21.2 3D dynamics of a rigid body

Momenta and energy of rigid bodies

The kinematics formulas we have developed allow us to calculate velocities and accelerations of points on a rigid body. We can therefore calculate the motion quantities: the momenta, their rates of change, and kinetic energy (and its rate of change). The three basic laws of mechanics that are at our disposal are:

$$
\begin{aligned}
\sum \vec{\boldsymbol{F}} &= \dot{\vec{\boldsymbol{L}}}; && \text{Linear momentum balance} \\
\sum \vec{\boldsymbol{M}}_{\mathrm{C}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}; && \text{Angular momentum balance} \\
\sum(\text{Power in}) &= \frac{d}{dt}(E_{\mathrm{K}}) + \frac{d}{dt}(E_{\mathrm{P}}) && \\
&\quad + \text{Dissipation rate}; && \text{Energy/power balance}
\end{aligned}
$$

In order to be a master of mechanics, all you need is to be able to effectively use these three equations for a variety of systems. One who knows how to efficiently and accurately evaluate both sides of these equations in terms of knowns and reasonably defined unknowns is an ace mechanic. One who, in addition, knows how to solve these equations is an ace programmer and/or mathematician.

Our emphasis in this book is on systems that can be idealized as a particle or a rigid body or a system of rigid bodies. So, once you know how to get the forces and moments from the free-body diagram and once you know the masses and moments of inertia, then you need to find the accelerations of the centers of mass, the angular velocities, and angular accelerations of the various bodies.

For each body,

$$
\begin{aligned}
\dot{\vec{\boldsymbol{L}}} &= m\vec{\boldsymbol{a}}_{cm}, \\
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} &= \vec{\boldsymbol{r}}_{cm/C} \times \mathrm{m}\vec{\boldsymbol{a}}_{cm} + \overbrace{\vec{\boldsymbol{\omega}} \times \underbrace{\vec{\boldsymbol{H}}_{\mathrm{cm}}}_{\boxed{\vec{\boldsymbol{H}}_{\mathrm{cm}} = [\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}}} + [\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\alpha}}}^{\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}}, \text{ and} \\
E_{\mathrm{K}} &= \frac{1}{2} m \vec{\boldsymbol{v}}_{cm} \cdot \vec{\boldsymbol{v}}_{cm} + \frac{1}{2} \vec{\boldsymbol{\omega}} \cdot \left[[\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}} \right]
\end{aligned}
$$

So, as repeatedly mentioned in this chapter, the trick is to evaluate $\vec{\boldsymbol{a}}_{cm}$, $\vec{\boldsymbol{\omega}}$, and $\vec{\boldsymbol{\alpha}}$ in terms of the natural variables in the problem.

The keys to make appropriate use of these formulae are:

$$
\begin{aligned}
\vec{\omega}_{\mathcal{C}/\mathcal{F}} &= \vec{\omega}_{\mathcal{B}/\mathcal{F}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}},\\
\vec{\alpha}_{\mathcal{C}/\mathcal{F}} &= \vec{\alpha}_{\mathcal{B}/\mathcal{F}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{\omega}_{\mathcal{C}/\mathcal{B}},\\
\vec{v}_{P/\mathcal{F}} &= \vec{v}_{0'/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}}, \text{ and}\\
\vec{a}_{P/\mathcal{F}} &= \vec{a}_{0'/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times (\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'}) + {}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'}\\
&\quad + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{v}_{P/\mathcal{B}}.
\end{aligned}
$$

Alternatively, the second, third, and fourth formulae may be effectively derived on an ad-hoc basis by use of the more fundamental formulae

$$ {}^{\mathcal{F}}\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q} \qquad (\text{for any } \vec{Q} \text{ fixed in } \mathcal{B}) $$

on various base vectors.

Now, we look at finding the linear and angular momenta for a rigid body using the 3-D example of the crane in fig. 21.19. The crane is a system of connected bodies moving in well defined ways relative to each other and to the ground.

21.4 $\dot{\vec{H}}$ for a rigid body

We can now derive the formula for rate of change of angular momentum of a rigid body $\mathcal{B}$ about some point C using the '$\dot{\vec{Q}}$ *formula*'

