6.4 Frames and structures

Although trusses are good, they are not good enough for all purposes. Nor are they even always good-enough models of very truss-looking structures. *Frames* are structures that are more general than trusses. In a truss, every bar is a two-force body. In a more general structure or frame, one or more components is not a two-force body. The analysis of non-truss frames is generally less formulaic, and thus more subtle, than is the method of joints for trusses.

Example: A-frame ladder.
The two-diagonal parts of an A-frame ladder are not two-force bodies and thus the ladder, triangular as it looks, is not a truss. And truss analysis is not appropriate.

A 2D frame is a collection of rigid objects connected by hinges. A special case is a truss, where each bar has exactly two hinges.

One could generalize this definition slightly to include also sliding joints. And one could narrow the definition, to exclude mechanisms, by insisting that the structure be rigid. Being precise about these cases is not important because the analysis methods are the same in all cases.

The overall mechanics recipe applies to frames, of course: a) draw free-body diagrams, b) apply the laws of mechanics to each free-body diagram, and c) solve the mechanics equations for unknowns of interest. For trusses, the free-body diagrams of each bar, with the 3 equilibrium equations (6 in 3D) just yield the “two-force” body result that the bar has equal tensions at the two ends. Because there was no more to learn from the bar free-body diagrams we didn’t even draw them. Instead we used the bar tensions as forces on free-body diagrams of the joints. It’s as if the bars were just a means to mediate action-reaction pairs between joints.

For more general frameworks we have to pay full respect to the free-body diagrams of all of the parts, not just the pins. At least for all of the parts that are not two-force bodies. Here is the analysis of frameworks recipe:

- Draw free-body diagrams of
  - the whole structure; and
  - the separate parts of the structure; and
  - collections of parts of the structure if such seems likely to be fruitful;
  - Use the principle of action and reaction in the free-body diagrams so that one action-reaction pair has only one unknown;

Figure 6.52: A ladder is well-modeled as a collection of rigid pieces connected by pin joints ...

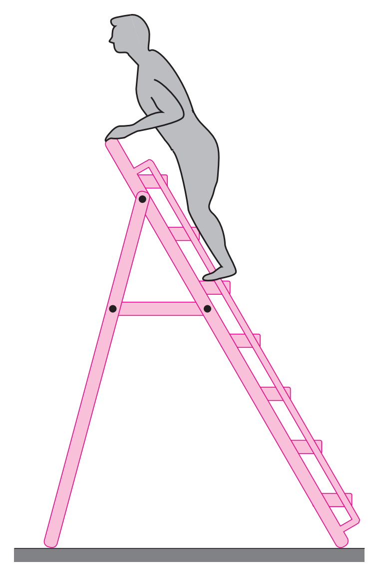


Figure 6.53: ... but A-frame ladder is not a truss because the bars are not all two-force members. The piece the person is standing on has forces at more than two places: the hands, the top pin, the middle brace, the feet and the ground. The left diagonal leg also has three ($3 > 2$) forces acting on it.

- For each free-body diagram, write equilibrium conditions (force and moment balance).
- Solve the equilibrium equations for desired unknowns.

For the parts that are not two-force bodies, we will not know the directions of the interaction forces *a priori*, that's why the method of joints is not used for frames ①. Naturally one can be on the look out for shortcuts:

- for any two-force bodies assign an equal-valued tension to each end (thus eliminating any need or use for equilibrium equations for that object)
- consider each pin as part of one of the bodies to which it is connected (*i.e.,* there is no need to draw a separate FBD of the pin).
- To minimize calculation, look for a subset of the equilibrium equations that
  - contains your unknowns of interest, and
  - has as many unknowns as scalar equations, and
  - contains as few equations as possible.

① Conversely, we could have analyzed trusses the way we are now going to analyze frames. This seldom-used approach to trusses, the 'method of bars and pins', is discussed in box 6.1 on page 4.

Our general goal here is to find the reaction forces, the interaction forces and the 'internal' forces in the components of a statically determinate structure.

Example: An X structure
Two bars are joined in an 'X' by a pin at J. Neither of the bars is a two-force body so a free-body diagram of the 'joint' at J, made by cutting and leaving stubs as we did with trusses, has 12 unknown force and moment components.

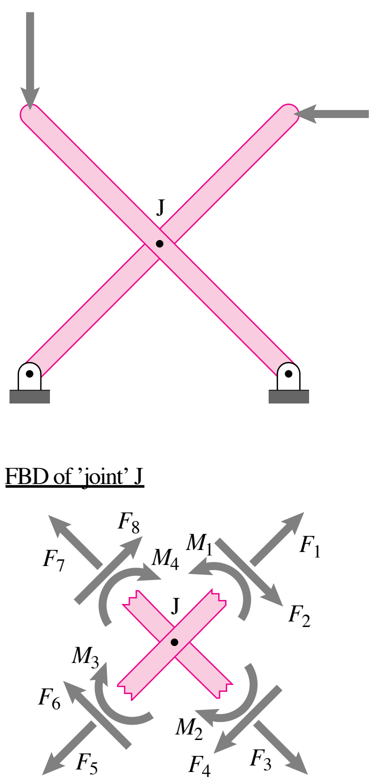


Figure 6.54: A simple frame and a free-body diagram of a joint. Unlike the case for trusses, free-body diagrams of joints (pins) are nearly useless in the study of more general frames. There are too many unknown forces and moments acting at the pin.

Instead of drawing free-body diagrams of the connections, our approach here is to draw free-body diagrams of each of the structure's or machine's parts. Sometimes, as was the case with trusses, it is also useful to draw a free-body diagram of a whole structure or of some multi-piece part of the structure

Static determinacy

A statically determinate structure has

- a solution for all possible applied loads, and
- only one solution, and
- this solution can be found by using equilibrium equations applied to each of the pieces.

