5.4 Internal forces

The vague concept of 'forces inside' a structure is superficially in conflict with the subject of mechanics. Why? Because mechanics equations only concern the forces *on* an object shown in a free-body diagram;'internal forces' have no place on a free-body diagram and thus no place in mechanics.

Example: Pulling on the ends of a rope; nothing internal
Consider two people pulling apart the frayed rope of fig. 5.66a. A free-body diagram of the rope is shown in fig. 5.66b. The laws of mechanics use the *external* forces on an isolated system. These are the forces that show on a free-body diagram. For the rope, these are the forces at the ends. The free-body diagram does not include internal forces. Thus nothing about the 'internal forces' at the fraying part of the rope shows up in the mechanics equations describing the rope.

Mechanics has nothing to say about so called 'internal forces' and thus nothing to say about the rope breaking in the middle. 'Internal forces' are meaningless in mechanics. The section title describes a non-existent subject.

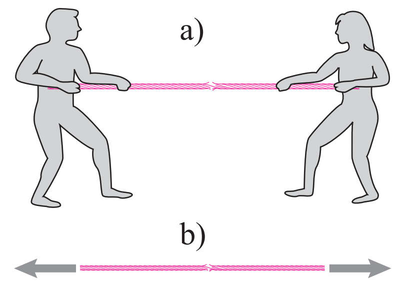


Figure 5.66: a) Two people pulling on a rope that is likely to break in the middle, b) A free-body diagram of the rope.

Something's wrong. The problem is somewhat one of language: 'internal forces' are not really internal and they are not really forces!

'Internal forces' represent external forces on a smaller body

On page 27 we advertised mechanics as being useful for predicting when things will break. And our intuitions strongly tell us that there is something about the forces *in* the rope that make it break. Yet mechanics equations are based on the forces that show on free-body diagrams. And free-body diagrams only show external forces. How can we use mechanics based on external forces to describe the 'forces' inside a body? We use an idea whose simplicity hides its incredible utility:

> You cut the body, and what was inside is now on the outside of a smaller body.

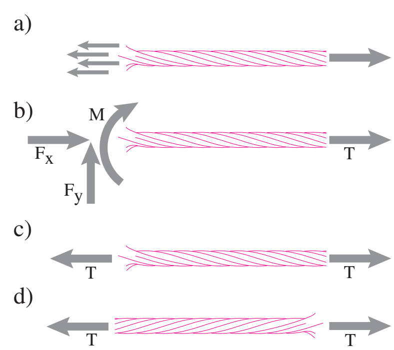


Figure 5.67: a) a free-body diagram of the right part of the rope, b) the same free-body diagram, with the force distribution at the cut replaced with an equivalent force couple system, c) further simplified by using the laws of mechanics, and d) a free-body diagram of the left portion of the rope.

In the case of the rope, we cut it in the middle. Then we fool the rope into thinking it wasn't cut using forces (remember, 'forces are *the* measure of mechanical interaction'), one force, say, at each fiber that is cut. Then we get the free-body diagram of fig. 5.67a. We can simplify this to the free-body diagram of fig. 5.67b because we know that every force system is equivalent to a force and couple at any point, in this case the middle of the rope. If we apply the equilibrium conditions to this cut rope, we see that

$$\begin{array}{rcrcl}
\text{Sum of vertical forces is zero} & \Rightarrow & F_y & = & 0 \\
\text{Sum of horizontal forces is zero} & \Rightarrow & F_x & = & -T \\
\text{Sum of moments about the cut is zero} & \Rightarrow & M & = & 0.
\end{array}$$

Thus we get the simpler free-body diagram of fig. 5.67c as you probably already guessed without using the equilibrium equations explicitly.

Tension

We have just discovered the concept of 'tension *in* a rope', also sometimes called the 'axial force'. The tension is the pulling force on a free-body diagram of the cut rope. If we had used the same cut for a free-body diagram of the left half of the rope we would see the free-body diagram of fig. 5.67d. Either by the principle of action and reaction, or by the equilibrium equations for the left half of the rope, you see also a tension T. The force vector is the opposite of the force vector on the right half of the rope. So it doesn't make sense to talk about the tension force vector *in* the rope since different (opposite) force vectors manifest themselves on the two sides of the cut ($-T\hat{\boldsymbol{\imath}}$ on the left end of the right half and $T\hat{\boldsymbol{\imath}}$ on the right end of the left half). Instead we talk about the scalar tension T which expresses the force vector at the cut as

$$\vec{\boldsymbol{F}} = T\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector pointing out from the free-body diagram cut. Because $\hat{\boldsymbol{\lambda}}$ switches direction depending on which half rope you are looking at, the same scalar T works for both pieces.

