

5.4 Internal forces

The vague concept of ‘forces inside’ a structure is superficially in conflict with the subject of mechanics. Why? Because mechanics equations only concern the forces *on* an object shown in a free-body diagram; ‘internal forces’ have no place on a free-body diagram and thus no place in mechanics.

Example: Pulling on the ends of a rope; nothing internal

Consider two people pulling apart the frayed rope of fig. 5.66a. A free-body diagram of the rope is shown in fig. 5.66b. The laws of mechanics use the *external* forces on an isolated system. These are the forces that show on a free-body diagram. For the rope, these are the forces at the ends. The free-body diagram does not include internal forces. Thus nothing about the ‘internal forces’ at the fraying part of the rope shows up in the mechanics equations describing the rope.

Mechanics has nothing to say about so called ‘internal forces’ and thus nothing to say about the rope breaking in the middle. ‘Internal forces’ are meaningless in mechanics. The section title describes a non-existent subject.

Something’s wrong. The problem is somewhat one of language: ‘internal forces’ are not really internal and they are not really forces!

‘Internal forces’ represent external forces on a smaller body

On page 27 we advertised mechanics as being useful for predicting when things will break. And our intuitions strongly tell us that there is something about the forces *in* the rope that make it break. Yet mechanics equations are based on the forces that show on free-body diagrams. And free-body diagrams only show external forces. How can we use mechanics based on external forces to describe the ‘forces’ inside a body? We use an idea whose simplicity hides its incredible utility:

You cut the body, and what was inside is now on the outside of a smaller body.

In the case of the rope, we cut it in the middle. Then we fool the rope into thinking it wasn’t cut using forces (remember, ‘forces are *the* measure of mechanical interaction’), one force, say, at each fiber that is cut. Then we get the free-body diagram of fig. 5.67a. We can simplify this to the free-body diagram of fig. 5.67b because we know that every force system is equivalent to a force and couple at any point, in this case the middle of the rope. If we apply the equilibrium conditions to this cut rope, we see that

$$\begin{aligned} \text{Sum of vertical forces is zero} &\Rightarrow F_y = 0 \\ \text{Sum of horizontal forces is zero} &\Rightarrow F_x = -T \\ \text{Sum of moments about the cut is zero} &\Rightarrow M = 0. \end{aligned}$$

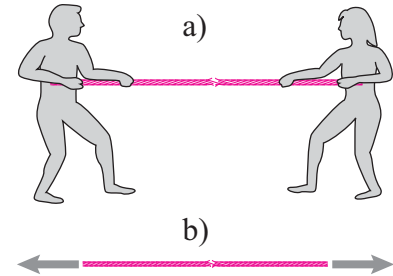


Figure 5.66: a) Two people pulling on a rope that is likely to break in the middle, b) A free-body diagram of the rope.

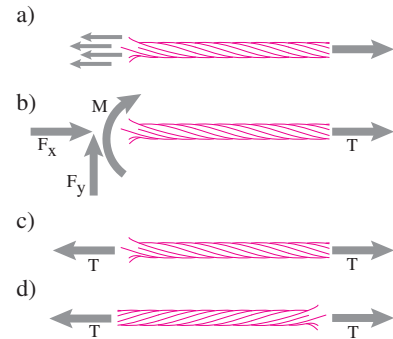


Figure 5.67: a) a free-body diagram of the right part of the rope, b) the same free-body diagram, with the force distribution at the cut replaced with an equivalent force couple system, c) further simplified by using the laws of mechanics, and d) a free-body diagram of the left portion of the rope.

Thus we get the simpler free-body diagram of fig. 5.67c as you probably already guessed without using the equilibrium equations explicitly.

