SAMPLE 5.6 Find force F for equilibrium of the L-shaped rigid object. The dimensions of the angle are $d = 0.3\,\text{m}$ and $a = 0.2\,\text{m}$.

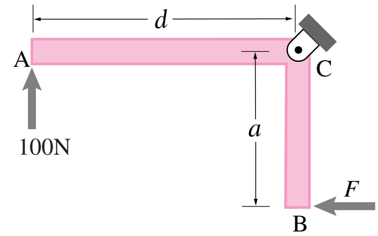

Figure 5.29

Solution The free-body diagram of the angle is shown in fig. 5.30. Since we are interested in force F, we can write the scalar moment-balance equation (in $\hat{\boldsymbol{k}}$ direction) about point C (and thus get rid of the other unknown force $\vec{\boldsymbol{R}}$):

$$\begin{aligned} -Fa - (100\,\text{N})d &= 0 \\ \Rightarrow \quad F &= -(100\,\text{N})\frac{d}{a} \\ &= -100\,\text{N} \cdot \frac{0.3\,\text{m}}{0.2\,\text{m}} \\ &= -150\,\text{N}. \end{aligned}$$

$\vec{\boldsymbol{F}} = -(150\,\text{N})\hat{\boldsymbol{\imath}}$

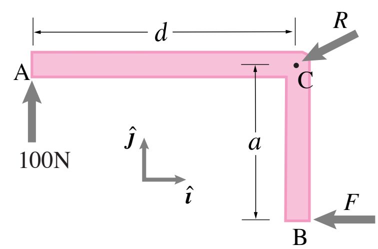

Figure 5.30

SAMPLE 5.7 Consider the angle shown in the figure with the applied forces. Can the angle be in equilibrium for some value of F? Explain.

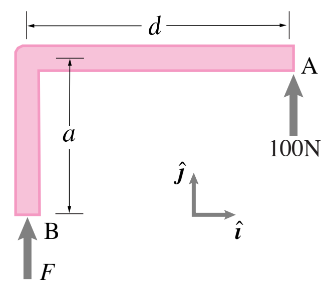

Figure 5.31

Solution Let us assume that the angle is in equilibrium. Then the forces acting on the angle must satisfy the force and moment-balance equations. Now the force balance in the $\hat{\boldsymbol{\jmath}}$ direction gives

$$\begin{aligned} F + (100\,\text{N}) &= 0 \\ \Rightarrow \quad F &= -100\,\text{N}. \end{aligned}$$

The moment balance about point A gives

$$\begin{aligned} Fd &= 0 \\ \Rightarrow \quad F &= 0 \end{aligned}$$

Thus,

$$-100\,\text{N} = 0$$

which is a contradiction. Thus the angle cannot be in equilibrium with the applied forces.

It is easy to see that no matter which way F acts (up or down), it cannot simultaneously balance the applied force at A and its moment. If $F = 100\,\text{N}$ acts upwards at B, the angle will accelerate up because it has a net force in the $\hat{\boldsymbol{\jmath}}$ direction. If $F = 100\,\text{N}$ acts downwards at B, the two equal and opposite forces at A and B produce a net moment on the angle and therefore the angle will start spinning about the $\hat{\boldsymbol{k}}$ direction. In fact, no matter what the value or direction of F is, as long as it acts at point B, the angle cannot be in equilibrium. This is because the angle, as given, is a two-force body, and for equilibrium, the two applied forces must be equal, opposite and collinear.

Equilibrium not possible.

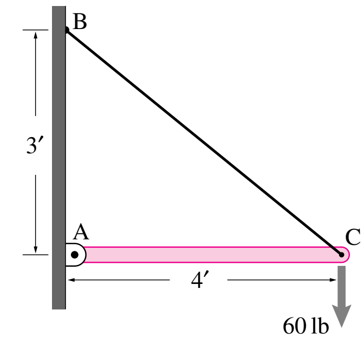

Figure 5.32

SAMPLE 5.8 A bar as a 2-force body: A 4 ft long horizontal bar AC supports a load of 60 lbf at one end and is pinned to a wall at the other end. The bar is also supported by a string BC as shown in the figure. Find the forces applied by the pin and the string on the bar.

Solution Let us do this problem two ways — using equilibrium equations without much thought, and using those equations with some insight.