$$
\begin{aligned}
\dot{\vec{H}}_{/C} &= \frac{d}{dt}(\vec{H}_{/C})\\
&= \frac{d}{dt}\left[\vec{r}_{cm/C} \times m\vec{v}_{cm} + \overbrace{\sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i}^{[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}}\right]\\
&= \underbrace{\dot{\vec{r}}_{cm/C} \times m\vec{v}_{cm}}_{\vec{0}} + \vec{r}_{cm/C} \times m\vec{a}_{cm} + \frac{d}{dt}\left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right]\\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} \ldots\\
&\quad \underbrace{+\vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] \quad + {}^{\mathcal{B}}\frac{d}{dt}\left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right]}_{\boxed{\frac{d}{dt}\left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] \text{ using the } \dot{\vec{Q}} \text{ formula}}}\\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} + \vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right]\\
&\quad + \underbrace{{}^{\mathcal{B}}\frac{d}{dt}[I^{cm}]}_{\vec{0}\ \leftarrow\ \boxed{[I^{cm}] \text{ is constant in } \mathcal{B}}} + [I^{cm}]\cdot {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}}\\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} + \vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] + [I^{cm}]\cdot {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}}\\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} + \vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] + [I^{cm}]\cdot \dot{\vec{\omega}}_{\mathcal{B}}
\end{aligned}
$$

In the last step of calculation, we have used the important fact about angular acceleration of a body:

$$
\begin{aligned}
{}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} &= {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} + \underbrace{\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{\omega}_{\mathcal{B}/\mathcal{F}}}_{\vec{0}}\\
&= {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}}.
\end{aligned}
$$

That is, the derivative with respect to time of $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$ is the same in $\mathcal{F}$ as in $\mathcal{B}$.

Example: Mechanics of a rigid body in a mechanism (3-D): A crane, again

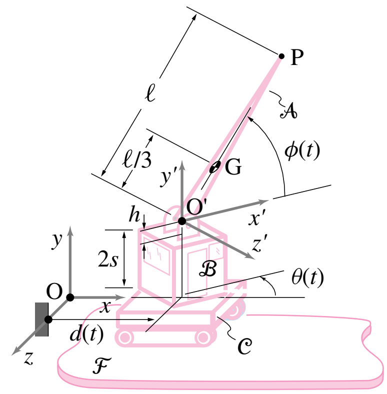


Figure 21.19

We would like to find the rate of change of linear momentum of the boom, $\dot{\vec{L}}_{\mathcal{A}}$, and the rate of change of angular momentum of the boom about point O, $\dot{\vec{H}}_{\mathcal{A}/O}$. Assume the mass $m_{\mathcal{A}}$ and the moment of inertia matrix $[I^{\text{cm}}]_{\mathcal{A}}$ are both known. We have

$$\begin{aligned}
\dot{\vec{L}}_{\mathcal{A}} &= m_{\mathcal{A}}\vec{a}_G, \text{ and} \\
\dot{\vec{H}}_{\mathcal{A}O} &= \vec{r}_{G/O} \times m_{\mathcal{A}}\vec{a}_G + [I^{\text{cm}}]_{\mathcal{A}} \cdot \vec{\alpha}_{\mathcal{A}} + \vec{\omega}_{\mathcal{A}} \times \left[[I^{\text{cm}}]_{\mathcal{A}} \cdot \vec{\omega}_{\mathcal{A}} \right].
\end{aligned}$$

So, we must find $\vec{r}_{G/O}$, $\vec{a}_G$, $\vec{\omega}_{\mathcal{A}}$, and $\vec{\alpha}_{\mathcal{A}}$. Here we go. First, we find $\vec{r}_{G/O}$

$$\begin{aligned}
\vec{r}_{G/O} &= \vec{r}_{O'/O} + \vec{r}_{G/O'} \\
&= \overbrace{d\hat{\imath} + (2s+h)\hat{\jmath}}^{\vec{r}_{O'/O}} + \overbrace{\frac{\ell}{3}(\cos\phi\,\hat{\imath}' + \sin\phi\,\hat{\jmath}')}^{\vec{r}_{G/O'}}.
\end{aligned}$$

Next, we find $\vec{\omega}_{\mathcal{A}}$

$$\vec{\omega}_{\mathcal{A}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{A}/\mathcal{B}} = \dot{\theta}\,\hat{\jmath}' + \dot{\phi}\,\hat{k}'.$$

Then, we find $\vec{\alpha}_{\mathcal{A}}$

$$\vec{\alpha}_{\mathcal{A}} = \vec{\alpha}_{\mathcal{B}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{C}/\mathcal{B}} = \ddot{\theta}\,\hat{\jmath}' + \ddot{\phi}\,\hat{k}' + \dot{\theta}\dot{\phi}\,\hat{\imath}'.$$

Finally, we find $\vec{a}_G$

$$\vec{a}_G = \vec{a}_{O'} + \vec{\omega}_{\mathcal{A}} \times (\vec{\omega}_{\mathcal{A}} \times \vec{r}_{G/O'}) + \vec{\alpha}_{\mathcal{A}/\mathcal{F}} \times \vec{r}_{G/O'} + \vec{a}_{G/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{G/\mathcal{B}}.$$