Tension

We have just discovered the concept of ‘tension *in* a rope’, also sometimes called the ‘axial force’. The tension is the pulling force on a free-body diagram of the cut rope. If we had used the same cut for a free-body diagram of the left half of the rope we would see the free-body diagram of fig. 5.67d. Either by the principle of action and reaction, or by the equilibrium equations for the left half of the rope, you see also a tension T . The force vector is the opposite of the force vector on the right half of the rope. So it doesn’t make sense to talk about the tension force vector *in* the rope since different (opposite) force vectors manifest themselves on the two sides of the cut ($-T\hat{i}$ on the left end of the right half and $T\hat{i}$ on the right end of the left half). Instead we talk about the scalar tension T which expresses the force vector at the cut as

$$\vec{F} = T\hat{\lambda}$$

where $\hat{\lambda}$ is a unit vector pointing out from the free-body diagram cut. Because $\hat{\lambda}$ switches direction depending on which half rope you are looking at, the same scalar T works for both pieces.

The *tension* in a rope, cable, or bar is the amount of force pulling out on a free-body diagram of the cut rope, cable, or bar. Tension is a scalar.

① Calling tension a scalar is a deception for pedagogical purposes. The best representation of ‘internal forces’ is with tensors which are too mathematically advanced for this book. But it is fun to notice that the concept of a tensor, something prominent in Einstein’s theory of general relativity for example, has its origin in tension, our object of study here. Note the non-coincidental similarity of the words *tensor* and *tension*. What is a tensor? Loosely, a tensor is a quantity that helps you find a vector (the force at a cut) once you are told another vector (the unit vector pointing outwards from the cut). [Aside for hyper-experts: The relation between the tension tensor and tension scalar can be expressed by the dyadic representation $\underline{T} = T\hat{e}_1\hat{e}_1$.]

Internal ‘forces’ are not force vectors

Note our abuse of language: force is a vector, tension is an ‘internal force’ and tension is a scalar. What we call ‘internal forces’ are not really forces. We can’t talk about the internal force vector at a point in the string because there are two different vectors for each cut, one for left half of string and one for the right. An ‘internal force’ isn’t a force vector. Rather it is a quantity from which we can find a force vector once we have made a cut and picked which side of the cut we care about. We use this confusing language because of its firm place in the engineering workplace.

The common phrase *internal force* means ‘a scalar with dimensions of force from which you can find the force on one side of a free-body diagram cut’.

① Summarizing:

- Internal forces are not internal. Rather they describe the forces on the boundary of a smaller system that has a free-body diagram cut that is inside the system of previous interest.
- Internal forces are not force vectors. Rather they are scalars from which you can find the force vector acting on one side of a free-body diagram cut.

What is the strength of a structural piece?

Getting back to the question of whether or not the rope will break, we can now characterize the rope by the tension it can carry. A 10kN cable can carry a tension of $10,000\text{ N}$ all along its length. This means a free-body diagram of the rope, cut anywhere along its length, could show forces up to but not bigger than $10,000\text{ N}$. If the rope is frayed it may break at, say, a tension of $2,000\text{ N}$, meaning a free-body diagram with a cut at the fray can only show forces up to $2,000\text{ N}$.

Note that tension is not always positive. A negative tension (negative pulling out from the ends) is also called a positive compression (positive pushing in at the ends). For ropes we don't see much negative tension, the rope bends with just a hint of compression. But for metal and wood bars, and bones, compression is as important as tension.

Shear force and bending moment

To characterize the strength of more than just 2-force bodies we need to generalize the concept of tension. The main idea, which was emphasized in Chapter 3, is this:

You can make a free-body diagram cut anywhere on any body no matter how it is loaded.

As for tension, we define internal forces in terms of the forces (and moments) that show up on a free-body diagram cut. Again we consider things (bars) that are rather longer than they are wide or thick because

- Long narrow pieces are commonly used in construction of buildings, machines, plants and animals.
- Internal forces in long narrow things are easier to understand than in bulkier objects.