> The *tension* in a rope, cable, or bar is the amount of force pulling out on a free-body diagram of the cut rope, cable, or bar. Tension is a scalar.

①Calling tension a scalar is a deception for pedagogical purposes. The best representation of 'internal forces' is with tensors which are too mathematically advanced for this book. But it is fun to notice that the concept of a tensor, something prominent in Einstein's theory of general relativity for example, has its origin in tension, our object of study here. Note the non-coincidental similarity of the words *tensor* and *tension*. What is a tensor? Loosely, a tensor is a quantity that helps you find a vector (the force at a cut) once you are told another vector (the unit vector pointing outwards from the cut). [Aside for hyper-experts: The relation between the tension tensor and tension scalar can be expressed by the dyadic representation $\underline{\underline{\boldsymbol{T}}} = T\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1$.]

Internal 'forces' are not force vectors

Note our abuse of language: force is a vector, tension is an 'internal force' and tension is a scalar. What we call 'internal forces' are not really forces. We can't talk about the internal force vector at a point in the string because there are two different vectors for each cut, one for left half of string and one for the right. An 'internal force' isn't a force vector. Rather it is a quantity from which we can find a force vector once we have made a cut and picked which side of the cut we care about. We use this confusing language because of its firm place in the engineering workplace.

> The common phrase *internal force* means 'a scalar with dimensions of force from which you can find the force on one side of a free-body diagram cut'.

① Summarizing:

- Internal forces are not internal. Rather they describe the forces on the boundary of a smaller system that has a free-body diagram cut that is inside the system of previous interest.
- Internal forces are not force vectors. Rather they are scalars from which you can find the force vector acting on one side of a free-body diagram cut.

What is the strength of a structural piece?

Getting back to the question of whether or not the rope will break, we can now characterize the rope by the tension it can carry. A $10kN$ cable can carry a tension of $10,000\,\mathrm{N}$ all along its length. This means a free-body diagram of the rope, cut anywhere along its length, could show forces up to but not bigger than $10,000\,\mathrm{N}$. If the rope is frayed it may break at, say, a tension of $2,000\,\mathrm{N}$, meaning a free-body diagram with a cut at the fray can only show forces up to $2,000\,\mathrm{N}$.

Note that tension is not always positive. A negative tension (negative pulling out from the ends) is also called a positive compression (positive pushing in at the ends). For ropes we don't see much negative tension, the rope bends with just a hint of compression. But for metal and wood bars, and bones, compression is as important as tension.

Shear force and bending moment

To characterize the strength of more than just 2-force bodies we need to generalize the concept of tension. The main idea, which was emphasized in Chapter 3, is this:

> You can make a free-body diagram cut anywhere on any body no matter how it is loaded.

As for tension, we define internal forces in terms of the forces (and moments) that show up on a free-body diagram cut. Again we consider things (bars) that are rather longer than they are wide or thick because

- Long narrow pieces are commonly used in construction of buildings, machines, plants and animals.
- Internal forces in long narrow things are easier to understand than in bulkier objects.

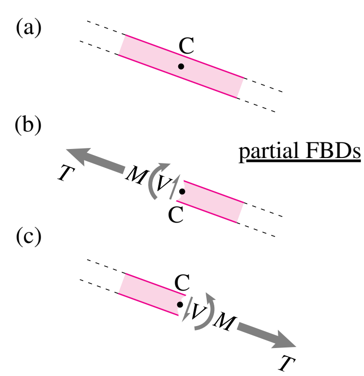


Figure 5.68: a) A piece of a structure, loads not shown; b) a partial free-body diagram of the right part of the bar; c) a partial free-body diagram of the left part of the bar.