At this point, we see several terms that we need to evaluate to plug into this formula. Here they are:

$$\begin{aligned}
\vec{a}_{O'} &= \ddot{d}\hat{\imath}, \\
\vec{v}_{G/\mathcal{B}} &= \vec{\omega}_{\mathcal{A}/\mathcal{B}} \times \vec{r}_{G/O'}, \\
&= \frac{\ell}{3}\dot{\phi}(\cos\phi\,\hat{\jmath}' - \sin\phi\,\hat{\imath}'), \text{ and} \\
\vec{a}_{G/\mathcal{B}} &= \vec{\omega}_{\mathcal{A}/\mathcal{B}} \times (\vec{\omega}_{\mathcal{A}/\mathcal{B}} \times \vec{r}_{G/O'}) + \vec{\alpha}_{\mathcal{A}/\mathcal{B}} \times \vec{r}_{G/O'}, \\
&= \frac{\ell}{3}(-\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi)\hat{\imath}' + \frac{\ell}{3}(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)\hat{\jmath}'.
\end{aligned}$$

So, we can plug in and find the absolute acceleration of the center-of-mass of body $\mathcal{A}$

$$\begin{aligned}
\vec{a}_G = \ddot{d}\hat{\imath} + \frac{\ell}{3}\Big[&-(\ddot{\phi}\sin\phi + (\dot{\phi}^2 + \dot{\theta}^2)\cos\phi)\hat{\imath}' \\
&+ (\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)\hat{\jmath}' + (2\dot{\phi}\dot{\theta}\sin\phi - \ddot{\theta}\cos\phi)\hat{k}' \Big].
\end{aligned}$$

Thus, for the crane boom, now we have $\vec{r}_{G/O}$, $\vec{\omega}_{\mathcal{A}}$, $\dot{\vec{\omega}}_{\mathcal{A}}$, and $\vec{a}_G$ in terms of known dimensions, position vectors, and the angular rates $\dot{\phi}$, and $\dot{\theta}$, and their derivatives $\ddot{\phi}$ and $\ddot{\theta}$. So, now we can evaluate $\dot{\vec{L}}_{\mathcal{A}}$ and $\dot{\vec{H}}_{\mathcal{A}/O}$.

Yes – this situation is a mess! Rigid body calculations in three dimensions lead to big messy equations. There are a few special cases where things simplify. Some of these situations are shown in the sample problems and the homework.

A note of caution: When back substituting with the formulae given, be sure that the same base vectors are used for $[I^{\text{cm}}]$ as for $\vec{\omega}_{\mathcal{A}}$ and $\vec{\alpha}_{\mathcal{A}}$.

SAMPLE 21.7 An axisymmetric body with precession and spin. A uniform circular disk of mass $m = 2\,\text{kg}$ is mounted on gimbals which in turn is mounted on a horizontal shaft. The disk spins about its normal axis of symmetry. The spin axis makes an angle $\phi = 15^\circ$ with the horizontal shaft. The shaft rotates at a constant speed $\omega_1 = 60\,\text{rpm}$. The disk spins with respect to the gimbals at a constant speed $\omega_2 = 120\,\text{rpm}$. The moment of inertia of the disk about its diameter is $A = 10\,\text{kg}\cdot\text{m}^2$ and the moment of inertia of the disk about its normal is $B = 20\,\text{kg}\cdot\text{m}^2$. At the instant shown, find

1. the angular momentum $\vec{\boldsymbol{H}}_{/G}$ of the disk,
2. the rate of change of the angular momentum $\dot{\vec{\boldsymbol{H}}}_{/G}$ of the disk.

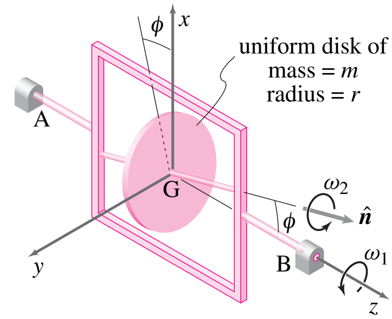
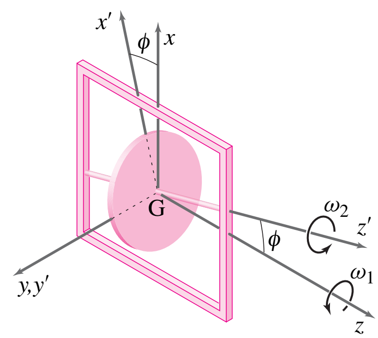