The free-body diagram of the bar is shown in Fig. 5.33. The moment balance about point A, $\sum \vec{M}_A = 0$, gives

$$\begin{aligned}
\vec{r}_{C/A} \times T\hat{\boldsymbol{\lambda}} + \vec{r}_{C/A} \times (-P\hat{\boldsymbol{\jmath}}) &= \vec{\mathbf{0}} \\
\underbrace{\ell\hat{\boldsymbol{\imath}} \times T(-\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})}_{\ell T \sin\theta \hat{\boldsymbol{k}}} + \underbrace{\ell\hat{\boldsymbol{\imath}} \times (-P\hat{\boldsymbol{\jmath}})}_{-\ell P \hat{\boldsymbol{k}}} &= \vec{\mathbf{0}} \\
(T\ell\sin\theta - P\ell)\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \qquad (5.13)\\
\Rightarrow \qquad T = \frac{P}{\sin\theta} = \frac{60\,\text{lbf}}{3/5} &= 100\,\text{lbf}.
\end{aligned}$$

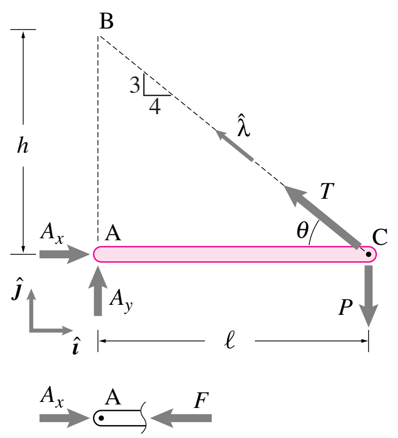

Figure 5.33

The force equilibrium, $\sum \vec{F} = 0$, gives

$$(A_x - T\cos\theta)\hat{\boldsymbol{\imath}} + (A_y + T\sin\theta - P)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \qquad (5.14)$$

Separating out x and y components of this equation, we get

$$\begin{aligned}
A_x = T\cos\theta = (100\,\text{lbf}) \cdot \frac{4}{5} &= 80\,\text{lbf} \\
A_y = P - T\sin\theta &= 0
\end{aligned}$$

where the last equation, $A_y = P - T\sin\theta = 0$ follows from eqn. (5.13) or directly from moment balance about point C. Thus, the force in the rod is $\vec{A} = (80\,\text{lbf})\hat{\boldsymbol{\imath}}$, *i.e.*, a purely compressive force, and the tension in the string is 100 lbf.

$$\boxed{\vec{A} = (80\,\text{lbf})\hat{\boldsymbol{\imath}}, \quad T = 100\,\text{lbf}}$$

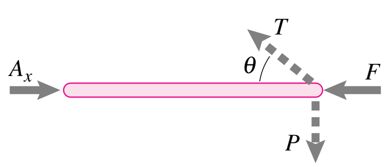

Figure 5.34

Alternate Solution: From the free-body diagram of the rod (see fig. 5.34), we realize that the rod is a *two-force body*, since the forces act at only two points of the body, A and C. The reaction force at A is a single force $\vec{A}$, and the forces at end C, the tension $\vec{T}$ and the load $\vec{P}$, sum up to a single net force, say $\vec{F}$. So, now using the fact that the rod is a two-force body, the equilibrium equation requires that $\vec{F}$ and $\vec{A}$ be equal, opposite, and collinear (along the longitudinal axis of the bar). Thus,

$$\vec{A} = -\vec{F} = -F\hat{\boldsymbol{\imath}}.$$

Now,

$$\begin{aligned}
\vec{F} &= \vec{P} + \vec{T} \\
-F\hat{\boldsymbol{\imath}} &= -P\hat{\boldsymbol{\jmath}} + T\sin\theta\hat{\boldsymbol{\jmath}} - T\cos\theta\hat{\boldsymbol{\imath}} \qquad (5.15)
\end{aligned}$$

Separating out x and y components of this equation, we get

$$\begin{aligned}
-F + T\cos\theta &= 0 \qquad (5.16)\\
P - T\sin\theta &= 0. \qquad (5.17)
\end{aligned}$$

Solving these two equations simultaneously, we get $T = P/\sin\theta = 100\,\text{lbf}$ and $F = T\cos\theta = 80\,\text{lbf}$. The answers, of course, are the same.

SAMPLE 5.9 **A bottle holder:** A clever design of a bottle holder (a plank with a hole) is shown in the figure. Note that the holder is not fixed to the support; it stands freely, but only when the bottle is in. Assume that the mass of the bottle is 1 kg and that the center-of-mass of the bottle is at 3/5th of its length ($h = 35\,\text{cm}$) from the neck support point. The bottle in its rest position is slightly tipped down ($\alpha = 15^\circ$). Assuming the mass of the stand to be negligible and $\ell = 30\,\text{cm}$, find the angle θ of the stand so that the bottle and the stand can be in equilibrium together as shown.

