5.2 Equilibrium of one object

For particle statics, we used that the forces acting on an object in equilibrium have no net push or pull; the forces add to zero. Now we will use that the forces have no tendency to cause rotation; the moments add to zero. These are not two in a long list of facts about equilibrium, but the whole story. As stated on the inside front cover and this chapter's introduction (page 242)

An object is in static equilibrium if and only if the force balance and moment-balance equations hold.

$$\underbrace{\sum_{\substack{\text{All external}\\\text{forces}}} \vec{F}_i = \vec{0}}_{\text{force balance}} \qquad \text{and} \qquad \underbrace{\sum_{\substack{\text{All external}\\\text{torques}}} \vec{M}_{i/C} = \vec{0}}_{\text{moment balance}} \qquad \text{(Ic,IIc)}$$

The total force system acting on the object is then equivalent to a zero force and zero moment acting at C.

By supplementing the force-balance equation with moment balance we can determine more about the forces that act on an object.

Rigid-body statics

To start with, one often thinks of the object of interest as one piece, for example a whole car, a wheel, a person, a limb, a chair or a derrick. We often think of such an object as rigid, meaning that the object's shape and size only change negligibly due to the forces of interest. Thus the phrase *rigid-body mechanics*. Actually, however, the equilibrium equations apply just as well to all things with little acceleration, whether or not they are stiff and solid. For a first pass at the subject, one thinks of applying the principles of statics to single, rather-solid, simply-defined objects. And such will be our main initial concern in this section. But really the delineation of an 'object' is up to you. And in later chapters we apply the same statics equations to clearly-non-rigid systems like water and rope. For statics the only concern is the delineation of the system at the instant of interest.

Once you know its shape, whether an object is rigid or not is irrelevant for statics.

The reference point C in moment balance.

The moment-balance equation is calculated by calculating the moments of forces relative to a point C using

$$\vec{\boldsymbol{M}}_{\mathrm{i}} = \vec{\boldsymbol{r}}_{\mathrm{i/C}} \times \vec{\boldsymbol{F}}_{i}.$$

C is any convenient point, possibly the origin O of your coordinate system. C is not a special point. As discussed in Section 2.1, if a force system is equivalent to zero force and zero couple at C, it is equivalent to a zero force and zero couple at any and every point D, E, Q, *etc.*

Example. As you sit still reading, gravity is pulling you down and forces from the floor on your feet, the chair on your seat, and the table on your elbows hold you up. All of these forces add to zero. The net moment of these forces about the front-left corner of your desk adds to zero. And the net moment of these forces about the mole near your left elbow is also zero.

The freedom to use any point you like for moment balance provides an oft-used shortcut.

Number of equations and number of unknowns

In two dimensions the equilibrium equations make up 3 independent scalar equations. These could be:

- 2 components of force balance and the one non-trivial component of moment balance; or
- moment balance about any two points and force balance in any direction (except in the direction orthogonal to the line connecting the two moment-balance points).
- moment balance about 3 points (any three points not on a straight line suffice).

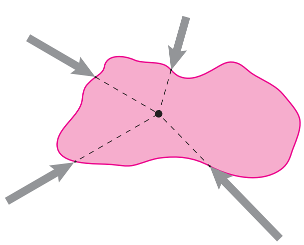


Figure 5.20: A set of forces acting *concurrently* on an object. All force lines of action intersect at one point.

Note that moment balance necessarily is part of the equilibrium equations, but that force balance can be finessed. With one 2D free-body diagram, the equilibrium equations can be solved to find three unknown scalars, for example,

- The magnitudes of three forces whose directions are known *a priori*; or
- One unknown force vector (two components, or angle and magnitude) and one unknown magnitude; or
- Some other three scalars associated with the forces on the free-body diagram. Besides force components and magnitudes these could include a force angle θ, a friction coefficient μ, or the location of force application.

Once you have three independent equations any additional equations you write, say moment about still another point, contains no new information(1). In some problems the forces shown on a free-body diagram au-

(1) A fourth equilibrium equation may superficially look different from an equation already written, but it can always be derived from the other equations.

tomatically satisfy one or more of the equilibrium equations; in making the drawing you may have implicitly solved some equilibrium equations. The equilibrium equations then offer less new information, and sometimes none at all (see 2-force bodies below).