Figure 21.20

Solution Let xyz be the coordinate axes attached to the inertial frame and $x'y'z'$ be the coordinate axes attached to the gimbals $\mathcal{G}$ such that z' is aligned with the spin axis of the disk. Therefore, at the instant shown,

$$\hat{\boldsymbol{\imath}}' = \cos\phi\hat{\boldsymbol{\imath}} - \sin\phi\hat{\boldsymbol{k}}, \quad \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}, \quad \hat{\boldsymbol{k}}' = \sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{k}}. \tag{21.12}$$



Basis vectors when viewed along the y-axis:

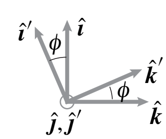


Figure 21.21

1. **Angular momentum:** The angular momentum of the disk is given by

$$\vec{\boldsymbol{H}}_{/G} = [\boldsymbol{I}^{\text{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}}$$

where $[\boldsymbol{I}^{\text{cm}}]$ is the moment of inertia matrix of the disk about its center-of-mass and $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ is the absolute angular velocity of the disk.
Now the absolute angular velocity of the disk

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \vec{\boldsymbol{\omega}}_{\mathcal{G}} + \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{G}}$$

$$\begin{aligned} \text{where} \quad \vec{\boldsymbol{\omega}}_{\mathcal{G}} &= \omega_1\hat{\boldsymbol{k}} = 60\,\text{rpm}\hat{\boldsymbol{k}} = 2\pi\,\text{rad/s}\hat{\boldsymbol{k}}, \\ \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{G}} &= \omega_2\hat{\boldsymbol{k}}' = 120\,\text{rpm}\hat{\boldsymbol{k}}' = 4\pi\,\text{rad/s}\hat{\boldsymbol{k}}'. \end{aligned}$$

The principal moments of inertia for the disk are given. We have chosen the body axes $x'y'z'$ along the principal axes (i.e. $x'y'$ along the two diameters of the disk and z' along the normal of the disk). Therefore in the $x'y'z'$ coordinate system

$$[\boldsymbol{I}^{\text{cm}}]_{x'y'z'} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix}$$

Where $A = 10\,\text{kg}\cdot\text{m}^2$ and $B = 20\,\text{kg}\cdot\text{m}^2$. Since the inertia matrix is written in the body coordinate system, we need to express $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ also in the body coordinate system to carry out the product $[\boldsymbol{I}^{\text{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}}$.

$$\text{Now} \quad \vec{\boldsymbol{\omega}}_{\mathcal{D}} = \omega_1\hat{\boldsymbol{k}} + \omega_2\hat{\boldsymbol{k}}',$$

and from Fig. 21.20 we can write $\hat{\boldsymbol{k}} = -\sin\phi\hat{\boldsymbol{\imath}}' + \cos\phi\hat{\boldsymbol{k}}'$, therefore,

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = (-\omega_1\sin\phi)\hat{\boldsymbol{\imath}}' + (\omega_2 + \omega_1\cos\phi)\hat{\boldsymbol{k}}'.$$

So,

$$\begin{aligned} \vec{\boldsymbol{H}}_{/G} &= [\boldsymbol{I}^{\text{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} \begin{Bmatrix} -\omega_1\sin\phi \\ 0 \\ \omega_2 + \omega_1\cos\phi \end{Bmatrix} \\ &= (-A\omega_1\sin\phi)\hat{\boldsymbol{\imath}}' + B(\omega_2 + \omega_1\cos\phi)\hat{\boldsymbol{k}}'. \end{aligned} \tag{21.13}$$

Substituting the given values of $\omega_1,\ \omega_2,\ A,\ B$ and ϕ we get

$$\vec{\boldsymbol{H}}_{/G} = [-16.26\hat{\boldsymbol{\imath}}' + 372.71\hat{\boldsymbol{k}}']\,\mathrm{kg\cdot m^2/s}$$

Although we have the answer we need, it is expressed in terms of the body coordinate system's basis vectors. We can easily express $\vec{\boldsymbol{H}}_{/G}$ in the fixed coordinate system using Eq 21.12:

$$\vec{\boldsymbol{H}}_{/G} = [-16.26(\cos\phi\hat{\boldsymbol{\imath}} - \sin\phi\hat{\boldsymbol{k}}) + 372.71(\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{k}})]\,\mathrm{N\cdot m\cdot s}$$

$$\boxed{\vec{\boldsymbol{H}}_{/G} = (80.76\hat{\boldsymbol{\imath}} + 364.22\hat{\boldsymbol{k}})\,\mathrm{N\cdot m\cdot s}}$$

2. **Rate of change of angular momentum:** The rate of change of angular momentum of the disk about point G is given by

$$\dot{\vec{\boldsymbol{H}}}_{/G} = [\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} + \vec{\boldsymbol{\omega}}_{\mathcal{D}} \times [\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}} \qquad (21.14)$$

