SAMPLE 6.12 The braced X-frame shown in the figure carries two vertical loads $F_1 = 2\,\text{kN}$ and $F_2 = 3\,\text{kN}$. Points G and H are directly above points A and B respectively. If $d = h = 2\,\text{m}$, find the tension in the brace CD.

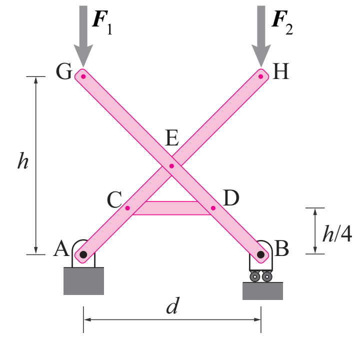

Figure 6.56

Solution The brace CD is pinned to the X-frame at C and D. The only loads acting on the brace are at its ends C and D. Therefore, it is a two-force body. Let us assume that the tension in brace is C_x. We need to find C_x under the given loads.

The free-body diagram of the whole frame is shown in fig. 6.57. Since the frame is supported by a hinge at A and a roller at B, there are three scalar support reactions acting on the frame. We can now determine all the three reactions from the static analysis of the frame:

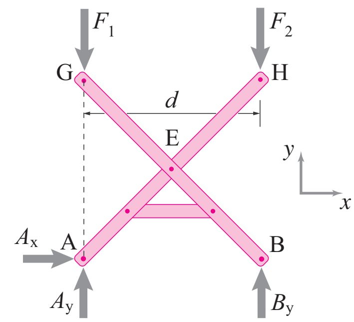

Figure 6.57

$$\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\
\sum M_A = 0 \quad &\Rightarrow \quad B_y d - F_2 d = 0 \\
&\Rightarrow \quad B_y = F_2 \\
\sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - B_y = F_1.
\end{aligned}$$

Thus all the reactions are known. Now we can analyze either bar AH or bar BG (the analysis is identical) to determine the tension C_x in the brace. The free-body diagram of bar AH is shown in fig. 6.58. Since we are only interested in C_x, we can carry out moment balance about point E ($\sum M_E = 0$) to give

$$\begin{aligned}
C_x \frac{d}{4} - F_2 \frac{d}{2} - A_y \frac{d}{2} &= 0 \\
\Rightarrow \quad C_x &= 2(F_2 + A_y) \\
&= 2(F_2 + F_1) \\
&= 2(3k\,\text{N} + 2\,\text{kN}) \\
&= 10\,\text{kN}.
\end{aligned}$$

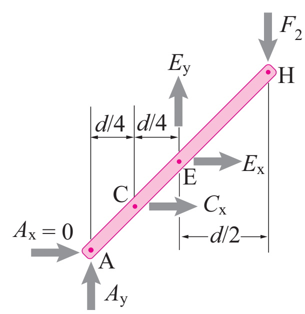

Figure 6.58

Thus the tension in the brace is twice the total load on the structure.

Tension in brace CD = 10 kN

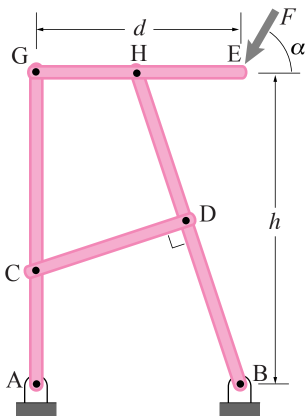

Figure 6.59

SAMPLE 6.13 The frame shown in the figure is supported by hinges at both A and B. Bar GE is as long as the base AB and bar BH is pinned to GE at the mid point H. Brace CD is pinned at D, the mid-point of bar BH, and is orthogonal to bar BH. The load on the structure, $F = 1\,\text{kN}$, is applied at E, at an angle $\alpha = 60°$. Given that $d = 2\,\text{m}$, $h = 3\,\text{m}$, find the forces on the inclined bar BH and the support reactions at A and B.
[Note: Usually, determinate framed structures are made up of overhangs and extensions on a rigid triangle. This structure is an example of a frame that does not contain any rigid triangle.]