In 3 dimensions, the equilibrium equations make up 6 independent scalar equations. Most directly these are 3 components of force and 3 components of moment. But there are many combinations of equilibrium equations that yield 6 independent scalar equations.

Special cases: concurrent forces, two-force bodies, three-force bodies

We now discuss some special loading situations for which there are special insights or problem-solution tricks. In principle, you don't need to know any of them because force balance and moment balance spell out the whole statics story. In practice it is best to know these special cases.

Concurrent forces

In the special case when the lines of action of all applied forces intersect at one point, moment balance is trivially satisfied (because none of the forces has a moment about the intersection point). Such a system of forces is called *concurrent* (fig. 5.20) and the particle model is appropriate[2]. In such a case, the 2D equilibrium equations only provide two independent scalar equations and one can only use them to solve for two unknown scalars. In 3D one gets three independent scalar equations for a concurrent force system.

One-force body

Let's first treat "one-force" bodies. Consider a finite body with only one force acting on it. Assume it is in equilibrium. Force balance says that the sum of forces must be zero. So that one force must be zero.

> If only one force is acting on a body in equilibrium, that force is zero.

That was too easy. But a count to 3 wouldn't feel complete if it didn't start at 1.

Two-force body

When only two forces act on an object, the situation is also simplified, though not so drastically as the case with one force. An object with only two forces acting on it is called a *two-force body* or *two-force member*.

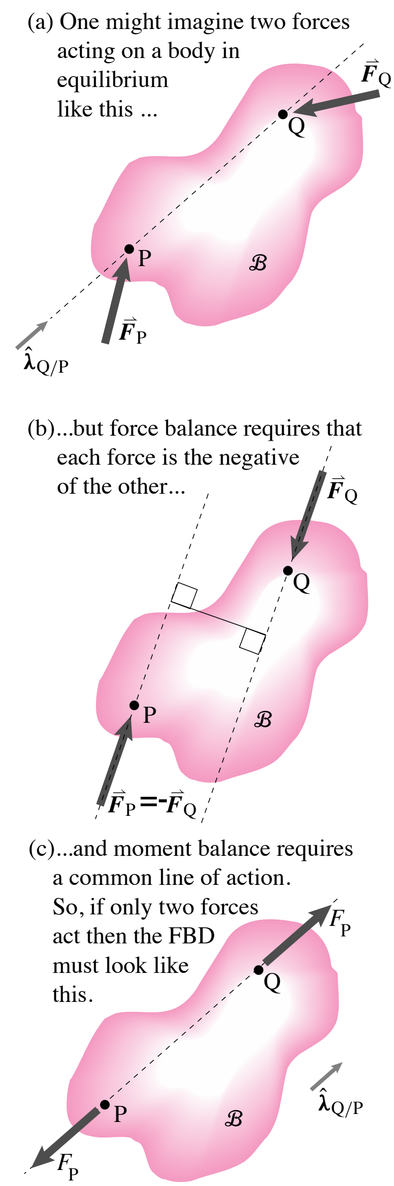


Figure 5.21: (a) Two forces acting on a body $\mathcal{B}$. (b) force balance implies that the forces are equal in magnitude and opposite in direction ($\vec{F}_P = -\vec{F}_Q$. (c) moment balance implies that the forces are collinear. Body $\mathcal{B}$ is a two-force member; the equilibrium equations imply that the two forces must be equal in magnitude, opposite in direction, and collinear. If the free-body diagram shows equal-in-magnitude, opposite-in-direction and collinear forces, then the equilibrium equations add no new information.

② If forces are not concurrent the particle model may still be useful, as demonstrated in the previous section.

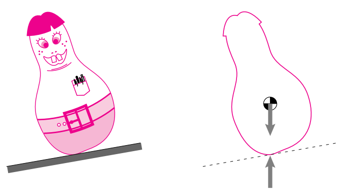


Figure 5.22: **An object balancing on one point of support** is a *two-force body* so has to have its center-of-mass over the support point.