For now we limit ourselves to 2D statics. At an arbitrary cut we can find the force and moment on the remaining piece in the same manner as in Section 5.2. And we could look at the x and y components of the force. Fine. The problem is that the force and moment we find do not just depend on the cut, but on which body we look at. On the right side of the cut a force and moment act. On the left side of the cut, the opposite force

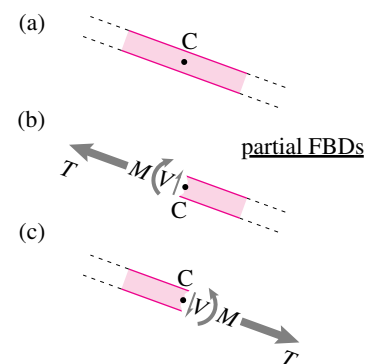


Figure 5.68: a) A piece of a structure, loads not shown; b) a partial free-body diagram of the right part of the bar; c) a partial free-body diagram of the left part of the bar.

and moment act on the other object. Another problem with xy components is that they don't necessarily line up with the natural directions for the structural part. So, for the purposes of thinking about internal forces we break the force into two components (see fig. 5.68) lined up with the part. And we measure the internal forces with scalars that are the same for both sides of the cut:

- The *tension* T is the scalar part of the force directed along the bar as assumed positive when pulling away from the free-body diagram cut.
- The *shear force* V is the force perpendicular to the bar (tangent to the free-body diagram cut). Our sign convention is that shear is positive if it tends to rotate the cut object clockwise. An equivalent statement of the sign convention is that shear is positive if down on cuts at the right of a bar and positive if up on a cut on the left of bar (and to the right on top and to the left on the bottom).

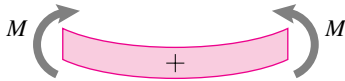


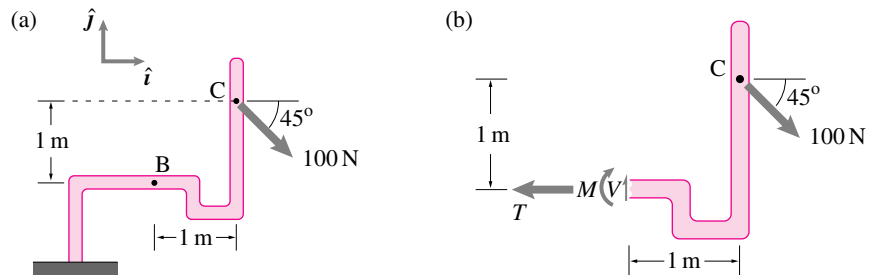
Figure 5.69: The smiling beam sign convention for bending moment. For a horizontal beam, moments which tend to make the beam smile (curve up) are called positive.

② Note that neither V nor T changes if you rotate your paper until the picture is upside down. However, the definition for the sign convention for M has the disadvantage that the bending moment changes sign if you turn your paper upside down. (This ambiguity can only be avoided if one picks a favored side or end of the beam).

Since we are just doing 2D problems now, the moment is always in the out-of-plane (typically $\hat{k}$) direction.

- The *bending moment* M is the scalar part of the bending moment. The sign convention is that for a smiling beam (fig. 5.69): A clockwise ($-\hat{k}$) couple is positive on a left cut and a counterclockwise ($\hat{k}$) couple is positive on a right cut^②.

The tension T , shear V , and bending moment M on fig. 5.68 follow these sign conventions.



Example: Internal forces in a bent rod

The internal forces at B can be found by making a free-body diagram of a portion of the structure with a cut at B.

$$\begin{aligned}
 \text{Sum of vertical forces is zero} &\Rightarrow V = (100/\sqrt{2}) \text{ N} \\
 \text{Sum of horizontal forces is zero} &\Rightarrow T = (100/\sqrt{2}) \text{ N} \\
 \text{Sum of moments about the cut at B is zero} &\Rightarrow M = -100\sqrt{2} \text{ N m.}
 \end{aligned}$$

You may have noticed that we did get ahead of ourselves and used the concept of tension in a rope or rod as a source of loading with known direction on a particle and rigid body. We will use the concept of tension extensively in our analysis of trusses. Calculating how internal forces vary from point to point in a structure is picked up in Section 8 on page 437.