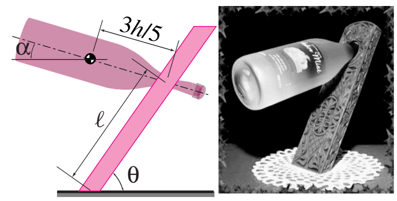

Figure 5.35

Solution Let us draw the free-body diagram of the bottle and the stand together as one system. The forces acting are shown in fig. 5.36. Since the only forces acting on the system are R and mg, they must be equal, opposite and collinear. Thus the line of action of the weight, mg, must pass through the center of the stand's footprint. From the given geometry, then, we must have,

$$\begin{aligned}
\ell\cos\theta &= \frac{3h}{5}\cos\alpha \\
\Rightarrow \quad \theta &= \cos^{-1}\left(\frac{3h}{5\ell}\cos\alpha\right) \\
&= \cos^{-1}\left(\frac{3\cdot 35\,\text{cm}}{5\cdot 30\,\text{cm}}\cos 15^\circ\right) \\
&= 47.5^\circ.
\end{aligned}$$

$\boxed{\theta = 47.5^\circ}$

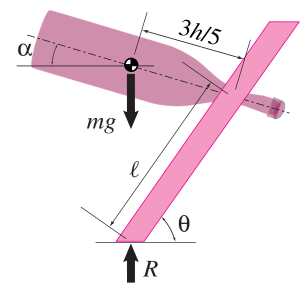

Figure 5.36

Note: The latitude in design of the angle θ depends on the width of the base of the stand. The two forces acting on the system must be collinear and must pass through the base. Therefore, a wider base (perhaps at the expense of elegance) provides more freedom for the forces to move sideways, giving a range of θ and α for design. (see fig. 5.37.)


Figure 5.37: A bottle holder with a wider base.

SAMPLE 5.10 **Reactions at fixed ends.** For the bent bar shown in the figure, find the reaction forces at the fixed end for $F = 10\,\text{kN}$.

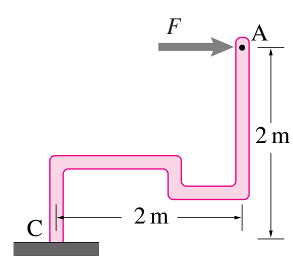

Figure 5.38

Solution The free-body diagram of the rod is shown in fig. 5.39. Note that in addition to the reaction force $\vec{\boldsymbol{R}}$, there is a reaction moment $\vec{\boldsymbol{M}} = M\hat{\boldsymbol{k}}$ acting on the rod because of the fixed support.

The force-balance equation, $\sum\vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives us

$$\begin{aligned}
F\hat{\boldsymbol{\imath}} + \vec{\boldsymbol{R}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad \vec{\boldsymbol{R}} &= -F\hat{\boldsymbol{\imath}} = -(10\,\text{kN})\hat{\boldsymbol{\imath}}.
\end{aligned}$$

Now, we can write the moment-balance equation about point C, $\sum\vec{\boldsymbol{M}}_C = \vec{\boldsymbol{0}}$, to give

$$\begin{aligned}
M\hat{\boldsymbol{k}} - Fd\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad M &= Fd = 20\,\text{kN}\cdot\text{m}.
\end{aligned}$$

$\boxed{\vec{\boldsymbol{R}} = -10\,\text{kN}\,\hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{M}} = 20\,\text{kN}\cdot\text{m}\,\hat{\boldsymbol{k}}}$

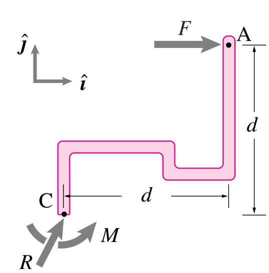

Figure 5.39

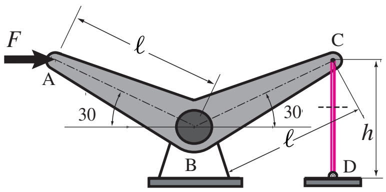

Figure 5.40

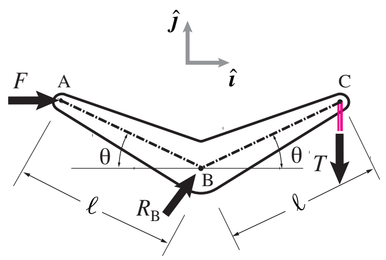

Figure 5.41: Free-body diagram of the structure with rod CD cut just below C.

SAMPLE 5.11 Consider the structure (a rocker arm) shown in the figure. Assume that bar CD can only take axial load (tension or compression). If a horizontal force, $F = 2\,\mathrm{kN}$ is applied at point A, what is the tension in rod CD?