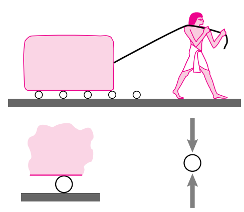


Figure 5.23: **Dragging a heavy stone on rolling round logs.** If there is only point contact between the logs and the ground and stone, each log is a *two-force body*. If the logs are also round there is no resistance to forward motion. The situation with ball bearings is identical.

> If a body in static equilibrium is acted on by two forces, then those forces are equal in magnitude, opposite in direction, and have a common line of action (the line connecting the two points of application).

This result is shown in fig. 5.21 and explained in box 5.3. If you recognize a two-force body you can draw it in a free-body diagram as in fig. 5.21c and the equations of force and moment balance provide no new information. The two-force-body shortcut is especially useful for systems with several parts, some of which are two-force members. Springs, dashpots, struts, and strings are generally idealized as two-force bodies.

Example: Tower and strut
Consider an accelerating cart (fig. 5.24) holding up massive tower AB which is pinned at A and braced by the light strut BC. The rod BC qualifies as a two-force member. The rod AB does not because it has three forces and is also not in static equilibrium (non-negligible accelerating mass). Thus, the free body diagram of rod BC shows the two equal and opposite collinear forces at each end parallel to the rod and the tower AB does not.

Example: Logs as bearings
Consider an ancient Egyptian dragging a big stone (fig. 5.23). If the stone and ground are flat and rigid, and the log is round, rigid and much lighter than the stone we are led to the free-body diagram of the log shown. *With these assumptions there can't be any resistance to rolling.* Note that this effectively frictionless rolling occurs no matter how big the friction coefficient between the contacting surfaces. That the Egyptian got tired comes from logs not being perfectly round, the ground or stone not being perfectly flat, and, most importantly, the ground, log or stone not being perfectly rigid. In any case, it takes effort to pick up the logs in the back and move them to the front.

Example: Pliers
The pliers of fig. 2.39a on page 168, when considered as a whole (with the pencil they are squeezing), are a two-force body. Thus $F_{HE} = F_{EH}$ and these two forces must act on a common line. Assuming the forces are large enough that gravity can be neglected, and the motions are slow enough that statics is accurate, the person has no choice but to apply the forces in this way.

Example: One point of support
If an object with weight is supported at just one point (fig. 5.22), that point must be directly above or below the center-of-mass. Why? The gravity forces are equivalent to a single force at the center-of-mass. The body is then a two-force body. Since the direction of the gravity force is down, the support point and center-of-mass must be above one another.

Similarly,

> If a body is suspended from one point, the center of gravity must be directly above or below that point.

Three-force body

If a body in equilibrium has only three forces on it, the equilibrium equations again restrict the forces in a geometrically describable manner. The simplification is not as great as for two-force bodies but is remarkably useful for both calculation and intuition. In box 5.4 on page 7 moment balance about various axes is used to prove that

> If exactly three forces act on a body (2D and 3D), the body is in equilibrium only if
>
> 1. the three force vectors are coplanar,
> 2. and either
>   a) have lines of action which intersect at a single point (*i.e.,* they are concurrent), or
>   b) they are parallel.

One could imagine three random forces acting on a body. But, for equilibrium they must be coplanar and either concurrent or parallel. Unlike the case for 2-force bodies where the 2-force-body conditions imply the satisfaction of all equilibrium equations, for 3-force bodies planar concurrency still leaves two independent equilibrium equations possibly unsatisfied (for both 2D and 3D). That is, one still needs the equations of force balance in the plane (or, in the special case of three parallel forces, one scalar force balance and one moment-balance equation).

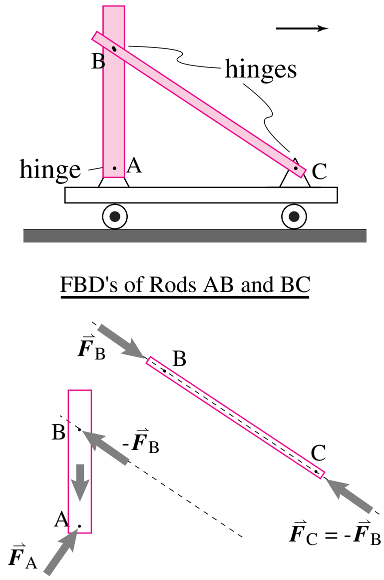


Figure 5.24: A cart is propelled by forces not shown. We are concerned about light strut BC and heavy tower AB. The strut is probably well-modeled as a *two-force body.*

Example: Hanging book box

A box with a book inside is hung by two strings so that it is in equilibrium when level. The system is a *three-force body* so the lines of action of the two strings must intersect on the vertical line that goes through the center-of-mass of the box/book system.