For now we limit ourselves to 2D statics. At an arbitrary cut we can find the force and moment on the remaining piece in the same manner as in Section 5.2. And we could look at the x and y components of the force. Fine. The problem is that the force and moment we find do not just depend on the cut, but on which body we look at. On the right side of the cut a force and moment act. On the left side of the cut, the opposite force and moment act on the other object. Another problem with xy components is that they don't necessarily line up with the natural directions for the structural part. So, for the purposes of thinking about internal forces we break the force into two components (see fig. 5.68) lined up with the part. And we measure the internal forces with scalars that are the same for both sides of the cut:

- The *tension* T is the scalar part of the force directed along the bar assumed positive when pulling away from the free-body diagram cut.
- The *shear force* V is the force perpendicular to the bar (tangent to the free-body diagram cut). Our sign convention is that shear is positive if it tends to rotate the cut object clockwise. An equivalent statement of the sign convention is that shear is positive if down on cuts at the right of a bar and positive if up on a cut on the left of bar (and to the right on top and to the left on the bottom).

Since we are just doing 2D problems now, the moment is always in the out-of-plane (typically $\hat{\boldsymbol{k}}$) direction.

- *The bending moment* M is the scalar part of the bending moment. The sign convention is that for a smiling beam (fig. 5.69): A clockwise ($-\hat{\boldsymbol{k}}$) couple is positive on a left cut and a counterclockwise ($\hat{\boldsymbol{k}}$) couple is positive on a right cut②.

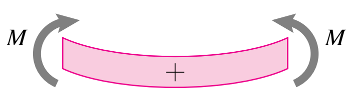
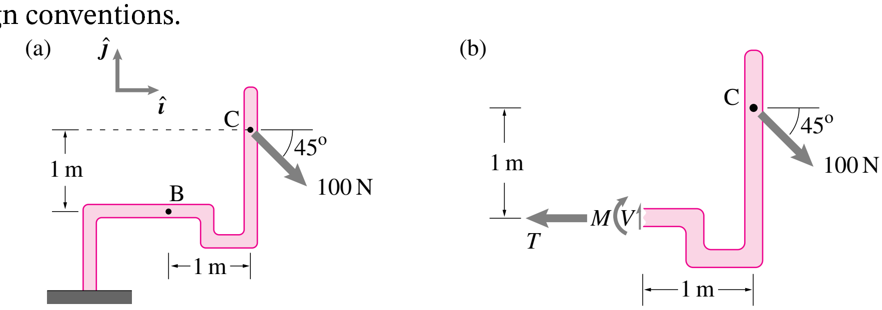


Figure 5.69: The smiling beam sign convention for bending moment. For a horizontal beam, moments which tend to make the beam smile (curve up) are called positive.

②Note that neither V nor T changes if you rotate your paper until the picture is upside down. However, the definition for the sign convention for M has the disadvantage that the bending moment changes sign if you turn your paper upside down. (This ambiguity can only be avoided if one picks a favored side or end of the beam).

The tension T, shear V, and bending moment M on fig. 5.68 follow these sign conventions.



Example: Internal forces in a bent rod
The internal forces at B can be found by making a free-body diagram of a portion of the structure with a cut at B.

$$\begin{array}{rcrcl}
\text{Sum of vertical forces is zero} & \Rightarrow & V & = & (100/\sqrt{2})\,\mathrm{N} \\
\text{Sum of horizontal forces is zero} & \Rightarrow & T & = & (100/\sqrt{2})\,\mathrm{N} \\
\text{Sum of moments about the cut at B is zero} & \Rightarrow & M & = & -100\sqrt{2}\,\mathrm{N\,m}.
\end{array}$$

You may have noticed that we did get ahead of ourselves and used the concept of tension in a rope or rod as a source of loading with known direction on a particle and rigid body. We will use the concept of tension extensively in our analysis of trusses. Calculating how internal forces vary from point to point in a structure is picked up in Section 8 on page 437.

SAMPLE 5.18 A structure is made up of two bars – a thick bent bar ABC and a thin rod CE. Point C is halfway between B and D, $\ell = 0.8\,\mathrm{m}$ and $\theta = 60^{\circ}$. Bar ABC is pulled up by a force $F = 500\,\mathrm{N}$ at point A.

1. Find the internal forces in the bar ABC just to the left of point B.
2. Find the force in bar CE at the section s-s shown in the figure.