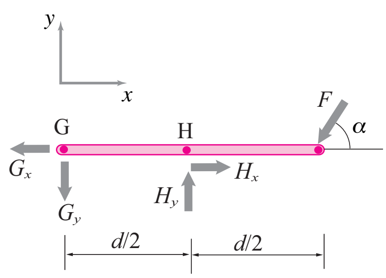

Figure 6.60

Solution The given structure has hinges at both A and B. Therefore, there are four scalar support reactions, two each at A and B. So, from the free-body diagram of the whole structure, we cannot determine all support reactions. In fact, the free-body diagram of each rod will have more than three unknown forces (you can check this mentally). Thus, we are not likely to find all unknown forces on a bar without analyzing other bars. Since bar CD is a two-force member bar, it only contributes one scalar force, the tension in this rod. Now, there are two unknown scalar forces at each pin joint, A, B, G, and H, and one force at C and D (the same force). Thus we have nine unknown scalar forces. We have three bars AG, GE, and BH, each with three independent scalar equations of static equilibrium. Thus we have nine independent equations in nine unknowns. Therefore, we can solve for all the unknown forces.

Consider the free-body diagram of bar GE. The static equilibrium of this bar requires

$$
\begin{aligned}
\sum M_H = 0 \quad &\Rightarrow \quad G_y(d/2) - F\sin\alpha(d/2) = 0 \\
&\Rightarrow \quad G_y = F\sin\alpha \\
\sum F_y = 0 \quad &\Rightarrow \quad -G_y + H_y - F\sin\alpha = 0 \\
&\Rightarrow \quad H_y = F\sin\alpha + G_y = 2F\sin\alpha \\
\sum F_x = 0 \quad &\Rightarrow \quad H_x - G_x - F\cos\alpha = 0.
\end{aligned}
\tag{6.21}
$$

Thus we have found G_y and H_y but only a relationship between G_x and H_x. Since G_x and H_x are colinear, we cannot solve for them from the static analysis of bar GE alone. Now, let us consider bar AG (or bar BH; does not make a difference). The equilibrium analysis of this bar gives

$$\sum F_y = 0 \quad \Rightarrow \quad A_y + G_y + R_{CD}\sin\theta = 0 \tag{6.22}$$

$$\sum M_C = 0 \quad \Rightarrow \quad A_x h_1 - G_x h_2 = 0 \tag{6.23}$$

$$\sum F_x = 0 \quad \Rightarrow \quad A_x + G_x + R_{CD}\cos\theta = 0. \tag{6.24}$$

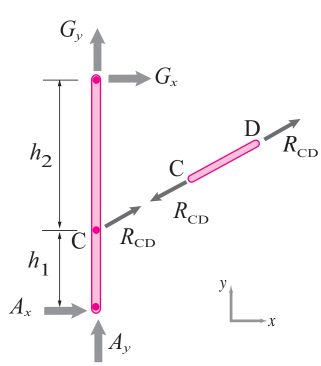

Figure 6.61

Since none of these equations contains only one unknown, we cannot solve for these forces from the equilibrium equations of bar AG alone. Note that we have written these equations in terms of h_1, h_2, and θ, thus far, undetermined geometric variables. However, we can easily find them from the given geometry. Now let us analyze bar BH.

$$\sum F_y = 0 \quad \Rightarrow \quad B_y - H_y - R_{CD}\sin\theta = 0 \tag{6.25}$$

$$\sum F_x = 0 \quad \Rightarrow \quad B_x - H_x - R_{CD}\cos\theta = 0 \tag{6.26}$$

$$\sum M_D = 0 \quad \Rightarrow \quad (H_x + B_x)\frac{h}{2} + (H_y + B_y)\frac{d}{4} = 0 \tag{6.27}$$

So, now we have seven independent equations, eqns. (6.21)–(6.27), in seven unknowns — $A_x, A_y, B_x, B_y, R_{CD}, G_x$, and H_x (we have already solved for G_y and H_y). We can solve these seven equations on a computer.

