SAMPLE 5.20 **3-D moment at the support:** A 'T'-shaped cantilever beam is loaded as shown in the figure. Find all the support reactions at A.

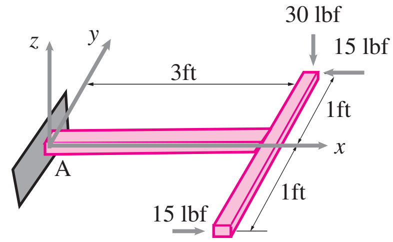

Figure 5.79

Solution The free-body diagram of the beam is shown in Fig. 5.80. Note that the forces acting on the beam can produce in-plane as well as out-of-plane moments. Therefore, we show the unknown reactions $\vec{R}$ and $\vec{M}_A$ as general 3-D vectors at A. The moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, gives

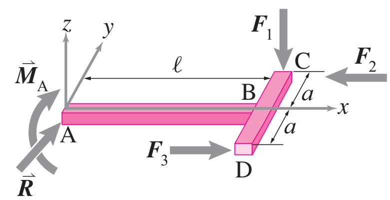

Figure 5.80: Free-body diagram of the cantilever.

$$\vec{M}_A + \vec{r}_{C/A} \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_{D/A} \times \vec{F}_3 = \vec{0}.$$

$$\begin{aligned} \Rightarrow \quad \vec{M}_A &= -(\vec{r}_{B/A} + \vec{r}_{C/B}) \times (\vec{F}_1 + \vec{F}_2) - (\vec{r}_{B/A} + \vec{r}_{D/B}) \times \vec{F}_3 \\ &= -(\ell\hat{\imath} + a\hat{\jmath}) \times (-F_1\hat{k} - F_2\hat{\imath}) - (\ell\hat{\imath} - a\hat{\jmath}) \times F_3\hat{\imath}. \end{aligned}$$

But $F_3 = F_2 = F$ (say). Therefore,

$$\begin{aligned} &= -(\ell\hat{\imath} + a\hat{\jmath}) \times (-F_1\hat{k} - F\hat{\imath}) - (\ell\hat{\imath} - a\hat{\jmath}) \times F\hat{\imath} \\ &= -F_1\ell\hat{\jmath} + F_1 a\hat{\imath} - 2Fa\hat{k} \\ &= -30\,\text{lbf} \cdot 3\,\text{ft}\,\hat{\jmath} + 30\,\text{lbf} \cdot 1\,\text{ft}\,\hat{\imath} - 2(15\,\text{lbf} \cdot 1\,\text{ft})\hat{k} \\ &= (30\hat{\imath} - 90\hat{\jmath} - 30\hat{k})\,\text{lbf} \cdot \text{ft}. \end{aligned}$$

Force equilibrium, $\sum \vec{F} = \vec{R} + \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$, gives

$$\begin{aligned} \Rightarrow \quad \vec{R} &= -\vec{F}_1 - \vec{F}_2 - \vec{F}_3 \\ &= -\vec{F}_1 - \vec{F} + \vec{F} \\ &= -(-F_1\hat{k}) = F_1\hat{k} \\ &= 30\,\text{lbf}\,\hat{k}. \end{aligned}$$

$$\boxed{\vec{A} = 30\,\text{lbf}\,\hat{k}, \qquad \text{and} \qquad \vec{M}_A = (30\hat{\imath} - 90\hat{\jmath} - 30\hat{k})\,\text{lbf} \cdot \text{ft}}$$

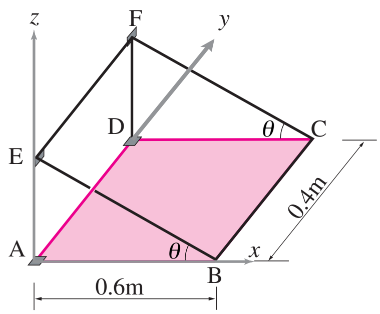

Figure 5.81

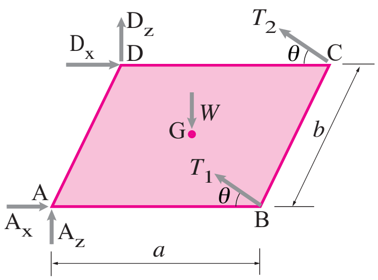

Figure 5.82

SAMPLE 5.21 An unsolvable problem? A $0.6\,\mathrm{m} \times 0.4\,\mathrm{m}$ uniform rectangular plate of mass $m = 4\,\mathrm{kg}$ is held horizontally with two strings BE and CF and linear hinges at A and D as shown in the figure. The plate is loaded uniformly with books of total mass 6 kg. If the maximum tension the strings can hold is 100 N, how much more load can be applied to the plate?

Solution The free-body diagram of the plate is shown in fig. 5.82. Note that we model the hinges at A and D with no resistance in the y-direction. Since the plate has uniformly distributed load (including its own weight), we replace the distributed load with an equivalent concentrated load $\vec{\boldsymbol{W}}$ acting vertically through point G.