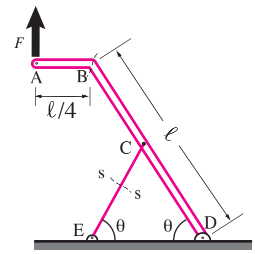


Figure 5.70

Solution We cut the bar ABC at point B. The free-body diagram of the left part AB is shown in fig. 5.71. The internal forces acting at the cut section are tension T, shear force V and the bending moment M. From force balance of part AB in x and y directions, we have

$$T = 0, \quad \text{and} \quad V = F = 500\,\mathrm{N}.$$

From the moment balance about point B, we have

$$M - F\ell/4 = 0 \quad \Rightarrow \quad M = F\ell/4 = 100\,\mathrm{N{\cdot}m}.$$

$$\boxed{T = 0, \quad V = 500\,\mathrm{N}, \quad M = 100\,\mathrm{N{\cdot}m}}$$

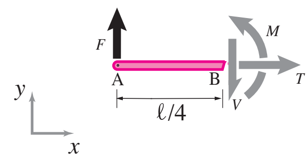


Figure 5.71

For finding the tension in rod CE at the given section, we cut the rod at s-s and draw the free-body diagram of the structure along with the upper part of the rod attached at point C. The tension in bar CE is T and the reaction of the support at pin D is R. We need to find T.

We can write the moment-balance equation about point D, $\sum \vec{\boldsymbol{M}}_{\mathrm{D}} = \vec{\boldsymbol{0}}$, so that the unknown force R (that we are not interested in) disappears from the equation:

$$\vec{\boldsymbol{r}}_{\mathrm{A/D}} \times \vec{\boldsymbol{F}} + \vec{\boldsymbol{r}}_{\mathrm{C/D}} \times \vec{\boldsymbol{T}} = \vec{\boldsymbol{0}}.$$

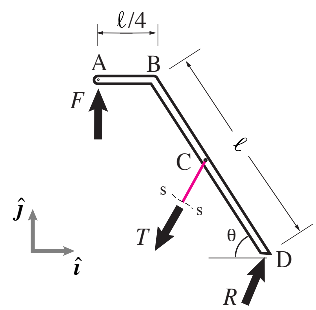


Figure 5.72

The moments of F and T about point D can be easily evaluated using the scalar formula 'force times the lever arm' (see fig. 5.73). Thus, the moment-balance equation in $\hat{\boldsymbol{k}}$ direction is:

$$-F\ell(1/4 + \cos\theta) + T\frac{\ell}{2}\sin 2\theta = 0$$

$$\Rightarrow \quad T = \frac{2(1/4 + \cos\theta)}{\sin 2\theta}F.$$

Substituting the given values, $F = 500\,\mathrm{N}$ and $\theta = 60^{\circ}$, we get

$$T = 866\,\mathrm{N}.$$

$$\boxed{T = 866\,\mathrm{N}}$$

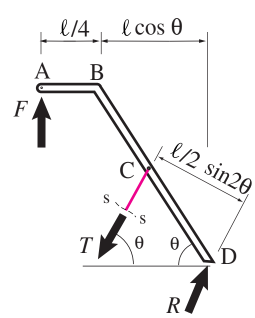


Figure 5.73: Moments of F and T about point D can be evaluated by multiplying the forces with their respective lever arms $\ell/4 + \ell\cos\theta$ and $\ell/2\sin 2\theta$, respectively.

Note: Evaluation of the moment equation about point D using vectors and cross products is as follows. Since $\vec{\boldsymbol{r}}_{\mathrm{A/D}} = \vec{\boldsymbol{r}}_{\mathrm{A/B}} + \vec{\boldsymbol{r}}_{\mathrm{B/D}} = -\frac{\ell}{4}\hat{\boldsymbol{\imath}} + \ell(-\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})$, $\vec{\boldsymbol{r}}_{\mathrm{C/D}} = \frac{\ell}{2}(-\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})$, $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{T}} = T(-\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}})$,

$$\vec{\boldsymbol{r}}_{\mathrm{A/D}} \times \vec{\boldsymbol{F}} = -F\left(\frac{\ell}{4} + \ell\cos\theta\right)\hat{\boldsymbol{k}}, \text{ and } \vec{\boldsymbol{r}}_{\mathrm{C/D}} \times \vec{\boldsymbol{T}} = T\ell\cos\theta\sin\theta\hat{\boldsymbol{k}}.$$

