# 5.5 3D statics

The structures and machines we study are most-often adequately modeled as 2D. In those cases a 2D analysis gives about the same answer as a 3D analysis would have given. However, sometimes a 2D model is inadequate, and a 3D analysis is needed. Here we use 3D statics to find various unknown aspects of forces acting on one part. By learning the 3D approach you can get a better sense of when to use a 2D model (which is most of the time for most engineers).

3D statics is conceptually the same as 2D: draw a free-body diagram and use the force and moment-balance equations. However, the geometry can be more of a challenge, the moment balance equation becomes a full vector equation (instead of just having one non-zero component, it has three)[1], and the number of scalar equations from one free-body diagram increases from 3 to 6. In 3D, issues related to static-indeterminacy arise more often and more subtly.

[1]For 2D problems we used the phrase ‘moment about a point’ to be short for ‘moment about an axis in the $z$ direction that passes through the point. In 3D moment about a point is a 3-component vector.

## The statics-of-a-3D-object recipe

Our recipe here:

1) **Draw a free-body diagram (FBD)** of the part of interest. Use knowledge of the contact conditions (see Chapter 3) to draw known and unknown aspects of the forces appropriately (see fig. 2.42 on page 169) [hint: use of the form $F\hat{\boldsymbol{\lambda}}$ is often appropriate];
2) **Write equilibrium equations** in terms of the forces (and couples) shown on the FBD;
3) **Solve the equilibrium equations** for unknowns.

## The brute-force approach to statically determinate problems

A problem is *statically determinate* when all as-yet-unknown forces can be found using the equilibrium equations. In 3D statics, this generally means that the two vector equilibrium equations

$$\sum \vec{\boldsymbol{F}}_i = \vec{\mathbf{0}} \qquad \text{and} \qquad \sum \vec{\boldsymbol{M}}_{i/C} = \vec{\mathbf{0}}$$

(where C is any point that you like) make up 6 independent scalar equations which you can solve for 6 unknown aspects of the applied forces (say the magnitudes of 6 forces whose directions are known *a priori*)[2].

[2]Most elementary text-book problems are statically determinate. Unfortunately most real-world problems, when you first model them, are not statically determinate.

## Alternative equation sets

In 2D single-part statics, we noted various alternatives to using vector force balance and moment about one point (see page 257). Similarly, here there are also an infinite number of true equilibrium equations, for example,

- $\left(\sum \vec{F}_i\right) \cdot \hat{\lambda} = 0$ where $\hat{\lambda}$ is a vector in any direction you please; and
- $\left(\sum \vec{M}_{/C}\right) \cdot \hat{\lambda} = 0$. This is moment balance about an axis through C in the $\hat{\lambda}$ direction.

From these there are various ways to extract 6 independent scalar equations, including:

- Cartesian components of force balance and moment balance about any point C: $\sum F_x = 0, \sum F_y = 0, \sum F_z = 0, \sum M_{Cx} = 0, \sum M_{Cy} = 0$, and $\sum M_{Cz} = 0$. This always works, although it does not necessarily minimize algebra.
- Force balance in any 3 non-coplanar directions and moment balance about point C resolved in any three non-coplanar directions.
- Moment balance about 6 independent axes. There seems to be no simple description of independent axes but that they give independent equilibrium equations ③. Practically speaking, six moment-about-an-axis equations are likely to be independent if not too many axes are parallel with each other, not too many are coplanar, and not too many intersect at one point.

③ One can test the sufficiency of the equations by this check: if a force at the origin and a couple are the only forces applied to a system, do the equations demand that they must both be zero? If so, then you have an independent set of equations.

In any case, force balance contributes at most 3 independent equations and moment balance can contribute up to 6 (thus rendering force balance a non-essential tool).

Solving 6 equations in 6 unknowns, or even setting up such for computer solution, is relatively time-consuming and error-prone. Thus one looks for shortcuts when one can, namely:

Useful shortcuts:

- Use moment balance about an axis that intersects, or is parallel to, as many unknown force lines-of-action as possible (thus those forces do not show up in that equilibrium equation);
- Use force balance in a direction orthogonal to as many of the unknown forces as possible (so those forces don't show up in that equation).

# Special loadings

## Two- and three-force bodies

The concepts of two-force (page 258) and three-force (page 260) bodies are identical in 3D.

- If there are only two forces applied to a body in equilibrium, they must be equal and opposite and acting along the line connecting the points of application. The full set of six equations tells you no more.
- If there are only three forces applied to an object, they must all be in the plane of the points of application and the three forces must have lines of action that intersect at one point. The three equations of force balance are an additional restriction on these three forces.

There are other special loadings where the equilibrium equations offer less than 6 independent equations:

- **2D**. If all of the forces have **lines of action in one plane**, then there are only three independent scalar equations and thus one can solve for 3 unknowns. For example, if all the forces lie in the $xy$ plane, then automatically $\sum F_{iz} = 0$, $\sum M_{ix/C} = 0$, and $\sum M_{iy/C} = 0$.
- **Concurrent forces**. If all the lines of action intersect in one point, say D, then $\sum \vec{\boldsymbol{M}}_{/\mathrm{D}} = \vec{\mathbf{0}}$ is automatically satisfied and only the 3 equations of force balance are independent.
- If all the forces are **parallel** in, say, the $\hat{\boldsymbol{k}}$ direction, then force balance in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions as well as moment balance about any axis in the $\hat{\boldsymbol{k}}$ direction, is automatically satisfied and there are only three independent equilibrium equations (say $\sum F_z = 0$, $\sum M_x = 0$ and $\sum M_y = 0$.