Thus, to compute $\dot{\vec{\boldsymbol{H}}}_{/G}$, we need the angular acceleration of the disk $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$.

$$\begin{aligned}
\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} &= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{G}} + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{G}} \\
&= \underbrace{\dot{\vec{\omega}}_1}_{0}\hat{\boldsymbol{k}} + \underbrace{\underbrace{\dot{\vec{\omega}}_2}_{0}\hat{\boldsymbol{k}}' + \vec{\omega}_1\hat{\boldsymbol{k}} \times \vec{\omega}_2\hat{\boldsymbol{k}}'}_{\dot{\vec{\omega}}_{\mathcal{D}/\mathcal{G}}} \\
&= \omega_1\omega_2(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}') = \omega_1\omega_2(-\sin\phi\hat{\boldsymbol{\imath}}' + \cos\phi\hat{\boldsymbol{k}}') \times \hat{\boldsymbol{k}}' = \omega_1\omega_2\sin\phi\,\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

Here, we calculated $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$ in the body coordinate system because we need to calculate $[\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$ and we know $[\boldsymbol{I}^{\mathrm{cm}}]$ in the body coordinate system. Thus the first term in $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ is

$$[\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1\omega_2\sin\phi \\ 0 \end{Bmatrix} = A\omega_1\omega_2\sin\phi\,\hat{\boldsymbol{\jmath}}'.$$

The second term in $\dot{\vec{\boldsymbol{H}}}_{/G}$ is obtained by taking the cross product of $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ expressed in the body coordinate system with the expression for $\vec{\boldsymbol{H}}_{/G}$ given by Eq 21.13:

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}} \times [\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}} \times \vec{\boldsymbol{H}}_{/G} \\
&= [(-\omega_1\sin\phi)\hat{\boldsymbol{\imath}}' + (\omega_2 + \omega_1\cos\phi)\hat{\boldsymbol{k}}'] \times [(-A\omega_1\sin\phi)\hat{\boldsymbol{\imath}}' + B(\omega_2 + \omega_1\cos\phi \\
&= (B - A)\omega_1\sin\phi[\omega_1\cos\phi + \omega_2]\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

Adding the two terms together we get

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/G} &= A\omega_1\omega_2\sin\phi\,\hat{\boldsymbol{\jmath}}' + (B - A)\omega_1\sin\phi[\omega_1\cos\phi + \omega_2]\hat{\boldsymbol{\jmath}}' \\
&= [(B - A)\omega_1\cos\phi + B\omega_2]\omega_1\sin\phi\,\hat{\boldsymbol{\jmath}}'. \qquad (21.15)
\end{aligned}$$

Substituting $B = 20\,\mathrm{kg\cdot m^2}$, $A = 10\,\mathrm{kg\cdot m^2}$, $\omega_1 = 2\pi\,\mathrm{rad/s}$, $\omega_2 = 4\pi\,\mathrm{rad/s}$ and $\phi = 15^\circ$ we get the answer

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{/G} = 507.41\,\mathrm{N\cdot m}\ \hat{\boldsymbol{\jmath}}.}$$

SAMPLE 21.8 Torque free motion of an axisymmetric rigid body

Consider the gimbal-mounted disk of sample 21.7 again. Assume that the net torque on the disk is zero. That is, for what motions could the gimbals be eliminated and have the motion proceed?

1. Derive an expression for the ratio of the two angular speeds, $\frac{\omega_1}{\omega_2}$, for the torque free motion of the disk.
2. Assume ϕ to be very small. What is the ratio of the angular speeds for the given disks (or any flat axisymmetric object with $B = 2A$)?

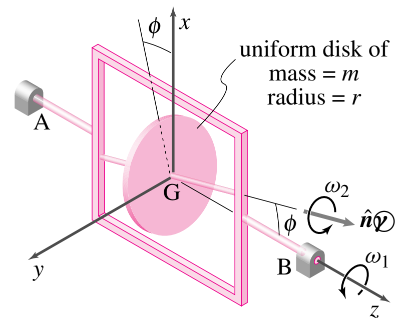


Figure 21.22: The origin G is at the center-of-mass of the disk in this problem.