Before we go to the computer, let us find the undetermined geometric quantities h_1 and h_2. From fig. 6.63, we see that

$$
\begin{aligned}
h_1 &= \frac{h}{2} - \Delta \\
h_2 &= \frac{h}{2} + \Delta
\end{aligned}
$$

where $\Delta = d' \tan\theta$, $d' = 3d/4$, and $\theta = \tan^{-1}(d/2h)$. Now, we are ready to solve the seven equations on a computer.

```
% input given quantities
d = 2; h = 3; F = 1; alpha = pi/3;
% Define other used quantities in the equations
Delta = 3*d^2/(8*h);
h1 = h/2 - Delta;  h2 = h/2 + Delta;
theta = arctan(0.5*d/h);
% Input equations
eqset = { Hx - Gx = F*cos(alpha)
          Ay + RCD*sin(theta) = -F*sin(alpha)
          Ax*h1 - Gx*h2 = 0
          Ax + Gx + RCD*cos(theta) = 0
          By - RCD*sin(theta) = 2*F*sin(alpha)
          Bx - Hx - RCD*cos(theta) = 0
          (Hx+Bx)*h/2 + By*d/4 = -F*d/2*sin(alpha) }

solve eqset for Ax, Ay, Bx, By, Gx, Hx, and RCD
```

Including the values of G_y and H_y obtained from the first two equations of equilibrium of bar GE, we get the following values for all unknown forces from the computer solution.

$$
\begin{aligned}
A_x &= 3.23\,\text{kN} \\
A_y &= 0.75\,\text{kN} \\
B_x &= -2.73\,\text{kN} \\
B_y &= 0.12\,\text{kN} \\
R_{CD} &= -5.11\,\text{kN} \\
G_x &= 1.62\,\text{kN} \\
G_y &= 0.87\,\text{kN} \\
H_x &= 2.12\,\text{kN} \\
H_y &= 1.73\,\text{kN}
\end{aligned}
$$

$R_{CD} = -5.11\,\text{kN}$

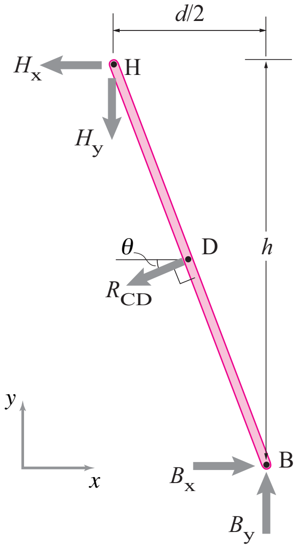

Figure 6.62

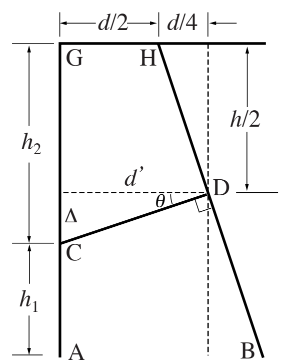

Figure 6.63: The geometry of h_1 and h_2. Since $d' = 3d/4$, $\Delta = d' \tan\theta$ and $\tan\theta = \frac{d/2}{h}$, we get $\Delta = \frac{3d}{4}\frac{d}{2h} = \frac{3d^2}{8h}$ and $h_1 = h/2 - \Delta$, $h_2 = h/2 + \Delta$.

SAMPLE 6.14 An easy-chair uses a curved frame as shown in the small picture in fig. 6.64. To simplify geometry, we can model the chair with straight bars as shown in the figure. Of special significance is the small pin at E that is rigidly attached to bar CDH and slides with negligible friction on bar ABD (see inset). This pin keeps the chair from collapsing and bears a large load. Assume the pin is $\epsilon = 2.5\,\text{cm}$ away from joint D towards B along bar ABD.

