be a unit vector along OB. The spring force can be written as $\vec{\boldsymbol{F}}_s = k\vec{\boldsymbol{r}}_{C/A}$ (since AC is a *zero-length* spring, the stretch in the spring is $|\vec{\boldsymbol{r}}_{C/A}|$). We need to determine θ and F_s.

Let us write moment equilibrium equation about point O, *i.e.*, $\sum \vec{\boldsymbol{M}}_O = \vec{\boldsymbol{0}}$,

$$\vec{\boldsymbol{r}}_{B/O} \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{r}}_{A/O} \times \vec{\boldsymbol{F}}_s = \vec{\boldsymbol{0}}.$$

Noting that

$$\begin{aligned}
\vec{\boldsymbol{r}}_{B/O} &= \ell\hat{\boldsymbol{\lambda}}, \quad \vec{\boldsymbol{r}}_{A/O} = \frac{\ell}{2}\hat{\boldsymbol{\lambda}}, \\
\vec{\boldsymbol{F}}_s &= k\vec{\boldsymbol{r}}_{C/A} = k(\vec{\boldsymbol{r}}_C - \vec{\boldsymbol{r}}_A) \\
&= k\left(h\hat{\boldsymbol{\jmath}} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}\right),
\end{aligned}$$

we get,

$$\begin{aligned}
\ell\hat{\boldsymbol{\lambda}} \times (-W\hat{\boldsymbol{\jmath}}) + \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times k\left(h\hat{\boldsymbol{\jmath}} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}\right) &= \vec{\boldsymbol{0}} \\
-W\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + kh\frac{\ell}{2}(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}}.
\end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$, we get,

$$\begin{aligned}
-W\ell + kh\frac{\ell}{2} &= 0 \\
\Rightarrow \quad kh &= 2W.
\end{aligned}$$

Thus the result is independent of θ! As long as the spring stiffness k and the height h of point C are such that their product equals $2W$, the system will be in equilibrium at any angle. This, however, is in general not possible if AC is not a zero-length spring.

Equilibrium is satisfied at any angle if $kh = 2W$

SAMPLE 7.5 **Deflection of an elastic structure:** For the two-spring structure shown in the figure, find the deflection of point C when

1. $\vec{F} = 1\,\text{N}\hat{\imath}$,
2. $\vec{F} = 1\,\text{N}\hat{\jmath}$,
3. $\vec{F} = 30\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}$,

The spring stiffnesses are $k_1 = 10\,\text{kN}/m$ and $k_2 = 20\,\text{kN}/m$.

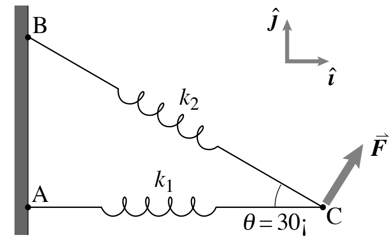

Figure 7.22

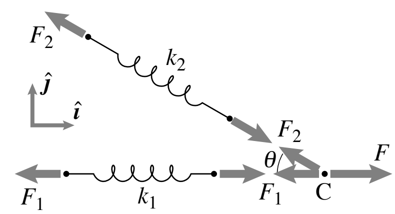

Figure 7.23

Solution Let $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$ be the displacement of point C of the structure due to the applied load. We can figure out the deflections in each spring as follows. Let $\hat{\lambda}_{AC}$ and $\hat{\lambda}_{BC}$ be the unit vectors along AC and BC, respectively (see fig. 7.24). Then, the change in the length of spring AC due to the (assumed small) displacement of point C is (see page 393 for a discussion)

$$\begin{aligned}\Delta_{AC} &= \hat{\lambda}_{AC}\cdot\Delta\vec{r} \quad \text{(this is the key equation)}\\ &= \hat{\imath}\cdot(\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x.\end{aligned}$$

Similarly, the change in the length of spring BC is

$$\begin{aligned}\Delta_{BC} &= \hat{\lambda}_{BC}\cdot\Delta\vec{r}\\ &= (\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})\cdot(\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x\cos\theta - \Delta y\sin\theta.\end{aligned}$$

Now we can find the force in each spring since we know the deflection in each spring.

$$\begin{aligned}\text{Force in spring AC} = F_1 &= k_1\Delta x \qquad (7.8)\\ \text{Force in spring BC} = F_2 &= k_2(\Delta x\cos\theta - \Delta y\sin\theta). \qquad (7.9)\end{aligned}$$

The forces in the springs, however, depend on the applied force, since they must satisfy static equilibrium. Thus, we can determine the deflection by first finding F_1 and F_2 in terms of the applied load and substituting in the equations above to solve for the deflection components.