## What does it mean for a problem to be '2D'?

The world we live in is three-dimensional: all the objects we wish to study mechanically are three-dimensional, and if they are in equilibrium, they satisfy the three-dimensional equilibrium equations. How then can an engineer justify doing 2D mechanics? There are a variety of overlapping justifications.

- The 2D equilibrium equations are a subset of the 3D equations. In both 2D and 3D, $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \vec{\boldsymbol{M}}_{/0} \cdot \hat{\boldsymbol{k}} = 0$. So, if when doing 2D mechanics, one just neglects the $z$ component of any applied forces and the $x$ and $y$ components of any applied couples, one is doing correct 3D mechanics, just not all of 3D mechanics. If the forces or conditions of interest to you are contained in the 2D equilibrium equations, then 2D mechanics is really 3D mechanics, ignoring equations you don't need.
- If the $xy$ plane is a plane of symmetry for the object and of any applied loading, then the 3D equilibrium equations not covered by the

2D equations are automatically satisfied. For a car, say, the assumption of symmetry implies that the forces in the $z$ direction will automatically add to zero, and the moments about the $x$ and $y$ axis will automatically be zero.

- If the object is thin and there are constraint forces holding it near the $xy$ plane, and these constraint forces are not of interest, then 2D statics is also appropriate. This last case is caricatured by all the poor mechanical objects you have drawn so. They are conceptually constrained to lie in your flat paper by invisible slippery glass in front of and behind the paper.

## "Internal forces" in 3D

At a free-body-diagram cut on a long narrow structural piece in 2D, we have two force components (tension and shear) and one scalar moment. In 3D, such a cut shows a force $\vec{\boldsymbol{F}}$ and a moment $\vec{\boldsymbol{M}}$ each with three components. If one picks a coordinate system with the $x$ axis aligned with the bar at the cut, the concept of tension remains the same. Tension is the force component along the bar.

$$T = F_x = \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{\imath}}.$$

The two other force components, $F_y$ and $F_z$, are two components of shear. The net shear force is a vector in the plane orthogonal to $\hat{\boldsymbol{\imath}}$.

The new concept, often called *torsion* is the component of $\vec{\boldsymbol{M}}$ along the axis:

$$\text{torsion} = M_x = \vec{\boldsymbol{M}} \cdot \hat{\boldsymbol{\imath}}$$

Torsion is the part of the moment that twists the shaft.

The remaining part of the $\vec{\boldsymbol{M}}$, in the $yz$ plane, is the bending moment. It has two components $M_y$ and $M_z$.

## The preponderance of statically indeterminate problems

As noted in box 5.1 on page 247, non-uniqueness of solutions is common in the real world. Especially in 3D, real world problems are, at first blush, rarely statically determinate. The statics equations are relevant and provide useful information, but are not sufficient for finding all unknowns of interest. Finding the forces then depends on knowing the deformation properties of the structures as well as details of their initial state.

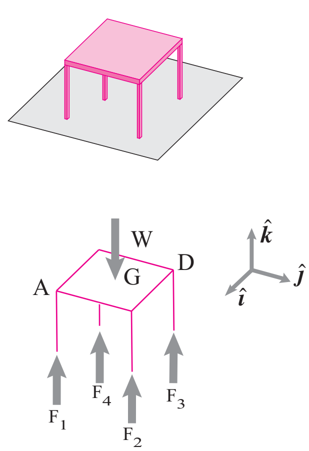



Figure 5.78: A symmetric 4 leg table on a flat floor. Statics is insufficient to find the 4 forces of the ground on the table. The full weight could be carried by either diagonal pair of legs.

**Example: Four-leg furniture**

Take the table, chair or bed you are now interacting with. It probably has 4 legs. To keep it simple, imagine the legs are on a slippery (negligible-friction) floor and the table is symmetric (left-right and front-back). What are the forces of the floor on the legs? The most we can get from the statics equations (force and moment balance) are that

$$F_1 = F_3, \quad F_2 = F_4, \qquad \text{and} \qquad F_1 + F_2 = W/2.$$

If we insist that there is no glue between the floor and table then $F_1 \geq 0, F_2 \geq 0, F_3 \geq 0, F_4 \geq 0$. But we still can't find the reactions. Here is the variety of solutions:

| | | | |
|---|---|---|---|
| | $F_1 = F_3 = W/2$ | and | $F_2 = F_4 = 0$ |
| or | $F_1 = F_3 = 0$ | and | $F_2 = F_4 = W/2$ |
| or | $F_1 = F_3 = W/4$ | and | $F_2 = F_4 = W/4$ |
| or | $F_1 = F_3 = C$ | and | $F_2 = F_4 = W/2 - C$ |
| | | | (with $C$ anything in the interval $0 \leq C \leq W/2$). |

It takes more than just statics to find the forces. One has to know the exact initial shape of the table and floor, and how the table and floor 'give way' in response to loads.