Therefore, the moment-balance equation is

$$-F\ell(1/4 + \cos\theta)\hat{\boldsymbol{k}} + T\frac{\ell}{2}\sin 2\theta\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}.$$

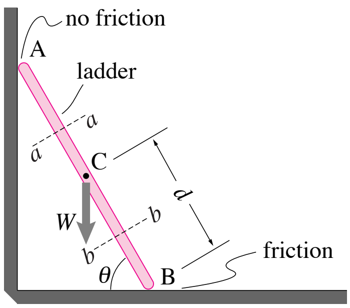


Figure 5.74

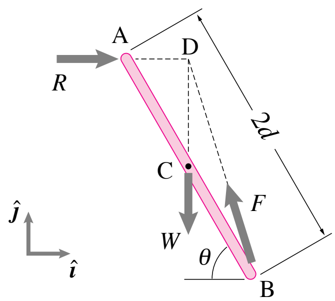


Figure 5.75

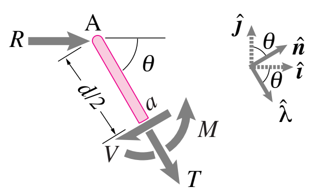


Figure 5.76: Free-body diagram of the upper part of the ladder cut at section a-a. Note that $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = \sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

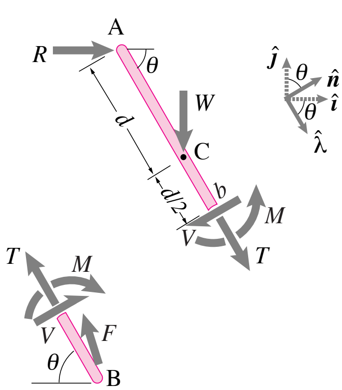


Figure 5.77

SAMPLE 5.19 A ladder of length $2d = 4\,\mathrm{m}$ rests against a wall as shown. A person of weight $W = 700\,\mathrm{N}$ stands at C. Assume that the ladder does not slip. Neglecting the weight of the ladder, find the internal forces in the ladder at sections a-a and b-b, at mid points of AC and AB, respectively. (See Sample 5.14.)

Solution To find the internal forces at the indicated sections, we need to cut the ladder at those sections, one at a time, draw the free-body diagram of each part and carry out the force and moment-balance equations. A little anticipation shows that we will need the support reactions at A and B in our calculations. So, let us first determine the support reactions. The free-body diagram of the ladder is shown in fig. 5.75. The moment balance about point B in $\hat{\boldsymbol{k}}$ direction gives

$$-R(2d\sin\theta) + W(d\cos\theta) = 0 \quad \Rightarrow \quad R = \frac{W}{2}\frac{\cos\theta}{\sin\theta}.$$

The force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$R\hat{\boldsymbol{\imath}} - W\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} = -R\hat{\boldsymbol{\imath}} + W\hat{\boldsymbol{\jmath}}.$$

Substituting the given values of $\theta(60^\circ)$ and $W(700\,\mathrm{N})$, we get,

$$\vec{\boldsymbol{R}} = (202\,\mathrm{N})\hat{\boldsymbol{\imath}}, \quad \text{and} \quad \vec{\boldsymbol{F}} = (-202\hat{\boldsymbol{\imath}} + 700\hat{\boldsymbol{\jmath}})\,\mathrm{N}.$$

Section a-a: Now, we cut the ladder at a-a and draw the free-body diagram of the upper part of the ladder as shown in fig. 5.76. The force balance for this part gives

$$\begin{aligned} T\hat{\boldsymbol{\lambda}} - V\hat{\boldsymbol{n}} + R\hat{\boldsymbol{\imath}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad T &= -R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\lambda}}) = -R\cos\theta \\ \text{and} \quad V &= R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{n}}) = R\sin\theta. \end{aligned}$$

Substituting the numerical values of R and θ, we get $T = -101\,\mathrm{N}$ and $V = 175\,\mathrm{N}$. Now, the moment-balance equation about a (the cut) gives

$$M - R(d/2)\sin\theta = 0 \quad \Rightarrow \quad M = (1/2)Rd\sin\theta$$

which, with numerical values, gives $M = 175\,\mathrm{N{\cdot}m}$.