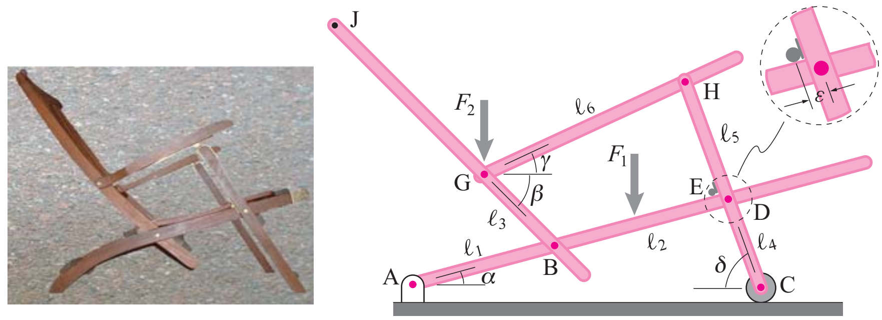

Figure 6.64: Dimensions: $\ell_1 = 45\,\text{cm}$, $\ell_2 = 60\,\text{cm}$, $\ell_3 = 30\,\text{cm}$, $\ell_5 = 40\,\text{cm}$, $\epsilon = 2.5\,\text{cm}$, $\alpha = 15°$, $\beta = 45°$ and $\delta = 70°$. Loads: $F_1 = 500\,\text{N}$ and $F_2 = 200\,\text{N}$. F_1 acts in the middle of bar segment BD and F_2 acts at G.

Solution First, referring to the figures, there is still some geometry needed. We can find γ and ℓ_6 by looking at the position of G relative to H two ways. One is directly from H to G. The other is going from H to E, then E to B, and then B to G. Writing this as a vector equation we get:

$$\vec{\boldsymbol{r}}_{\text{G/H}} = \ell_6 \begin{bmatrix} \cos(\gamma) \\ \sin(\gamma) \end{bmatrix} = \begin{bmatrix} \ell_3 \cos(\beta) + \ell_2 \cos(\alpha) - \ell_5 \cos(\delta) \\ -\ell_3 \sin(\beta) + \ell_2 \sin(\alpha) - \ell_5 \sin(\delta) \end{bmatrix}.$$

The right hand side has all given quantities. Look at this vector equation as two scalar equations. Square the first and add it to the square of the second gives $\ell_6 \approx 72.85\,\text{cm}$. Applying this to either of the two equations then gives $\gamma \approx 25.97°$.

Because the chair is supported by a hinge at A and a roller at B, there are three scalar support reactions. So, we can determine them from the static analysis of the whole chair frame. The free-body diagram of the chair is shown in fig. 6.65. The moment and force equilibrium equations give

$$\begin{aligned} \sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\ \sum M_A = 0 \quad &\Rightarrow \quad C_y(d_1 + d_2) - F_1(d_1) - F_2(d_3) = 0 \\ &\Rightarrow \quad C_y = \frac{F_1 d_1 + F_2 d_3}{d_1 + d_2} \\ \sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - C_y. \end{aligned}$$

From the given geometry,

$$\begin{aligned} d_1 &\approx 72.44\,\text{cm} \\ d_2 &\approx 38.87\,\text{cm} \\ d_3 &\approx 22.25\,\text{cm}. \end{aligned}$$

Substituting these dimensions above with their numerical values, we get

$$C_y = 365\,\text{N}, \quad \text{and} \quad A_y = 335\,\text{N}.$$

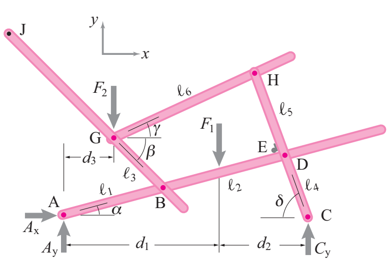

Figure 6.65: Free-body diagram of the whole chair.

Some geometry:

$d_1 = (\ell_1 + \ell_2/2)\cos\alpha$,

$d_2 = (\ell_2/2)\cos(\alpha) + \ell_4 \cos\delta$, and

$d_3 = \ell_1 \cos\alpha - \ell_3 \cos\beta$.