The lack of static determinacy of a table is not merely an academic curiosity. If you measured the forces of the floor on your table legs, they each could well differ noticeably from $W/4$, with, probably, two bigger and two smaller. Once friction is taken into account, the situation is near hopeless.

**Example: Statically determinate stool**
Is it even possible to make a stool in 3 dimensions that is statically determinate? Here's one way. Give it three legs. One leg can have a point frictional contact (3 reaction components), one leg can have a wheel (2 reaction components) and one can be frictionless (like with a castered wheel, 1 reaction component). $3 + 2 + 1 = 6$.

In general, it is hard to hold an object in place in three dimensions in a statically determinate manner. Here are some other ways (besides the unusual stool above):

- with six rods that have ball-and-socket joints at both the object-end and at the ground-end. The rods need to have a variety of orientations and attachment points (this idea is used in a 'Stewart Platform').
- With one ball-and-socket joint and three rods.
- A 3-leg stool with three wheels (at the contact points. For each wheel one can draw a line in the plane in the direction normal to rolling. These three lines must not intersect at a point).
- With one hinge and one two-force-member rod.
- With one axially sliding hinge and two rods.
- With a single welded connection.

Given that many things are held in place in a manner that seems statically indeterminate, what can one do in practice? A common approach is to remove reaction components that you think are relatively unimportant. Some examples:

- A door held by two hinges. That's 10 reaction components. Usually one replaces, in the analysis, the hinges with ball-and-socket joints. That makes 6 unknown reaction components but is still statically indeterminate no matter what the loading (the force along the line connecting the joints cannot be decomposed into parts acting at each joint). So one joint is allowed to slide along the nominal hinge axis.

- 4-leg furniture. Counting friction, there are 12 reaction components. If side loads are not an issue, then we can assume-away friction. Thus we have only 4 reaction components for 3 equations (see table example above). We can get a unique solution by assuming the forces share the symmetry of the table (thus $F_1 = F_2$).

Given this complex state of affairs in 3D, it is easy to see why engineers often resort to the more-easily-made determinate 2D world for their models and analyses.

# 5.8 Statically determinate ways to hold an object in 3D

An advanced aside.

It takes some thought to find ways to hold something in place in 3D that are statically determinate. If you hold it securely enough so that it can't move you will often do it in a way that there can be reaction forces even when there are no loads on the structure. These 'locked-in' forces are sometimes called 'pre-stresses', 'pre-loads' or 'self-stresses'. The values of these locked-in forces will depend on the construction. You can't find them from the equations of statics. Hence the indeterminacy.

Below we show some ways to hold things in a statically determinate way. Static determinacy is great for homework problems. Whether it is good or bad in structural design is not a simple question.

**The big tradeoff.** Before showing how to achieve determinacy, a warning.

> A statically determinate structure has no redundancy.
> A redundant structure can have locked-in forces.

And locked-in forces can contribute to structural failure. Two parallel rods holding something in place can break because they fight each other, even when there is no external load on the thing.

**Counting rules.** Counting dominates the discussion. Here's how to do that counting. The rule of thumb for determinacy is:

$$\underbrace{\text{Number of equations}}_{n_{eq}} = \underbrace{\text{number of unknowns}}_{n_{unkn}}$$

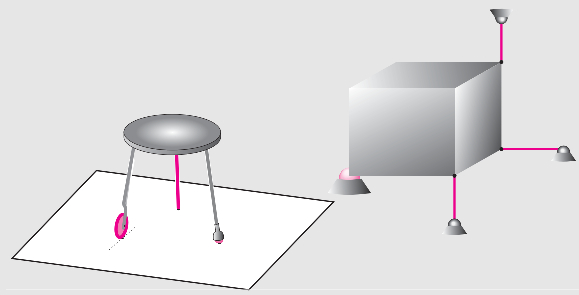
For one object in 3 dimensions we have 6 independent equilibrium equations, for example 3 components of force balance and 3 components of moment balance about a given reference point. Thus $n_{eq} = 6$. To achieve determinacy we thus want $n_{unkn} = 6$. That is, in a free-body diagram of the object we also want 6 unknown reaction components.

**Attachment schemes.** For each attachment point we can select an attachment type from the table of common connections on page 1219. Here's a compact list with the number (#) of unknown reaction components:

| 3D Connection | # | what |
|---|---|---|
| weld | 6 | 3 moment and 3 force comps |
| keyed slot | 5 | 3 moment & 2 force comps |
| hinge | 5 | 3 moment & 2 force comps |
| linear bearing | 4 | 2 moment & 2 force comps |
| ball &socket | 3 | 3 force components |
| no-slip point contact | 3 | 3 force comps |
| skate or wheel | 2 | normal force and side force |
| sloppy linear bearing | 2 | 2 force components |
| rod or string | 1 | tension along 2-force member |
| frictionless point contact | 1 | normal force |

**Mix and match.** At least to satisfy the counting rules, you can hold something in place in a statically determinate manner by selecting any number of connections from the table above just make the number of unknown components add to 6. Here are some options:

| Scheme | $\sum = 6$ |
|---|---|
| * One weld | 6=6 |
| * Hinge+rod | 5+1=6 |
| * Ball & socket + sloppy linear bearing + rod | 3+2 + 1 =6 |
| * 6 rods | 1+1+1+1+1+1=6 |
| * Ball & socket + 3 rods | 3 +1+1+1=6 |
| * No-slip point contact + wheel + frictionless point contact | 3 + 2+1 =6 |
| * 3 wheels or skates | 2+2+2 = 6 |
| * Etc | n + m + ... = 6 |



**A statically determinate stool** has one normal leg (with friction), one wheel and one frictionless leg (shown as a ball). **A box** is held in a statically determinate way with a ball & socket and three bars.