$$T = -101\,\mathrm{N}, \quad V = 175\,\mathrm{N}, \quad M = 175\,\mathrm{N{\cdot}m}$$

Section b-b: Now we consider the internal forces at section b-b. We cut the ladder at the given section. We can consider the free-body diagram of the upper part or the lower part of the ladder to find the internal forces. Considering the upper part, (see fig. 5.77) we get, from force balance,

$$T\hat{\boldsymbol{\lambda}} - V\hat{\boldsymbol{n}} + R\hat{\boldsymbol{\imath}} - W\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}$$

which, as the analysis above, gives

$$\begin{aligned} T &= -R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\lambda}}) + W(\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\lambda}}) = -R\cos\theta + W(-\sin\theta) = -707\,\mathrm{N} \\ V &= R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{n}}) - W(\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{n}}) = R\sin\theta - W\cos\theta = -175\,\mathrm{N}. \end{aligned}$$

Similarly, the scalar moment-balance equation about point b gives

$$M - R\frac{3d}{2}\sin\theta + W\frac{d}{2}\cos\theta = 0 \quad \Rightarrow \quad M = 175\,\mathrm{N{\cdot}m}.$$

$$T = -707\,\mathrm{N}, \quad V = -175\,\mathrm{N}, \quad M = 175\,\mathrm{N{\cdot}m}$$

5.4 Internal forces

Preparatory Problems

5.4.1 For the bar shown which ones of the following statements are true? *

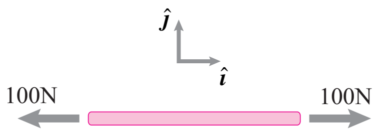
a) The two forces cancel so the tension is zero.

b) The two forces add so the tension is 200 N.

c) The tension is 100 N$\hat{\imath}$.

d) The tension is -100 N$\hat{\imath}$.

e) The tension is 100 N$\hat{\imath}$ on the right end and -100 N$\hat{\imath}$ on the left end.



Problem 5.4.1

5.4.2 What letters and case (upper or lower) are used in this book for tension, shear force, and bending moment?

5.4.3 Mechanics depends on free-body diagrams. And free-body diagrams only show the external forces on an object. So how can mechanical sense be made of the concept of "internal" force?

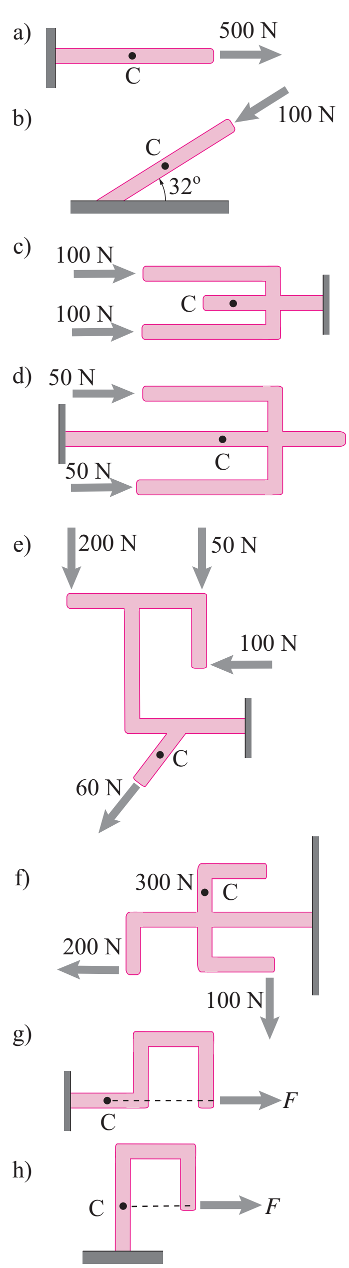
5.4.4 A string is conceptually cut in half by making free-body diagrams of the left and right halves of the string. At the cut on the left half of the string acts the force 50 N$\hat{\imath}$. At the cut on the right half of the string acts the force -50 N$\hat{\imath}$. With two different forces acting on the two halves how can one define a single 'tension'?