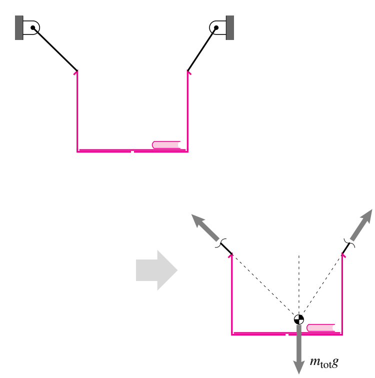


Figure 5.25: **A book is in a box** which hangs from two ropes: a *three-force body.*

Example: Which way do the forces go?

The spread of directions (the smallest angle that includes all force directions) in a 3-force body can be (a) greater than, (b) equal to, or (c) less than 180° (see the figure below). In each case we can know something about the directions of the forces. Call the point of force concurrency D.

(a) Forces spread over less than 180°. Force balance perpendicular to the middle force implies that the outer two forces are both directed toward D or both directed away from D. Force balance in the direction of the middle force shows that it has to have the opposite sense than the outer forces. If the others are pushing in then it is pulling away. If the outer forces are pulling away then it is pushing in.

(b) Forces spread exactly 180°. Force balance in the direction perpendicular to line ADC shows that the odd force must be zero. The other forces must obviously oppose each other.

(c) Forces spread over more than 180°. Force balance perpendicular to the force at C shows that the other two forces must both pull away towards D or both push in. Then force balance along C shows that all three forces must have the same sense. All three forces are pulling away from D or all three are pushing in.

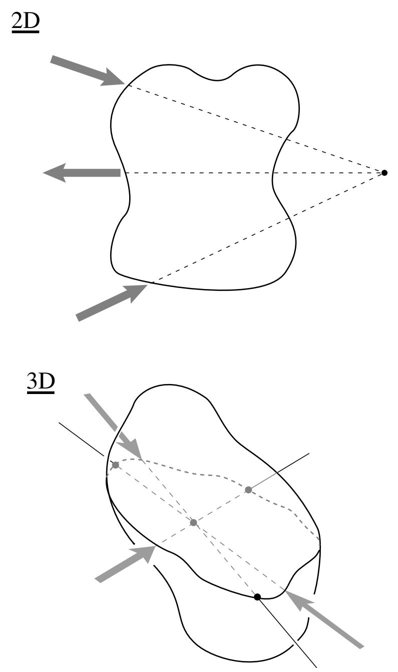
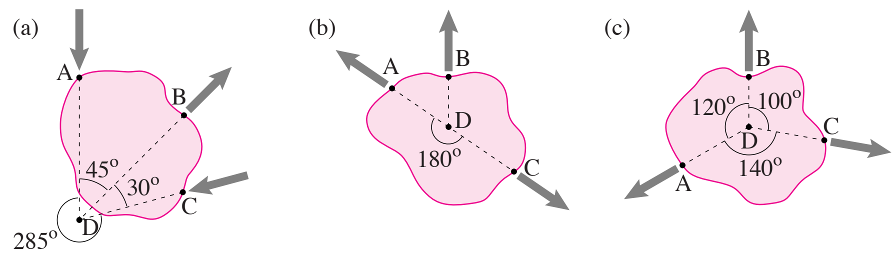


Figure 5.26: If exactly 3 forces act on a body the lines of action of the forces intersect at a single point and are coplanar. The point of intersection does not have to lie within the body. A special case is when that point is at infinity and the three forces are parallel.



The idealized massless pulley

Both real machines and mechanical models are built of various building blocks. One of the standards is a pulley. We often draw pulleys schematically something like in fig. 5.27a which shows that we believe that the tension in a string, line, cable, or rope that goes around an ideal pulley is the same on both sides, $T_1 = T_2 = T$. An *ideal pulley* is

(i.) Round,

(ii.) Has frictionless bearings,

(iii.) Has negligible inertia, and

(iv.) Is wrapped with a line which only carries forces along its length.