Solution

1. The angular momentum balance for the disk about point G gives:

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

We are given that the net torque on the disk must be zero, *i.e.*, $\sum \vec{M}_G = \dot{\vec{H}}_G = \vec{0}$. We found an expression for $\dot{\vec{H}}_G$ in the previous sample problem (see Eq 21.15 of Sample 21.7) to be:

$$\dot{\vec{H}}_G = [(B - A)\omega_1 \cos\phi + B\omega_2]\omega_1 \sin\phi\, \hat{\boldsymbol{j}}.$$

Now under the torque free condition,

$$\begin{aligned} \dot{\vec{H}}_G &= \vec{0} \\ \Rightarrow \quad (B - A)\omega_1 \cos\phi &= -B\omega_2 \\ \Rightarrow \quad \frac{\omega_1}{\omega_2} &= \frac{B}{(A - B)\cos\phi} \end{aligned} \tag{21.16}$$

2. for small ϕ, $\cos\phi \approx 1$. Also, for the disk, $B = 2A$. Substituting these values into the above expression for $\frac{\omega_1}{\omega_2}$, we get

$$\frac{\omega_1}{\omega_2} = \frac{2A}{A - 2A} = -2$$

That is, the rate of precession is twice the rate of relative spin. Also, if $\vec{\omega}_1 = \omega_1 \hat{\boldsymbol{k}}$, then $\vec{\omega}_2 = -\frac{\omega_1}{2}\hat{\boldsymbol{k}}'$.

$$\boxed{\frac{\omega_1}{\omega_2} = -2}$$

Comments: The results obtained here are more general than they seem. In particular, it can be shown that the most general motion of a torque free axisymmetric body satisfies Eq 21.16. If you throw a coin in the air it wobbles according to the motion described as constant rate motion about an axis that itself rotates at a constant rate.

In case (b), the absolute angular velocity is $\omega_1 \hat{\boldsymbol{k}} + \omega_2 \hat{\boldsymbol{k}}'$ which is $\omega_1 \boldsymbol{k} + (-\frac{\omega_1}{2})\boldsymbol{k}'$ which is approximately $\frac{\omega_1}{2}\boldsymbol{k}'$. So, the wobble rate ω_1 is twice the spin rate $\frac{\omega_1}{2}$ for flat round objects in torque free motion.

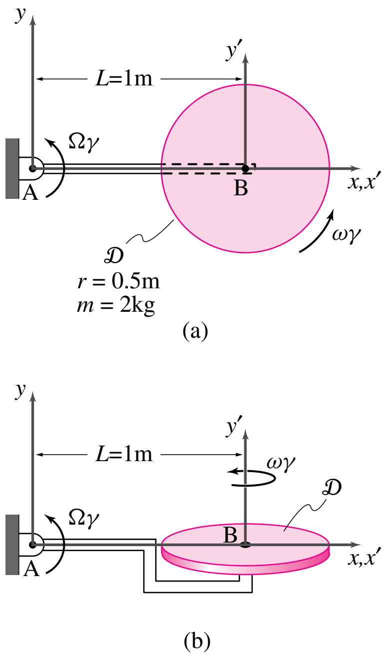


Figure 21.23

SAMPLE 21.9 Calculation of energy. In the figures shown, the rigid arm AB rotates with constant angular velocity $\Omega = 5\,\mathrm{rad/s}$ and the disk $\mathcal{D}$ rotates with respect to the arm at constant angular speed $\omega = 10\,\mathrm{rad/s}$. Assume that the rigid arm is massless. The uniform disk has a mass of 2 kg and measures 50 cm along the diameter. Find the kinetic energy of the system in each case.

Solution Case (a): The motion of the disk is in 2-D . Therefore, the kinetic energy of the disk is

$$E_{\mathrm{K}} = \frac{1}{2} m v_{\mathrm{cm}}^2 + \frac{1}{2} I_{zz}^{\mathrm{cm}} \omega^2$$

Where v_{cm} is the speed of the center-of-mass of the disk and ω is the magnitude of the absolute angular velocity of the disk. Since the center-of-mass of the disk executes constant-speed circular motion with radius $L = 1\,\mathrm{m}$,

$$v_{\mathrm{cm}} = \Omega L = 5\,\mathrm{rad/s} \cdot 1\,\mathrm{m} = 5\,\mathrm{m/s}.$$

The absolute angular velocity of the disk is

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{k}} = (\Omega + \omega)\hat{\boldsymbol{k}}.$$

Therefore, the angular speed of the disk is $\omega_{\mathcal{D}} = \Omega + \omega = 15\,\mathrm{rad/s}$. Substituting these values and $m = 2\,\mathrm{kg}$ and $I_{zz}^{\mathrm{cm}} = \frac{1}{2} m r^2 = 0.0625\,\mathrm{kg\,m^2}$, we get

$$E_{\mathrm{K}} = \frac{1}{2} \cdot 2\,\mathrm{kg} \cdot (5\,\mathrm{m/s})^2 + \frac{1}{2} \cdot (0.0625\,\mathrm{kg\,m^2}) \cdot (15\,\mathrm{rad/s})^2 = 32\,\mathrm{N{\cdot}m}$$

$$\boxed{E_{\mathrm{K}} = 32\,\mathrm{N{\cdot}m}}$$

Case (b): The motion of the disk is in 3-D. The kinetic energy is now given by

$$E_{\mathrm{K}} = \frac{1}{2} m v_{\mathrm{cm}}^2 + \frac{1}{2} \vec{\boldsymbol{\omega}}_{\mathcal{D}} \cdot ([\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}})$$

The translational part of the energy is the same as in case (a). For the rotational part, note that $\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{j}}' = \Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{j}}$, and (1)

$$[\boldsymbol{I}^{\mathrm{cm}}] = \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(1) Without calculation, could you have predicted that the kinetic energy in case (a) would be greater than in case (b)?