From the law of sines we also have:

$$\ell_4 = \frac{(\ell_1 + \ell_2)\sin(\alpha)}{\sin(\delta)} \approx 28.92\,\text{cm}.$$

The support reactions are thus determined. To find the force on the pin E, we can use either bar ABD or bar CDH. In either case however, we have more unknown forces on the bars than we can determine from the equilibrium equations of that bar alone. So, we will have to use equilibrium of some other bar as well. Note that bar GH is a two-force body. Therefore, the tension in this rod can be shown as a single scalar force R_{GH}. Let us now analyze the equilibrium of bar BGJ since it has only three unknown forces on it (see fig. 6.66). The moment and force equilibrium equations give

$$\begin{aligned}
\sum M_G = 0 \quad &\Rightarrow \quad F_2(\ell_3 \cos\beta) - R_{GH}(\ell_3 \sin(\gamma+\beta)) = 0 \\
&\Rightarrow \quad R_{GH} = \frac{F_2(\cos\beta)}{\sin(\gamma+\beta)} = 150\,\mathrm{N}. \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x = R_{GH}\cos\gamma = 134\,\mathrm{N} \\
\sum F_y = 0 \quad &\Rightarrow \quad B_y = -B_x = -134\,\mathrm{N} \quad (\text{Note } \beta = 45°)
\end{aligned}$$

Now that we know A_x, A_y, B_x and B_y, we can analyze bar ABD and determine the rest of the unknown forces on it including the force in the pin E, R_E (see the free-body diagram in fig. 6.67):

$$\begin{aligned}
\sum M_D = 0 \quad &\Rightarrow \quad -A_y(d_4+d_5) - B_y d_5 + B_x h + F_1 d_6 + R_E \epsilon = 0 \\
&\Rightarrow \quad R_E = \frac{A_y(d_4+d_5) + B_y d_5 - B_x h - F_1 d_6}{\epsilon} \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x + D_x + R_E \sin\alpha = 0 \\
&\Rightarrow \quad D_x = -B_x - R_E \sin\alpha \\
\sum F_y = 0 \quad &\Rightarrow \quad D_y = R_E\cos\alpha - A_y - B_y + F_y.
\end{aligned}$$

From geometry,

$$\begin{aligned}
d_4 &= \ell_1 \cos\alpha \\
d_5 &= \ell_2 \cos\alpha \\
d_6 &= d_5/2 = (\ell_2/2)\cos\alpha \\
h &= \ell_2 \sin\alpha.
\end{aligned}$$

Substituting these variables with their numerical values above, we get

$$R_E \approx 3826\,\mathrm{N}, \qquad D_x \approx -1125\,\mathrm{N}, \qquad \text{and} \qquad D_y \approx 3826\,\mathrm{N}.$$

$$A_x = 0, A_y = 335\,\mathrm{N}, C_y = 365\,\mathrm{N}, R_E = 3826\,\mathrm{N}$$

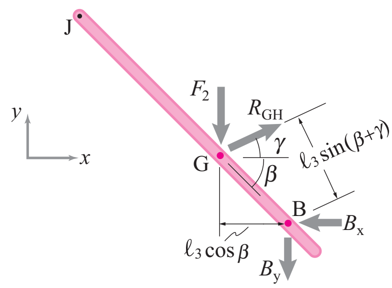

Figure 6.66

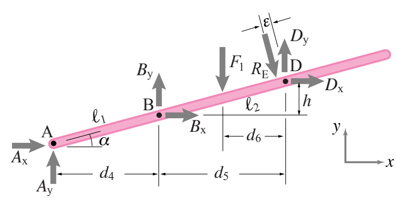

Figure 6.67: Free-body diagram of bar ABD. Note that $d_4 = \ell_1\cos\alpha$, $d_5 = \ell_2\cos\alpha$, $d_6 = d_5/2 = (\ell_2/2)\cos\alpha$, and $h = \ell_2\sin\alpha$.

Check: equilibrium of bar CDH

We can check for consistency by seeing if the equilibrium equations for bar CDH are satisfied, and they are (arithmetic not shown).

$$\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad -D_x - R_E\sin(\alpha) - R_{GH}\cos(\gamma) = 0 \qquad (\text{Checks!}) \\
\sum F_y = 0 \quad &\Rightarrow \quad C_y - D_y + R_E\cos(\alpha) - R_{GH}\sin(\gamma) = 0 \qquad (\text{Checks!}) \\
\sum M_D = 0 \quad &\Rightarrow \quad \ell_4\cos(\delta)C_y - \epsilon R_E + \ell_5\sin(\gamma+\beta)R_{GH} = 0 \qquad (\text{Checks!})
\end{aligned}$$

SAMPLE 6.15 A stack of three cylinders in static equilibrium? Three identical cylinders, each of mass m and radius R, are stacked such that the top cylinder rests on the lower two cylinders. The two cylinders at the bottom do not touch each other. The coefficient of friction, between the cylinders and between the cylinders and the ground, is μ. Find the minimum value of μ so that the three cylinders are in static equilibrium.