Not all practical structures are statically determinate. Some structures are rigid but redundant, thus precluding finding all unknowns from statics. Some structures cannot carry all loads, but can carry the loads of interest (e.g., a vertical cable that can usefully carry a weight but cannot carry a side load). Nonetheless, for starters here we emphasize determinate structures. The basic counting formula

$$\text{number of equations} = \text{number of unknowns}$$

is necessary for determinacy but does not guarantee determinacy. How to count? Frameworks in 2D have three equilibrium equations for each object that isn't just a point (in which case there are only two equilibrium equations). There are two unknown force components for every pin connection, whether to the ground or to another piece. And there is one unknown force component for every roller connection whether to the ground or between objects. Applied forces do not count in this determinacy check, even if they are unknown.

Example: 'X' structure counting
In the 'X' structure above we can count as follows.

$$\begin{aligned}
\text{number of equations} &\stackrel{?}{=} \text{number of unknowns}\\
(3 \text{ eqs per bar}) \cdot (2 \text{ bars}) &\stackrel{?}{=} (2 \text{ unknown force components per pin}) \cdot (3 \text{ pins})\\
6 \text{ eqs} &\stackrel{\surd}{=} 6 \text{ unknown force components}
\end{aligned}$$

So the 'X' structure passes the counting test for static determinacy.

Redundant structures

A redundant structure can carry whatever loads it can carry in more than one way. If not also indeterminate, a redundant structure has fewer equilibrium equations than unknown reaction or interaction force components. Finding all the reaction components is only possible if one models the deformation, a topic for more advanced structural mechanics. **Example: Over-braced 'X'**

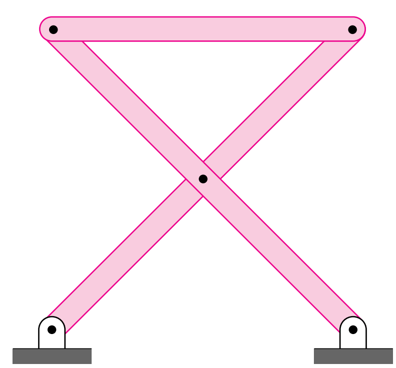


Figure 6.55: Overbraced X. A rigid frame that is not statically determinate.

The structure is evidently redundant. Why? Because it has a bar added to a structure that was already statically determinate. By counting we get

$$\begin{aligned}
\text{number of equations} &\stackrel{?}{=} \text{number of unknowns}\\
3 \cdot \underbrace{(\text{number of bars})}_{3} &\stackrel{?}{=} 2 \cdot \underbrace{(\text{number of joints})}_{5}\\
9 \text{ eqs} &< 10 \text{ unknown force components}
\end{aligned}$$

thus demonstrating redundancy.

6.1 The 'method of bars and pins' for trusses

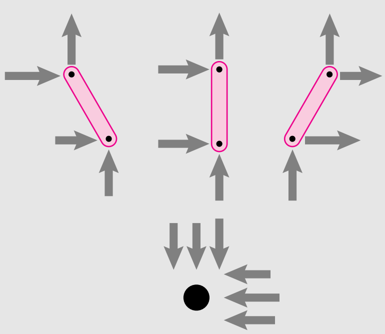
This is an aside for those who wonder why truss analysis seems so different than frame analysis.

Trusses are simple frameworks. So the methods used for more general frameworks should work for trusses. They do. The resulting method, which is essentially never used in such detail, we will call 'the method of bars and pins'.

In the method of bars and pins you treat a truss like any other structure. You draw a free-body diagram of each part.

One approach: treat the pins as parts. One approach is to draw free-body diagrams of each pin also. You use the principle of action and reaction to relate the forces on the different bars and pins. Then you solve the collection of equilibrium equations.

Consider one joint of a truss where three bars meet at a hinge (pin). Below are free-body diagrams of the three bars and of the pin. Assuming a frictionless round pin at the hinge, all the bar forces on the pin pass through its center.

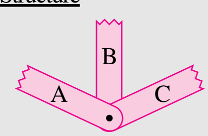


Thus, in 2D, you get two equilibrium equations for each pin and three for each bar. If you apply the three bar equations to a given bar you find that it obeys the two-force body relations. Namely, the reactions on the two bar ends are equal and opposite and along the connecting points. Now application of the pin equilibrium equations is identical to the joint equations we had previously. Thus, the 'method of bars and pins' reduces to the method of joints in the end.

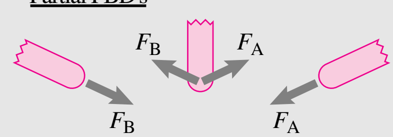
Approach two: draw FBDs of just the bars. Another approach is to associate each pin with one of the bars to which it is attached. Then just think of a truss as bars that are connected with forces and no moments. Draw free-body diagrams of each piece, use the principle of action and reaction, and write the equilibrium equations for each bar. This is the approach that is used in this section for other structures.

If three bars A, B, and C are connected to a pin, consider the pin as part of, say, A. Then consider action-reaction pairs between A and B, and between A and C, but not between B and C. Similarly if there are four or more bars, consider interactions between each bar and the one-bar that has the pin.

Partial Structure



Partial FBD's



Determinate equations. In all cases, if the truss is statically determinate the equilibrium equations generated from the free-body diagrams above will produce a solvable set of linear algebraic equations. But, in all cases above, these will not be the more minimal set of equations we generated in the method of joints in the truss analysis. The methods of this box work, they are just harder to implement.