We now show that these assumptions lead to the result that $T_1 = T_2 = T$. First, look at a free-body diagram of the pulley with a little bit of string at both ends (fig. 5.27b). Since we assume the bearing has no friction, the interaction between the pulley-bearing shaft and the pulley has no component tangent to the bearing.

To find the relation between tensions, we apply angular momentum

5.3 Two-force bodies

Here we derive the ubiquitously-used result that, if only two forces act on a body, the two forces must be equal in magnitude, opposite in direction, and on a common line of action. You can (and will) use this result even if you do not master the reasoning in this box. But learning this reasoning may help your intuition.

Consider the free-body diagram of a body $\mathcal{B}$ in fig. 5.21a. Forces $\vec{\boldsymbol{F}}_P$ and $\vec{\boldsymbol{F}}_Q$ are acting on $\mathcal{B}$ at points P and Q. Let's apply the equilibrium equations. First, we have that the sum of all forces on the body is zero,

$$\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$$

$$\vec{\boldsymbol{F}}_P + \vec{\boldsymbol{F}}_Q = \vec{\boldsymbol{0}} \quad \Rightarrow \quad \vec{\boldsymbol{F}}_P = -\vec{\boldsymbol{F}}_Q.$$

Thus, the two forces must be equal in magnitude and opposite in direction. So, thus far, we can conclude that the forces must be parallel as shown in fig. 5.21b. But the forces still seem to have a net turning effect, thus still violating the concept of static equilibrium. The sum of all external torques on the body about any point are zero. So, summing moments about point P, we get,

$$\sum_{\substack{\text{All external}\\ \text{torques}}} \vec{\boldsymbol{M}}_{/P} = \vec{\boldsymbol{0}}$$

$$\vec{\boldsymbol{r}}_{Q/P} \times \vec{\boldsymbol{F}}_Q = \vec{\boldsymbol{0}} \qquad (\vec{\boldsymbol{F}}_P \text{ produces no torque about } P)$$

$$\left|\vec{\boldsymbol{r}}_{Q/P}\right| \left(\hat{\boldsymbol{\lambda}}_{Q/P} \times \vec{\boldsymbol{F}}_Q\right) = \vec{\boldsymbol{0}} \qquad \left(\hat{\boldsymbol{\lambda}}_{Q/P} = \frac{\vec{\boldsymbol{r}}_{Q/P}}{|\vec{\boldsymbol{r}}_{Q/P}|} = -\frac{\vec{\boldsymbol{r}}_{P/Q}}{|\vec{\boldsymbol{r}}_{P/Q}|}\right)$$

So $\vec{\boldsymbol{F}}_Q$ has to be parallel to the line connecting P and Q. Similarly, taking the sum of moments about point Q, we get

$$-\hat{\boldsymbol{\lambda}}_{Q/P} \times \vec{\boldsymbol{F}}_Q = \vec{\boldsymbol{0}}$$

and $\vec{\boldsymbol{F}}_P$ also must be parallel to the line connecting P and Q. So, not only are $\vec{\boldsymbol{F}}_P$ and $\vec{\boldsymbol{F}}_Q$ equal and opposite, they are collinear as well since they are parallel to the axis passing through their points of action (see fig. 5.21c).

balance (equation II) about point O

$$\left\{\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{\mathrm{O}}\right\} \cdot \hat{k}. \tag{5.12}$$

Evaluating the left hand side of eqn. 5.12

$$\begin{aligned}\sum \vec{M}_O \cdot \hat{k} &= R_2T_2 - R_1T_1 + \underbrace{\text{bearing friction}}_{0} \\ &= R(T_2 - T_1), \text{ since } R_1 = R_2 = R.\end{aligned}$$

Because there is no friction, the bearing forces acting perpendicular to the round bearing shaft have no moment about point O (see also the short example on page 141). Because the pulley is round, $R_1 = R_2 = R$.

When mass is negligible, dynamics reduces to statics.