**Bad luck.** Although the rule

$$n_{eq} = n_{unkn}$$

is a good starting point, it is not enough. In the bad cases, two things happen together:

1) The object is not held in place (some loads can not be equilibrated), and
2) The constraints can fight each other.

Then, in the language of linear algebra, the $6 \times 6$ matrix for the linear equilibrium equations is singular. So, for some loadings no solution exists (the object is not held in place). And if a solution exists, it is not unique (the homogeneous equations have non-trivial solutions, there can be locked-in forces).

**Bad examples.** Some ways of holding where 6 reaction forces lead to *indeterminacy* include:

- A hinge + rod with the rod co-planar with the hinge.
- 6 rods where some 4 or more of them have lines of action that intersect in one point.
- A ball & socket with 3 rods where one or more of the rods has a line of action that goes through the ball & socket.
- A ball & socket with two rods that are coplanar with the ball.
- Three wheels on a plane where the normal-lines to the rolling direction all intersect in a single point (the wheels roll on a common circle).

**SAMPLE 5.20 3-D moment at the support:** A 'T'-shaped cantilever beam is loaded as shown in the figure. Find all the support reactions at A.

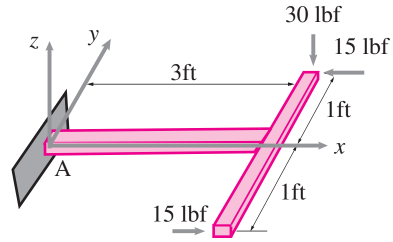



Figure 5.79

**Solution** The free-body diagram of the beam is shown in Fig. 5.80. Note that the forces acting on the beam can produce in-plane as well as out-of-plane moments. Therefore, we show the unknown reactions $\vec{R}$ and $\vec{M}_A$ as general 3-D vectors at A. The moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, gives

$$\vec{M}_A + \vec{r}_{C/A} \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_{D/A} \times \vec{F}_3 = \vec{0}.$$

$$\begin{aligned}
\Rightarrow \quad \vec{M}_A &= -(\vec{r}_{B/A} + \vec{r}_{C/B}) \times (\vec{F}_1 + \vec{F}_2) - (\vec{r}_{B/A} + \vec{r}_{D/B}) \times \vec{F}_3 \\
&= -(\ell \hat{\imath} + a \hat{\jmath}) \times (-F_1 \hat{k} - F_2 \hat{\imath}) - (\ell \hat{\imath} - a \hat{\jmath}) \times F_3 \hat{\imath}.
\end{aligned}$$

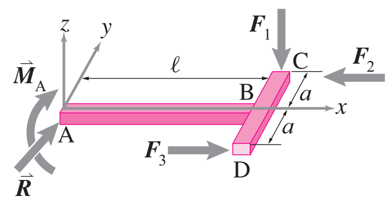



Figure 5.80: Free-body diagram of the cantilever.

But $F_3 = F_2 = F$ (say). Therefore,

$$\begin{aligned}
&= -(\ell \hat{\imath} + a \hat{\jmath}) \times (-F_1 \hat{k} - F \hat{\imath}) - (\ell \hat{\imath} - a \hat{\jmath}) \times F \hat{\imath} \\
&= -F_1 \ell \hat{\jmath} + F_1 a \hat{\imath} - 2Fa\hat{k} \\
&= -30\,\text{lbf} \cdot 3\,\text{ft}\,\hat{\jmath} + 30\,\text{lbf} \cdot 1\,\text{ft}\,\hat{\imath} - 2(15\,\text{lbf} \cdot 1\,\text{ft})\hat{k} \\
&= (30\hat{\imath} - 90\hat{\jmath} - 30\hat{k})\,\text{lbf} \cdot \text{ft}.
\end{aligned}$$

Force equilibrium, $\sum \vec{F} = \vec{R} + \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$, gives

$$\begin{aligned}
\Rightarrow \quad \vec{R} &= -\vec{F}_1 - \vec{F}_2 - \vec{F}_3 \\
&= -\vec{F}_1 - \vec{F} + \vec{F} \\
&= -(-F_1 \hat{k}) = F_1 \hat{k} \\
&= 30\,\text{lbf}\,\hat{k}.
\end{aligned}$$

$$\vec{A} = 30\,\text{lbf}\,\hat{k}, \qquad \text{and} \qquad \vec{M}_A = (30\hat{\imath} - 90\hat{\jmath} - 30\hat{k})\,\text{lbf} \cdot \text{ft}$$