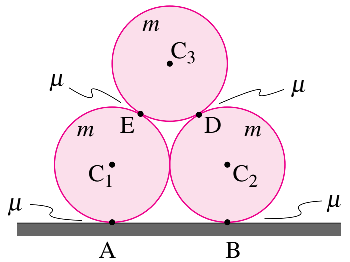

Figure 6.68

Solution We seek the forces for equilibrium. Knowing the forces, we can find the needed friction coefficient μ.

The free-body diagrams of the upper cylinder and the lower right cylinder (why the right cylinder? No particular reason.) are shown in fig. 6.69. The contact forces, $\vec{F}_E$ and $\vec{F}_D$, act on the upper cylinder at points E and D, respectively. Each contact force can be thought of as a force normal N to the contact surface plus a tangential friction force F_s. From the free-body diagrams, we see that each cylinder is a three-force body. Therefore, all the three forces — the two contact forces and the force of gravity — must be concurrent. This requires, for both cylinders, that the two contact forces must intersect on the vertical line passing through the center of the cylinder (the line of action of the force of gravity). Now, if we consider the free-body diagram of the lower right cylinder, we find that force $\vec{F}_D$ has to pass through point B since the other two forces intersect at point B. Thus, we know the direction of force $\vec{F}_D$. Note that ODB is a straight line.

Let ϕ be the angle between the contact force $\vec{F}_D$ and the normal to the cylinder surface at D. Now, from geometry, $\angle C_3DO + \angle C_3OD + OC_3D = 180°$. But, $\phi = \angle C_3DO = \angle C_3OD$. Therefore,

$$\begin{aligned}\phi &= \frac{1}{2}(180° - \angle OC_3D) = \frac{1}{2}(\angle GC_3D)\\ &= \frac{1}{2}30° = 15°\end{aligned}$$

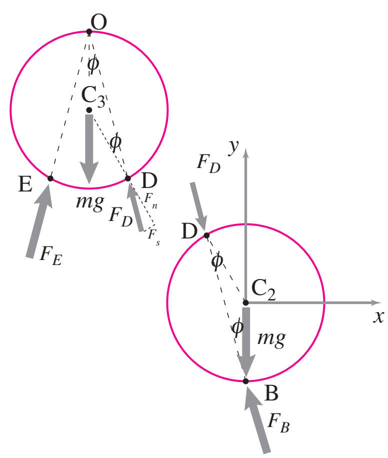

Figure 6.69

where $\angle GC_3D = 30°$ follows from the fact that $C_1C_2C_3$ is an equilateral triangle and C_3G bisects $\angle\, C_1C_3C_2$.

Note that ϕ is the friction angle between the surface normal and the friction force. Somewhat surprisingly, this same ϕ is the friction angle for both for F_D and for F_B. Now, from fig. 6.70, we see that

$$\tan\phi = \frac{F_{sB}}{F_{nB}} = \frac{F_{sD}}{F_{nD}}.$$

But, the force of friction $F_s \leq \mu N$. Therefore,

$$\mu \geq \tan\phi = \tan 15° = 0.27.$$

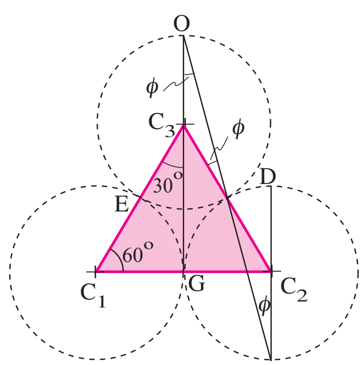

Figure 6.70

The friction coefficient must be at least 0.27 if the three cylinders have to be in static equilibrium.

$$\boxed{\mu \geq 0.27}$$