Putting these assumptions and results together gives

$$\begin{aligned}&\left\{\sum \vec{M}_{\mathrm{O}} = \vec{0}\right\} \cdot \hat{k} \\ &\Rightarrow R(T_2 - T_1) = 0 \\ &\qquad \Rightarrow T_1 = T_2\end{aligned}$$

Thus, the tensions on the two lines of an ideal massless pulley are equal.

Lopsided pulleys are not often encountered, so it is usually satisfactory to assume round pulleys. But, in engineering practice, the assumption of frictionless bearings is often suspect. In dynamics, you may not want to neglect pulley mass.

Lack of equilibrium as a sign of dynamics

Surprisingly, statics calculations often give useful information about dynamics. If, in a given problem, you find that forces or moments cannot be balanced this is a sign that the related physical system will accelerate in the direction of imbalance (See the example 'block on ramp' on page 270). For more about nonexistence of a statics solution, see box 5.1 on page 247.

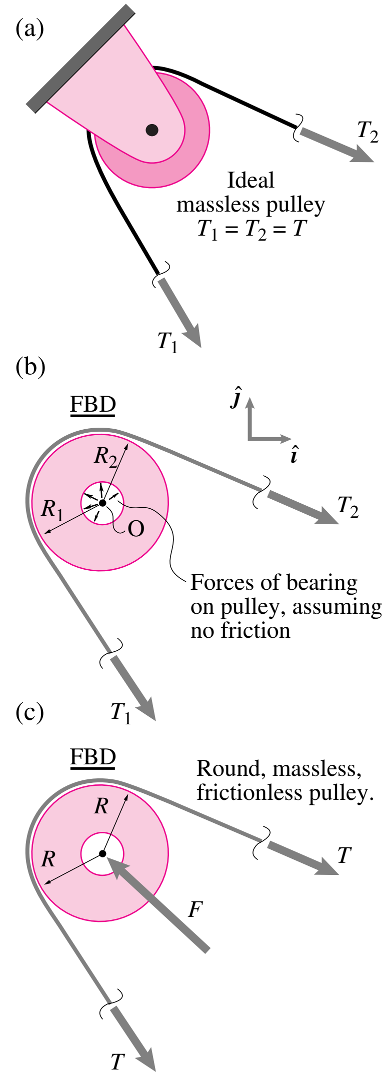


Figure 5.27: (a) An ideal massless pulley, (b) FBD of idealized massless pulley, detailing the frictionless bearing forces and showing forces at the cut strings, (c) final FBD after analysis.

5.4 Three-force bodies

Here is a brief derivation of the result for three-force bodies. The derivation is not needed for problem-solving. However understanding the derivation may help build intuition.

Consider a body in static equilibrium with just three forces on it; $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ acting at $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. Taking moment balance about the axis through points at $\vec{r}_2$ and $\vec{r}_3$ implies that the line of action of $\vec{F}_1$ must pass through that axis. Similarly, for equilibrium to hold, the line of action of $\vec{F}_2$ must intersect the axis through points at $\vec{r}_1$ and $\vec{r}_3$ and the line of action of $\vec{F}_3$ must intersect the axis through $\vec{r}_1$ and $\vec{r}_2$. So, the lines of action of all three forces are in the plane defined by the three points of action and the lines of action of $\vec{F}_2$ and $\vec{F}_3$ must intersect. Taking moment balance about this point of intersection implies that $\vec{F}_1$ has a line of action passing through the same point. A special case is when $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ are parallel and have a common plane of action (equivalent to the concurrency point being at infinity).

Linearity and superposition

For a given geometry, the equilibrium equations are *linear*: If you know a set of forces that is in equilibrium and you also know a second set of forces that is in equilibrium, then the sum of the two sets is also in equilibrium.

Example: A bicycle wheel

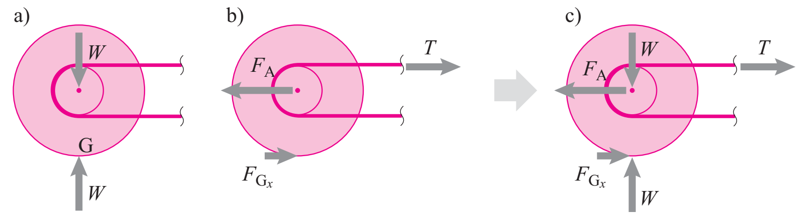


Figure 5.28: A bicycle wheel.