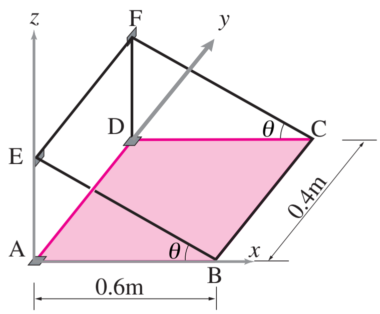



Figure 5.81

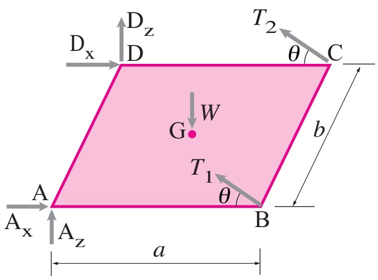



Figure 5.82

**SAMPLE 5.21 An unsolvable problem?** A $0.6\,\mathrm{m} \times 0.4\,\mathrm{m}$ uniform rectangular plate of mass $m = 4\,\mathrm{kg}$ is held horizontally with two strings BE and CF and linear hinges at A and D as shown in the figure. The plate is loaded uniformly with books of total mass 6 kg. If the maximum tension the strings can hold is 100 N, how much more load can be applied to the plate?

**Solution** The free-body diagram of the plate is shown in fig. 5.82. Note that we model the hinges at A and D with no resistance in the $y$-direction. Since the plate has uniformly distributed load (including its own weight), we replace the distributed load with an equivalent concentrated load $\vec{\boldsymbol{W}}$ acting vertically through point G.

The various forces acting on the plate are

$$\vec{\boldsymbol{W}} = -W\hat{\boldsymbol{k}},\ \vec{\boldsymbol{T}}_1 = T_1\hat{\boldsymbol{\lambda}}_{\mathrm{BE}},\ \vec{\boldsymbol{T}}_2 = T_2\hat{\boldsymbol{\lambda}}_{\mathrm{CF}},\ \vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{\imath}} + A_z\hat{\boldsymbol{k}},\ \vec{\boldsymbol{D}} = D_x\hat{\boldsymbol{\imath}} + D_z\hat{\boldsymbol{k}}.$$

Here, $\hat{\boldsymbol{\lambda}}_{\mathrm{BE}} = \hat{\boldsymbol{\lambda}}_{\mathrm{CF}} = -\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{k}} = \hat{\boldsymbol{\lambda}}$(let). Now, we apply moment equilibrium about point A, *i.e.*, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\boldsymbol{0}}$.

$$\vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{r}}_{\mathrm{C}} \times \vec{\boldsymbol{T}}_2 + \vec{\boldsymbol{r}}_{\mathrm{G}} \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{r}}_{\mathrm{D}} \times \vec{\boldsymbol{D}} = \vec{\boldsymbol{0}} \tag{5.36}$$

where,

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{T}}_1 &= a\hat{\boldsymbol{\imath}} \times T_1\hat{\boldsymbol{\lambda}} = -aT_1\sin\theta\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{\mathrm{C}} \times \vec{\boldsymbol{T}}_2 &= (a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) \times T_2\hat{\boldsymbol{\lambda}} = T_2 b\sin\theta\hat{\boldsymbol{\imath}} - T_2 a\sin\theta\hat{\boldsymbol{\jmath}} + T_2 b\cos\theta\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{r}}_{\mathrm{G}} \times \vec{\boldsymbol{W}} &= \frac{1}{2}(a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) \times (-W\hat{\boldsymbol{k}}) = -\frac{Wa}{2}\hat{\boldsymbol{\imath}} + \frac{Wa}{2}\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{\mathrm{D}} \times \vec{\boldsymbol{D}} &= b\hat{\boldsymbol{\jmath}} \times (D_x\hat{\boldsymbol{\imath}} + D_z\hat{\boldsymbol{k}}) = D_z b\hat{\boldsymbol{\imath}} - D_x b\hat{\boldsymbol{k}}.
\end{aligned}$$

Substituting these products in eqn. (5.36) and dotting with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, we get

$$T_2\sin\theta + D_z = \frac{W}{2} \tag{5.37}$$

$$T_2\cos\theta - D_x = 0 \tag{5.38}$$

$$(T_1 + T_2)\sin\theta = \frac{W}{2}. \tag{5.39}$$

The force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{D}} + \vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{T}}_2 + \vec{\boldsymbol{W}} = \vec{\boldsymbol{0}}.$$

Again, substituting the forces in their component form and dotting with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{k}}$ (there are no $\hat{\boldsymbol{\jmath}}$ components), we get

$$\begin{aligned}
A_x + D_x - (T_1 + T_2)\cos\theta &= 0 \\
\Rightarrow \quad A_x - T_1\cos\theta &= 0 \qquad (5.40) \\
A_z + D_z + (T_1 + T_2)\sin\theta &= 0 \\
\Rightarrow \quad A_z + T_1\sin\theta &= \frac{W}{2}. \qquad (5.41)
\end{aligned}$$