Therefore, the rotational kinetic energy is

$$\begin{aligned}
\frac{1}{2} \vec{\boldsymbol{\omega}}_{\mathcal{D}} \cdot ([\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}}) &= \frac{1}{2} (\Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{j}}) \cdot \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ \Omega \end{Bmatrix} \\
&= \frac{1}{2} (\Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{j}}) \cdot \frac{1}{4} m r^2 (\omega \hat{\boldsymbol{j}} + 2\Omega \hat{\boldsymbol{k}}) \\
&= \frac{1}{8} m r^2 (2\Omega^2 + \omega^2) \\
&= 2.34\,\mathrm{N{\cdot}m}
\end{aligned}$$

Therefore,

$$E_{\mathrm{K}} = \frac{1}{2} \cdot 2\,\mathrm{kg} \cdot (5\,\mathrm{m/s})^2 + 2.34\,\mathrm{N{\cdot}m} \approx 27\,\mathrm{N{\cdot}m}.$$

$$\boxed{E_{\mathrm{K}} = 27\,\mathrm{N{\cdot}m}}$$

SAMPLE 21.10 **Dynamic reactions.** A uniform disk of mass $m = 2\,\mathrm{kg}$ and radius $r = 0.5\,\mathrm{m}$ is mounted on frictionless bearings on a 'L' shaped massless shaft OBC. The disk spins about its centroidal axis at a constant rate $\omega_2 = 15\,\mathrm{rad/s}$ while the shaft rotates about O in the vertical plane at a constant rate $\omega_1 = 3\,\mathrm{rad/s}$. Given that $L = 1\,\mathrm{m}$, $R = 0.5\,\mathrm{m}$, find the dynamic reactions at the support O at the instant shown in Fig. 21.24.

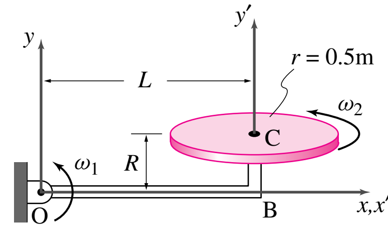


Figure 21.24:

Solution Let us take the disk together with the massless shaft as our system. The free-body diagram of this system is shown in Fig. 21.25. Since we are interested only in dynamic reactions, the force on the system due to gravity, *i.e.,* the weight of the disk is ignored (because this force induces static reactions). Now let us carry out the momentum balance for our system.

From the linear momentum balance for the system,

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}}_{\mathrm{cm}} \\ \vec{\boldsymbol{R}} &= m\vec{\boldsymbol{a}}_C. \end{aligned} \tag{21.17}$$

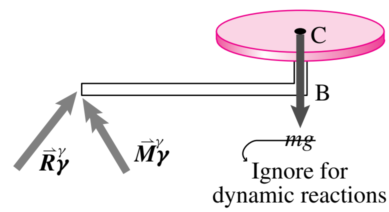


Figure 21.25: Free-body diagram of the shaft and the disk system. $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{M}}$ are the unknown dynamic reactions at the support O.

Thus, we need to find the acceleration of point C, the center-of-mass of the disk, to evaluate $\vec{\boldsymbol{R}}$. Since point C is at the end of the shaft OBC, it rotates about point O with constant speed ω_1, i.e. it executes constant rate circular motion with radius OC (see Figure 21.26) and its motion lies in the xy-plane. Therefore,

$$\begin{aligned} \vec{\boldsymbol{a}}_C &= \underbrace{\dot{\omega}_1}_{0}\,\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_C - \omega_1^2 \vec{\boldsymbol{r}}_C \\ &= -\omega_1^2 (L\hat{\boldsymbol{\imath}} + R\hat{\boldsymbol{\jmath}}) \\ &= -9(\,\mathrm{rad/s})^2 \cdot (1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 0.5\,\mathrm{m}\hat{\boldsymbol{\jmath}}) \\ &= -(9.0\hat{\boldsymbol{\imath}} + 4.5\hat{\boldsymbol{\jmath}})\,\mathrm{m/s^2}. \end{aligned}$$