The free-body diagram of an ideal massless bicycle wheel with a vertical load is shown in (a) above. The same wheel driven by a chain tension but with no weight is shown in equilibrium in (b) above. The sum of these two load sets (c) is therefore in equilibrium.

That you can add solutions of linear equations is called the *principle of superposition*, also called the principle of superimposition③. The principle of superposition provides a useful shortcut for some mechanics problems.

③**Superimposition.** Here's a bad pun to help your memory. When talkative Sam comes over, you get bored. When hungry Sally comes over, you reluctantly go get a snack for her. When Sam and Sally come over together, you get bored *and* reluctantly go get a snack. Each one of them is imposing. When they come over together, their effects add. They are super imposing.

5.5 Moment balance about 3 points is sufficient in 2D

This is a theoretical aside showing that moment balance can totally replace force balance.

In 2D one can solve any statically determinate problem using moment balance about any 3 non-collinear points. Force balance adds no information.

Here we show the math behind this useful trick. The derivation here is only for logical completeness, it does not help with problem solving.

Consider two points A and B. Moment balance about these two points gives

$$\sum \vec{r}_{i/A} \times \vec{F}_i = \vec{0} \qquad \text{and} \qquad \sum \vec{r}_{i/B} \times \vec{F}_i = \vec{0}.$$

Subtracting one of these equations from the other gives:

$$\begin{aligned}
\sum \vec{r}_{i/A} \times \vec{F}_i - \sum \vec{r}_{i/B} \times \vec{F}_i &= \vec{0} \\
\sum \left(\vec{r}_{i/A} - \vec{r}_{i/B}\right) \times \vec{F}_i &= \vec{0} \\
\sum \left((\vec{r}_i - \vec{r}_A) - (\vec{r}_i - \vec{r}_B)\right) \times \vec{F}_i &= \vec{0} \\
\sum (\vec{r}_B - \vec{r}_A) \times \vec{F}_i &= \vec{0} \\
(\vec{r}_B - \vec{r}_A) \times \sum \vec{F}_i &= \vec{0}
\end{aligned}$$

Dotting both sides with a vector $\hat{k}$ normal to the plane we get (recalling the mixed triple product identity from page 82 in Section 1.3 that $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}$) we can re-arrange terms to get

$$\begin{aligned}
\left((\vec{r}_B - \vec{r}_A) \times \sum \vec{F}_i\right) \cdot \hat{k} &= \vec{0} \cdot \hat{k} \\
(\hat{k} \times (\vec{r}_B - \vec{r}_A)) \cdot \sum \vec{F}_i &= 0.
\end{aligned}$$

Thus moment balance about the points A and B implies force balance in the direction $\hat{k} \times (\vec{r}_B - \vec{r}_A)$. This is force balance in the direction normal to the line AB (and in the plane).

Now consider a third point C. By the same reasoning moment balance about B and C implies force balance in the direction orthogonal to BC. So long as BC is not parallel to AB then we have force balance in two independent directions. So

$$\sum \vec{F}_i = \vec{0}.$$

The result only goes sour if the two directions are parallel, which occurs when two of the points A, B, and C are on a line. If A, B, and C are not on a line, moment balance about them implies force balance. So use of moment balance replaces the force-balance equilibrium equations.

Moment balance about convenient points A, B, and C can simplify the equilibrium equations if the points are picked so that, by inspection, some forces have no moment.

5.6 How to hold something in place statically determinately?

Often something is held in place, loads are applied and you are to calculate the constraint forces (the forces that hold the object in place). You do this with the equilibrium equations. In 2D, force and moment balance for one free-body diagram give you 3 independent scalar equations. So you can find 3 scalar forces and moments.

Some examples of how to hold an extended object statically determinately in 2D:

- A hinge joint and a rod or string (the line of the string cannot intersect the hinge);
- A hinge joint and a frictionless contact (the normal line at the contact point cannot intersect the hinge);
- Three rods (their three lines of action cannot intersect at one point);
- Three frictionless contacts (the three normals to the contact points cannot intersect at one point, and the applied load must be such that all of the contacts are compressive); and
- A welded joint, this carries two reaction forces and a moment.