These are all the equations that we can get. Now, note that we have five independent equations (eqns. (5.37) to (5.41)) but six unknowns. Thus we cannot solve for the unknowns uniquely. This is an indeterminate structure! No matter which point we use for our moment equilibrium equation, we will always have one more unknown than the number of independent equations. We can, however, solve the problem with an extra assumption (see comments below) — the structure is symmetric about the axis passing through G and parallel to $x$-axis. From this symmetry we conclude that $T_1 = T_2$. Then, from eqn. (5.40) we have

$$2T\sin\theta = \frac{W}{2} \quad \Rightarrow \quad T = \frac{W}{4\sin\theta}.$$

We can now find the maximum load that the plate can take subject to the maximum allowable tension in the strings.

$$\begin{aligned} W &= 4T\sin\theta \\ \Rightarrow \quad W_{max} &= 4T_{max}\sin\theta \\ &= 4(100\,\mathrm{N})\cdot\frac{1}{2} = 200\,\mathrm{N}. \end{aligned}$$

The total load as given is $(6+4)\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2} = 98.1\,\mathrm{N} \approx 100\,\mathrm{N}$. Thus we can double the load before the strings reach their break-points. Now the reactions at D and A follow from eqns. (5.37), (5.38), (5.40), and (5.41).

$$\begin{aligned} D_z = A_z = \frac{W}{2} - T\sin\theta &= \frac{W}{2} \\ D_x = A_x = T\cos\theta &= \frac{W}{4}\cot\theta. \end{aligned}$$

$$\boxed{W_{max} = 200\,\mathrm{N}}$$

**Comments:**

1. We got only five independent equations (instead of the usual 6) because the force equilibrium in the $y$-direction gives a zero identity $(0 = 0)$. There are no forces in the $y$-direction. The structure seems to be free in the $y$-direction — if you push a little, it will move. This freedom to move in the $y$-direction arose because we chose to model the hinges at A and D as allowing sliding, keeping in mind only the vertical loading. The actual hinges used on a bookshelf will not allow movement in the $y$-direction either. If we model the hinges as ball and socket joints, we introduce two more unknowns, one at each joint, and get just one more scalar equation. Thus we are back to square one. There is no way to determine $A_y$ and $D_y$ from equilibrium equations alone.
2. The assumption of symmetry and the consequent assumption of equality of the two string tensions is, mathematically, an extra independent equation based on deformations (strength of materials). At this point, you may not know any strength of material calculations or deformation theory, but your intuition is likely to lead you to make the same assumption. Note, however, that this assumption is sensitive to accuracy in fabrication of the structure. If the strings were slightly different in length, the angles were slightly off, or the wall was not perfectly vertical, the symmetry argument would not hold and the two tensions would not be the same.

Most real problems are like this — indeterminate. Our modelling, if good, makes them determinate, solvable and useful.

# 5.5 Advanced statics

## Preparatory Problems

**5.5.1** In 2D, the force balance and moment-balance equations for equilibrium of a body give three independent scalar equations that can be used to solve for three unknowns. How many independent scalar equations can you get from force and moment balance in 3D? Write down a set of such equations.

**5.5.2** In 3D, how many independent scalar equations can you write for equilibrium of a particle?

**5.5.3** How is moment-balance equation about an axis different from moment balance about a point? Illustrate your answer with an example.

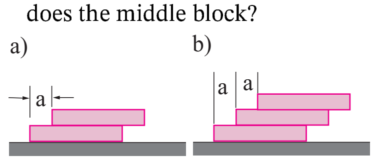
**5.5.4** How many independent scalar equations of equilibrium can you get by writing moment-balance equations about different lines or axes in 3D?

## More-Involved Problems

**5.5.5** Assume identical uniform rigid blocks with weight $W = 1\,\text{N}$, height $h = 1\,\text{cm}$, and length $\ell = 10\,\text{cm}$ are put one on top of the other. Assume there is no glue so blocks can only push against each other.

a) For two blocks what is the biggest overhang $a$ so that the top block does not tip over?

b) For three blocks what is the biggest total overhang $= 2a$ (the same overhang $a$ at each layer) so that the top block doesn't tip, nor does the middle block?




Problem 5.5.5

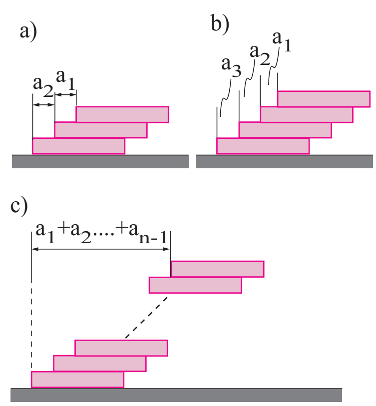
**5.5.6** See simpler problem 5.5.5. Stacking identical rigid blocks, one on top of the other, one wants to get the biggest overhang possible without the tower toppling. Each block has, say, $W = 1\,\text{N}$, height $h = 1\,\text{cm}$, and length $\ell = 10\,\text{cm}$.

a) For three blocks find the biggest $a_1$ and $a_2$ so there is no toppling. [First put the top block as far to the right as you can, $a_1$, for no toppling. Then put that pair as far to the right as possible for no toppling over the bottom block.] The total overhang is $a_1 + a_2$.

b) For 4 blocks find the largest possible overhang $a_1 + a_2 + a_3$ by placing the tower of three above as far to the right as possible relative to the bottom block. [Note that you place the center of mass of the top 3 blocks over the right edge of the fourth bottom block].

c) For $n$ blocks what is the biggest possible overhang $(= a_1 + a_2 + a_3 + \cdots + a_{n-1})$?

d) Using blocks with length $\ell = 10\,\text{cm}$ how many blocks $n$ are needed to get an overhang of 1 m? 2 m?