The various forces acting on the plate are

$$\vec{\boldsymbol{W}} = -W\hat{\boldsymbol{k}},\ \vec{\boldsymbol{T}}_1 = T_1\hat{\boldsymbol{\lambda}}_{\mathrm{BE}},\ \vec{\boldsymbol{T}}_2 = T_2\hat{\boldsymbol{\lambda}}_{\mathrm{CF}},\ \vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{\imath}} + A_z\hat{\boldsymbol{k}},\ \vec{\boldsymbol{D}} = D_x\hat{\boldsymbol{\imath}} + D_z\hat{\boldsymbol{k}}.$$

Here, $\hat{\boldsymbol{\lambda}}_{\mathrm{BE}} = \hat{\boldsymbol{\lambda}}_{\mathrm{CF}} = -\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{k}} = \hat{\boldsymbol{\lambda}}$(let). Now, we apply moment equilibrium about point A, *i.e.*, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\boldsymbol{0}}$.

$$\vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{r}}_{\mathrm{C}} \times \vec{\boldsymbol{T}}_2 + \vec{\boldsymbol{r}}_{\mathrm{G}} \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{r}}_{\mathrm{D}} \times \vec{\boldsymbol{D}} = \vec{\boldsymbol{0}} \tag{5.36}$$

where,

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{T}}_1 &= a\hat{\boldsymbol{\imath}} \times T_1\hat{\boldsymbol{\lambda}} = -aT_1\sin\theta\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{\mathrm{C}} \times \vec{\boldsymbol{T}}_2 &= (a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) \times T_2\hat{\boldsymbol{\lambda}} = T_2 b\sin\theta\hat{\boldsymbol{\imath}} - T_2 a\sin\theta\hat{\boldsymbol{\jmath}} + T_2 b\cos\theta\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{r}}_{\mathrm{G}} \times \vec{\boldsymbol{W}} &= \frac{1}{2}(a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) \times (-W\hat{\boldsymbol{k}}) = -\frac{Wa}{2}\hat{\boldsymbol{\imath}} + \frac{Wa}{2}\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{\mathrm{D}} \times \vec{\boldsymbol{D}} &= b\hat{\boldsymbol{\jmath}} \times (D_x\hat{\boldsymbol{\imath}} + D_z\hat{\boldsymbol{k}}) = D_z b\hat{\boldsymbol{\imath}} - D_x b\hat{\boldsymbol{k}}.
\end{aligned}$$

Substituting these products in eqn. (5.36) and dotting with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, we get

$$\begin{aligned}
T_2\sin\theta + D_z &= \frac{W}{2} &\quad (5.37)\\
T_2\cos\theta - D_x &= 0 &\quad (5.38)\\
(T_1 + T_2)\sin\theta &= \frac{W}{2}. &\quad (5.39)
\end{aligned}$$

The force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{D}} + \vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{T}}_2 + \vec{\boldsymbol{W}} = \vec{\boldsymbol{0}}.$$

Again, substituting the forces in their component form and dotting with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{k}}$ (there are no $\hat{\boldsymbol{\jmath}}$ components), we get

$$\begin{aligned}
A_x + D_x - (T_1 + T_2)\cos\theta &= 0 \\
\Rightarrow \quad A_x - T_1\cos\theta &= 0 &\quad (5.40)\\
A_z + D_z + (T_1 + T_2)\sin\theta &= 0 \\
\Rightarrow \quad A_z + T_1\sin\theta &= \frac{W}{2}. &\quad (5.41)
\end{aligned}$$

These are all the equations that we can get. Now, note that we have five independent equations (eqns. (5.37) to (5.41)) but six unknowns. Thus we cannot solve for the unknowns uniquely. This is an indeterminate structure! No matter which point we use for our moment equilibrium equation, we will always have one more unknown than the number of independent equations. We can, however, solve the problem with an extra assumption (see comments below) — the structure is symmetric about the axis passing through G and parallel to x-axis. From this symmetry we conclude that $T_1 = T_2$. Then, from eqn. (5.40) we have

$$2T\sin\theta = \frac{W}{2} \quad \Rightarrow \quad T = \frac{W}{4\sin\theta}.$$

We can now find the maximum load that the plate can take subject to the maximum allowable tension in the strings.

$$\begin{aligned} W &= 4T\sin\theta \\ \Rightarrow \quad W_{\max} &= 4T_{\max}\sin\theta \\ &= 4(100\,\mathrm{N})\cdot\frac{1}{2} = 200\,\mathrm{N}. \end{aligned}$$

The total load as given is $(6+4)\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2} = 98.1\,\mathrm{N} \approx 100\,\mathrm{N}$. Thus we can double the load before the strings reach their break-points. Now the reactions at D and A follow from eqns. (5.37), (5.38), (5.40), and (5.41).

$$\begin{aligned} D_z = A_z = \frac{W}{2} - T\sin\theta &= \frac{W}{2} \\ D_x = A_x = T\cos\theta &= \frac{W}{4}\cot\theta. \end{aligned}$$

$$\boxed{W_{\max} = 200\,\mathrm{N}}$$

Comments:

1. We got only five independent equations (instead of the usual 6) because the force equilibrium in the y-direction gives a zero identity ($0 = 0$). There are no forces in the y-direction. The structure seems to be free in the y-direction — if you push a little, it will move. This freedom to move in the y-direction arose because we chose to model the hinges at A and D as allowing sliding, keeping in mind only the vertical loading. The actual hinges used on a bookshelf will not allow movement in the y-direction either. If we model the hinges as ball and socket joints, we introduce two more unknowns, one at each joint, and get just one more scalar equation. Thus we are back to square one. There is no way to determine A_y and D_y from equilibrium equations alone.
2. The assumption of symmetry and the consequent assumption of equality of the two string tensions is, mathematically, an extra independent equation based on deformations (strength of materials). At this point, you may not know any strength of material calculations or deformation theory, but your intuition is likely to lead you to make the same assumption. Note, however, that this assumption is sensitive to accuracy in fabrication of the structure. If the strings were slightly different in length, the angles were slightly off, or the wall was not perfectly vertical, the symmetry argument would not hold and the two tensions would not be the same.

Most real problems are like this — indeterminate. Our modelling, if good, makes them determinate, solvable and useful.