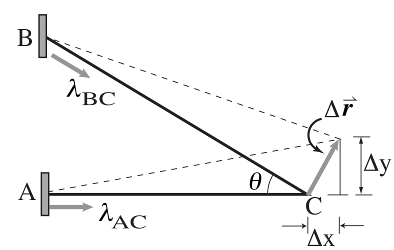

Figure 7.24

1. *Deflections with unit force in the x-direction:*
   Let $\vec{F} = f_x\hat{\imath} = 1\,\text{N}\hat{\imath}$, (we have adopted a special symbol f_x for the unit load). Then, from the free-body diagram of the springs and the end pin shown in fig. 7.23 and the force equilibrium ($\sum\vec{F} = \vec{0}$), we have,

   $$f_x\hat{\imath} - F_1\hat{\imath} + F_2(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) = \vec{0}.$$

   Dotting this eqn. with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get,

   $$\begin{aligned}F_2 &= 0\\ F_1 &= f_x = 1\,\text{N}.\end{aligned}$$

   Substituting these values of F_1 and F_2 in eqns. (7.8) and (7.9), and solving for Δx and Δy we get,

   $$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{F}=f_x\hat{\imath}} = \begin{pmatrix}\frac{1}{k_1}\\ \frac{1}{k_1}\cot\theta\end{pmatrix} f_x. \qquad (7.10)$$

   Substituting the given values of θ, k_1, and $f_x = 1\,\text{N}$, we get

   $$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = (100\hat{\imath} + 173\hat{\jmath})\times 10^{-6}\,\text{m}.$$

   $$\boxed{\Delta\vec{r} = (100\hat{\imath} + 173\hat{\jmath})\times 10^{-6}\,\text{m}}$$

2. *Deflections with unit force in the y-direction:* We carry out a similar analysis for this case. We again assume the displacement of point C to be $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$. Since the geometry of deformation and the associated results are the same, eqns. (7.8)

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and (7.9) remain valid. We only need to find the spring forces from the static equilibrium under the new load. From the free-body diagram in Fig. 7.25 we have,

$$(-F_1 - F_2\cos\theta)\hat{\boldsymbol{\imath}} + (F_2\sin\theta + F)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{7.11}$$

$$\begin{aligned} \{\text{eqn. (7.11)}\}\cdot\hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad F_2 = -\frac{F}{\sin\theta} \\ \{\text{eqn. (7.11)}\}\cdot\hat{\boldsymbol{\imath}} \quad &\Rightarrow \quad F_1 = -F_2\cos\theta = F\cot\theta. \end{aligned}$$

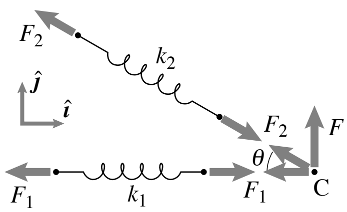

Figure 7.25

Substituting these values of F_1 and F_2 in terms of $F = f_y$ in eqns. (7.8) and (7.9), we get

$$\begin{aligned} f_y\cot\theta &= k_1\Delta x \quad \Rightarrow \quad \Delta x = \frac{f_y}{k_1}\cot\theta \\ -\frac{f_y}{\sin\theta} &= k_2(\Delta x\cos\theta - \Delta y\sin\theta) \\ \Rightarrow \quad \Delta y &= \frac{1}{\sin\theta}\left(\Delta x\cos\theta + \frac{f_y}{k_2\sin\theta}\right) \\ &= f_y\left(\frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\right). \end{aligned}$$

Thus,

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=f_y\hat{\boldsymbol{\jmath}}} = \begin{pmatrix} \frac{1}{k_1}\cot\theta \\ \frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta \end{pmatrix} f_y. \tag{7.12}$$

Substituting the values of θ, k_1, k_2, and $f_y = 1\,\text{N}$, we get

$$\Delta\vec{\boldsymbol{r}} = \Delta x\hat{\boldsymbol{\imath}} + \Delta y\hat{\boldsymbol{\jmath}} = (173\hat{\boldsymbol{\imath}} + 500\hat{\boldsymbol{\jmath}}) \times 10^{-6}\,\text{m}.$$

$\Delta\vec{\boldsymbol{r}} = (173\hat{\boldsymbol{\imath}} + 500\hat{\boldsymbol{\jmath}}) \times 10^{-6}\,\text{m}$

3. *Deflection under a general load:* Since we have already got expressions for deflections in the x and y-directions under unit loads in the x and y-directions, we can now combine the results (using superposition, see page 263) to find the deflection under any general load $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ as follows.

$$\begin{aligned} \Delta\vec{\boldsymbol{r}} &= \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = F_x \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=1\hat{\boldsymbol{\imath}}} + F_y \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=1\hat{\boldsymbol{\jmath}}} \\ &= \begin{bmatrix} k_1^{-1} & k_1^{-1}\cot\theta \\ k_1^{-1}\cot\theta & k_1^{-1}\cot^2\theta + k_2^{-1}\csc^2\theta \end{bmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}. \end{aligned}$$

Once again, substituting all given values and $F_x = 30\,\text{N}$ and $F_y = 20\,\text{N}$, we get

$$\Delta\vec{\boldsymbol{r}} = (6.4\hat{\boldsymbol{\imath}} + 15.2\hat{\boldsymbol{\jmath}}) \times 10^{-3}\,\text{m}.$$

$\Delta\vec{\boldsymbol{r}} = (6.4\hat{\boldsymbol{\imath}} + 15.3\hat{\boldsymbol{\jmath}}) \times 10^{-3}\,\text{m}$

Note: The matrix obtained above for finding the deflection under a general load is called the *compliance matrix* of the structure. Its inverse is the *stiffness matrix* K of the structure. K is used to find forces given deflections.

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