Solution Let T be the tension in the rod (although intuitively you can see that the rod must be under compression). Then, the free-body diagram of the rocker arm ABC is as shown in fig. 5.41. We need to find T. We can do so using either moment balance or force balance as shown below.

Method-1: Using moment balance The easiest way to solve this problem is to apply moment balance, $\sum \vec{\boldsymbol{M}}_{\mathrm{B}} = \vec{\mathbf{0}}$, about point B. Taking moments about this point gets rid of the unknown reaction force R_B and relates T to F directly:

$$\vec{\boldsymbol{r}}_{\mathrm{A/B}} \times \vec{\boldsymbol{F}} + \vec{\boldsymbol{r}}_{\mathrm{C/B}} \times \vec{\boldsymbol{T}} = \vec{\mathbf{0}}$$

We can evaluate the cross products vectorially or use the scalar form of the moment calculation (force times the lever arm) to give

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{A/B}} \times \vec{\boldsymbol{F}} &= -F\ell \sin\theta \hat{\boldsymbol{k}} \\
\vec{\boldsymbol{r}}_{\mathrm{C/B}} \times \vec{\boldsymbol{T}} &= -T\ell \cos\theta \hat{\boldsymbol{k}}.
\end{aligned}$$

So, the scalar moment-balance equation in the $\hat{\boldsymbol{k}}$ direction is

$$\begin{aligned}
-F\ell \sin\theta - T\ell\cos\theta &= 0 \\
\Rightarrow \quad T &= -F\tan\theta.
\end{aligned}$$

Now substituting the given values, $F = 2\,\mathrm{kN}$, and $\theta = 30^{\circ}$, we get

$$T = -(2\,\mathrm{kN}) \cdot (\tan 30^{\circ}) = -1.15\,\mathrm{kN}.$$

Thus the rod is under compression, not tension. It is also clear from the picture that if we push at A, ABC will try to rotate clockwise about B, thus pushing down on the rod at C.

$$\boxed{T = -1.15\,\mathrm{kN}}$$

Method-2: Using force balance We can also use the force-balance equation, $\sum \vec{\boldsymbol{F}} = \vec{\mathbf{0}}$ to find T. However, force balance will involve two unknown forces T and R. The force balance gives

$$\begin{aligned}
&\vec{\boldsymbol{F}} + \vec{\boldsymbol{T}} + \vec{\boldsymbol{R}}_B = 0 \\
\text{or} \quad &F\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + R_B\hat{\boldsymbol{\lambda}} = 0
\end{aligned} \tag{5.18}$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector in the direction of $\vec{\boldsymbol{R}}_B$ and is not known yet. However, we know that the rocker arm is a three-force body, and therefore, all the three forces must be concurrent (they cannot be parallel here). From geometry it is clear that the lines of action of all the three forces must pass through point C. This realization immediately gives us the direction of $\vec{\boldsymbol{R}}_B$, that is, $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$. So, now we can write out eqn. (5.18), separate out x and y components and solve the two scalar equations simultaneously to find both T and R_B. But we are not interested in finding R_B. So why not get use an appropriate dot product with eqn. (5.18) to get rid of R_B and get one scalar equation relating T to F. Let $\hat{\boldsymbol{n}}$ be normal to $\hat{\boldsymbol{\lambda}}$. Thus, $\hat{\boldsymbol{n}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$. Now, dotting with $\hat{\boldsymbol{n}}$ gives

$$\begin{aligned}
[\text{eqn. (5.18)}] \cdot \hat{\boldsymbol{n}} \quad &\Rightarrow \quad F \underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}}}_{-\sin\theta} - T \underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{n}}}_{\cos\theta} + R_B \underbrace{\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{n}}}_{0} = 0 \\
&\Rightarrow \quad -F\sin\theta - T\cos\theta = 0 \\
&\Rightarrow \quad T = -F\tan\theta = -1.15\,\mathrm{kN}
\end{aligned}$$

as obtained by moment balance.

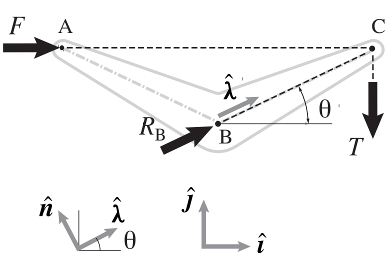

Figure 5.42: The rocker arm is a three-force body and, therefore, the three forces have to be concurrent (their lines of action intersect at one point, C). Thus the support reaction $\vec{\boldsymbol{R}}_B$ acts in the $\hat{\boldsymbol{\lambda}}$ direction. From geometry, $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$. A unit vector normal to $\hat{\boldsymbol{\lambda}}$ is, therefore, $\hat{\boldsymbol{n}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$ (you can guess it from geometry or find it from $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}$).