5.4.5 Define as precisely as you can:

a) Shear force

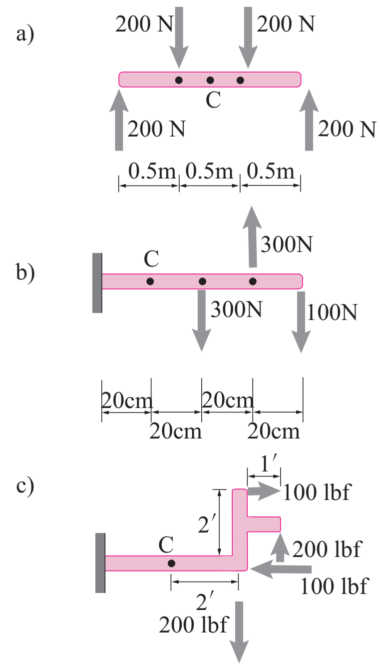
b) Bending moment

5.4.6 Find the tension, shear force and bending moment at C for each of the structures below. Neglect gravity. Assume dimensions as needed.



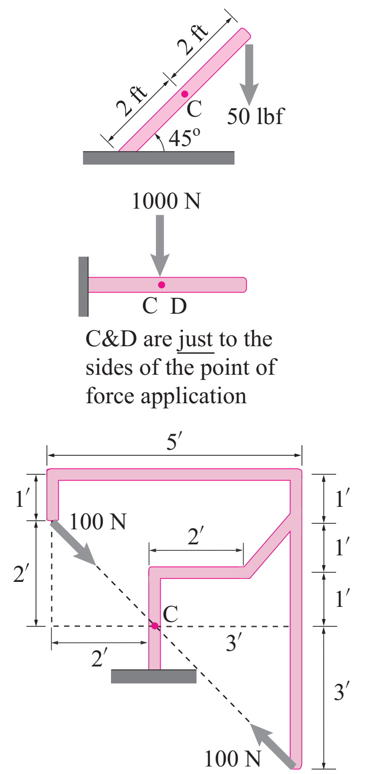
Problem 5.4.6

5.4.7 Find the tension, shear force and bending moment at C for each of the structures below. There is no gravity. Assume dimensions if needed.



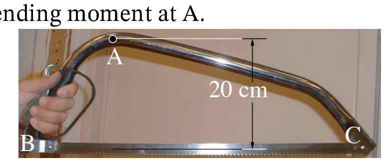
Problem 5.4.7

5.4.8 Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Assume reasonable dimensions as needed.



Problem 5.4.8

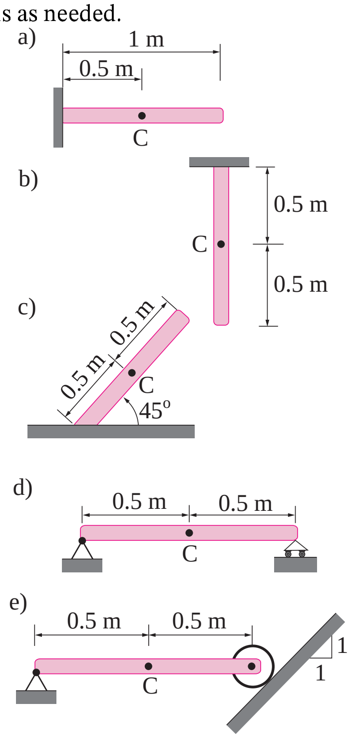
5.4.9 The tension in the bow-saw blade BC is 250 N. Find the tension, shear, and bending moment at A.



Problem 5.4.9

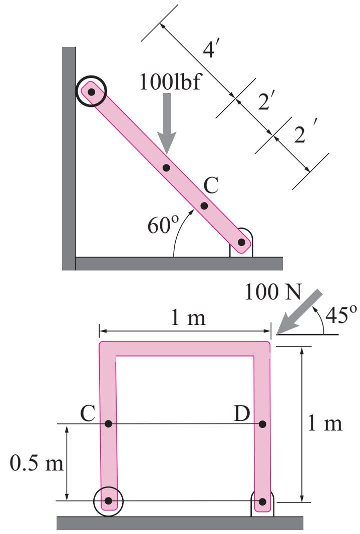
More-Involved Problems

5.4.10 Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Include gravity, assume all bars are uniform with density of (100 N/m). Assume reasonable dimensions as needed.



Problem 5.4.10

5.4.11 Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Neglect gravity. Assume reasonable dimensions as needed.



Problem 5.4.11