Problem 5.5.6

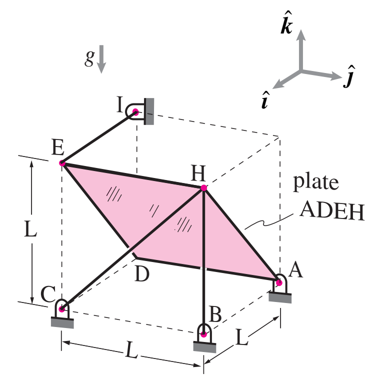
**5.5.7** Uniform plate ADEH with mass $m$ is connected to the ground with a ball and socket joint at A. It is also held by three massless bars (IE, CH and BH) that have ball and socket joints at each end, one end at the rigid ground (at I, C and B) and one end on the plate (at E and H).
In terms of some or all of $m$, $g$, and $L$ find

a) the reaction at A (the force of the ground on the plate),

b) $T_{\text{IE}}$,

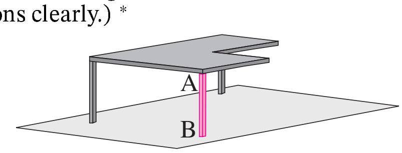
c) $T_{\text{CH}}$,

d) $T_{\text{BH}}$.



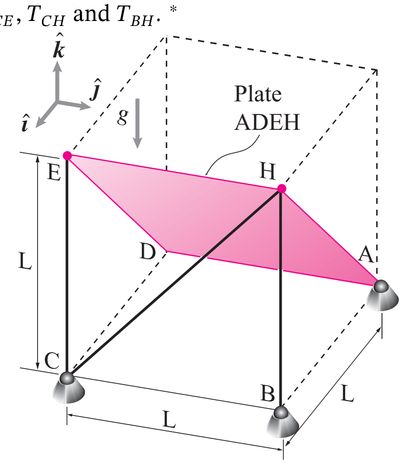

Problem 5.5.7

**5.5.8** An 80 kg square table has one quarter cut away. The remaining 60 kg are supported on 3 massless legs on a level floor. Use g = 10N/kg. What is the load carried by leg AB? (State your assumptions clearly.) *



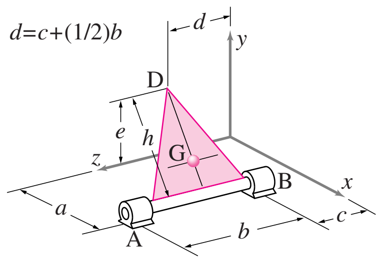

Problem 5.5.8

**5.5.9** Uniform plate ADEH with mass $m$ is connected to the ground with a ball and socket joint at A. It is also held by three massless bars (CE, CH and BH) that have ball and socket joints at each end, one end at the rigid ground (at C and B) and one end on the plate (at E and H). In terms of some or all of $m$, $g$, and $L$ find the reaction at A (the force of the ground on the plate) and the three bar tensions $T_{CE}$, $T_{CH}$ and $T_{BH}$. *




Problem 5.5.9

**5.5.10** A massless triangular plate rests against a frictionless wall at point $D$ and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor. A ball of mass $m$ is fixed to the centroid of the plate. There is gravity and the system is at rest. What is the reaction at point $D$ on the plate?

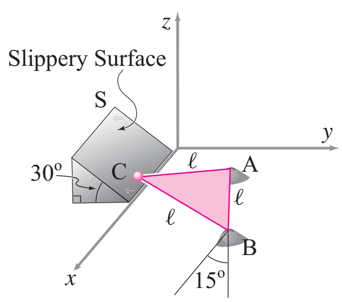



Problem 5.5.10

**5.5.11** A uniform equilateral triangular plate with weight $W = 1000N$ and sides $\ell = 2\,\text{m}$ rests against a slippery plane S. Point C is negligibly above the $xy$ plane. The bottom edge of the triangle has ball-and-socket joints at A and B, with the line AB on the $xy$ plane making an angle of $15^\circ$ with the $x$ direction.

a) Find the reaction at C

b) Find all you can about the reactions at A and B.




Problem 5.5.11

**5.5.12** A uniform 5 kg shelf is supported at one corner with a ball and socket joint and the other three corners with strings. At the instant of interest the shelf is at rest. Gravity acts in the $-\hat{\boldsymbol{k}}$ direction. The shelf is in the $xy$ plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try to find this force. If you can't, then come back to this question after you have done all the calculations.

c) Write down the equation of force equilibrium.

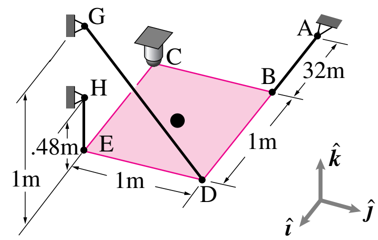
d) Write down the moment-balance equation using the center of mass as a reference point.

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns.

f) Solve these equations by hand or on the computer.

g) Instead of using a system of equations try to find a single equation which can be solved for $T_{EH}$. Solve it and compare to your result from before. *

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try moment balance about an appropriate axis as well as force balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (f)? Do they?




Problem 5.5.12

**5.5.13** The sign is held up by 6 rods. Find the tension in bars

a) BH *

b) EB *

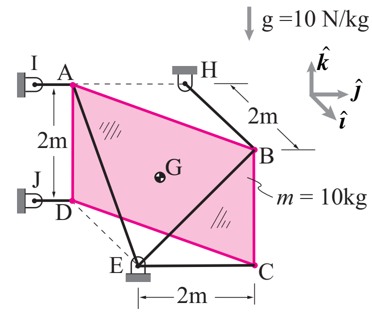
c) AE *

d) IA *

e) JD *

f) EC *

[One game you can play is to see how many of the tensions you can find *without* knowing any of the others. Another approach is to set up and solve 6 equations in 6 unknowns.]



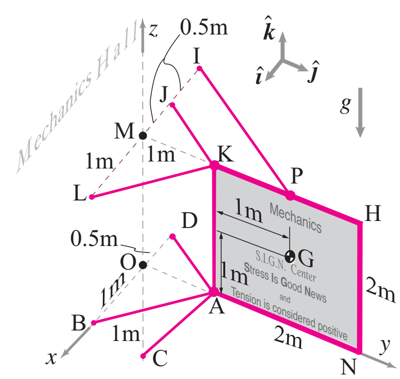

Problem 5.5.13

**5.5.14** The 100 kg, 2 m square, uniform sign KHNA is held up by 6 bars.
**Structure and geometry clarifications:** The sign is held vertically, 1m in front of, and orthogonal-to a vertical wall. Each bar holding the sign has a ball-and-socket joint both where it attaches to the sign and where it attaches to the wall. The points L, M, J, I, K, P and H lie in the same horizontal plane that includes the top edge of the sign. The points M, O, and C lie on a vertical line that is coplanar with the sign. Points B, O, D, A, and N lie in a horizontal plane shared with the bottom edge of the sign. The center of mass of the sign is at G. $g = 10\text{N/kg}$.

a) Find the "bar force" in bar AC.
[hint: $\Sigma F_z = \{\Sigma\vec{\boldsymbol{F}}\}\cdot\hat{\boldsymbol{k}} = 0$]. *

b) Find the "bar force" in bar IP.
[hint: $\Sigma M_{AK} = \{\Sigma\vec{\boldsymbol{M}}_{/A}\}\cdot\hat{\boldsymbol{k}} = 0$]. *

c) Find the "bar force" in bar KL. *




Problem 5.5.14

**5.5.15** Below is a highly schematic picture of a tricycle. The wheels are at C, B and A. The person-trike system has center of mass at G directly over the rear axle. The wheels at C and A are good free-turning, high friction wheels. The wheel at B is in a small ditch and can't move. Assume no slip and that $F, m, g, w, \ell$, and $h$ are given.

a) Of the 9 possible reaction components at A, B, and C, which do you know are zero *a priori*.

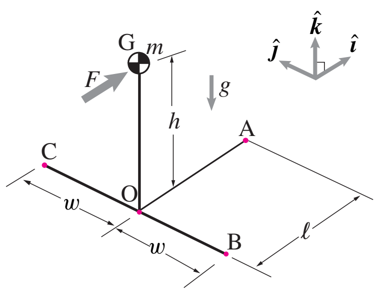
b) Find all the reaction components (the full reaction force) at A.

c) Find the vertical component of the reaction at C.

d) Find the $x$ and $z$ reaction components at B.

e) Find the sum of the $y$ components of the reactions at B and C.

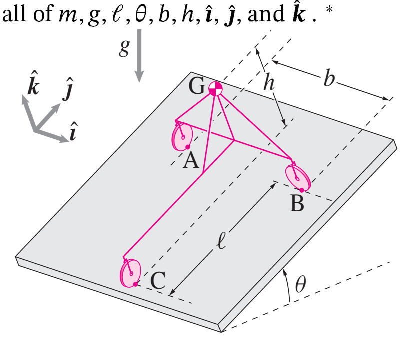
f) Can you find the $y$ component of the reaction at C? Why or why not?




Problem 5.5.15

**5.5.16** A 3-wheeled robot with mass $m$ is parked on a hill with slope $\theta$. The ideal massless robot wheels are free to roll but not to slip sideways. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\boldsymbol{\jmath}}$ direction and the wheel at B is free to roll in the $\hat{\boldsymbol{\imath}}$ direction. The center of mass of the robot at G is $h$ above (normal to the slope) the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance $b$ apart. The length of the robot is $\ell$.

Find the force vector $\vec{\boldsymbol{F}}_A$ of the ground on the robot at A in terms of some or all of $m, g, \ell, \theta, b, h, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. *




Problem 5.5.16