Substituting the result in Eqn.(21.17) we get

$$\begin{aligned} \vec{\boldsymbol{R}} &= 2\,\mathrm{kg} \cdot (-9.0\hat{\boldsymbol{\imath}} - 4.5\hat{\boldsymbol{\jmath}})\,\mathrm{m/s^2} \\ &= -18\,\mathrm{N}\hat{\boldsymbol{\imath}} - 9\,\mathrm{N}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

$$\vec{\boldsymbol{R}} = -18\,\mathrm{N}\hat{\boldsymbol{\imath}} - 9\,\mathrm{N}\hat{\boldsymbol{\jmath}}$$

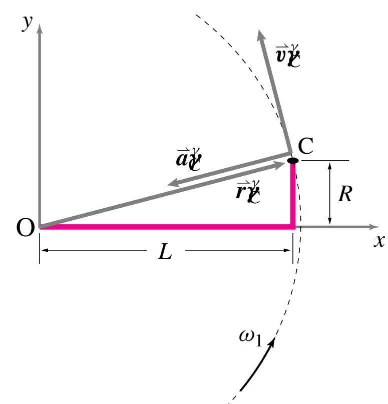


Figure 21.26: The center-of-mass of the disc, point C, executes constant rate circular motion about point O.

To find $\vec{\boldsymbol{M}}$ let us do angular momentum balance for the system about point O:

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \\ \vec{\boldsymbol{M}} &= [\boldsymbol{I}^{\mathrm{cm}}] \cdot \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} + \vec{\boldsymbol{\omega}}_{\mathcal{D}} \times ([\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}}) + \underbrace{\vec{\boldsymbol{r}}_C \times m\vec{\boldsymbol{a}}_C}_{=\vec{\boldsymbol{0}}} \end{aligned} \tag{21.18}$$

where the last term is zero because $\vec{\boldsymbol{r}}_C || \vec{\boldsymbol{a}}_C$ (see Fig .21.26). Thus, to find $\vec{\boldsymbol{M}}$ we need to find the absolute angular acceleration $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$, the absolute angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ and $[\boldsymbol{I}^{\mathrm{cm}}]$ of the disk.

Let us attach a frame $\mathcal{B}$ to the shaft OBC. Thus frame $\mathcal{B}$ rotates with the shaft with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_1\hat{\boldsymbol{k}}$. For calculations in the rotating frame, we attach a coordinate system $x'y'z'$ with basis vectors $\hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}', \hat{\boldsymbol{k}}'$ to the shaft (and *not* to the disk) at point C (see

Fig .21.27). Note that the moment of inertia matrix $[\boldsymbol{I}^{\text{cm}}]$ of the disk in the $x'y'z'$ coordinate system remains constant due to the symmetry of the disk about these axes and this matrix in the primed coordinate system is written as

$$[\boldsymbol{I}^{\text{cm}}] = \frac{mr^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

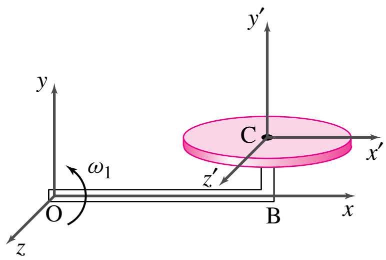


Figure 21.27: The primed coordinate system $x'y'z'$ is glued to the rotating frame $\mathcal{B}$ which rotates with the shaft OBC with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_1\hat{\boldsymbol{k}}$. At the instant shown, the two coordinate axes xyz and $x'y'z'$ are parallel to each other.

Now we find the absolute angular velocity and angular acceleration of the disk:

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\
&= \omega_1\hat{\boldsymbol{k}} + \omega_2\hat{\boldsymbol{j}}. \\
\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} &= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\
&= \underbrace{\dot{\omega}_1}_{0}\hat{\boldsymbol{k}} + \underbrace{\dot{\omega}_2}_{0}\hat{\boldsymbol{j}} + \omega_1\hat{\boldsymbol{k}} \times \omega_2\hat{\boldsymbol{j}} \\
&= -\omega_1\omega_2\hat{\boldsymbol{\imath}}.
\end{aligned}$$

But, for the matrix and vector products in Eqn. (21.18) to be defined both quantities *must* be expressed in the same coordinate system. Since $[\boldsymbol{I}^{\text{cm}}]$ is in the primed coordinate system, we must express $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ and $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$ in the primed coordinate system. Fortunately, at the instant of interest

$$\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}, \quad \hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}.$$

Therefore, the components of any vector in the two coordinate systems are trivially the same. Hence,

$$[\boldsymbol{I}^{\text{cm}}] \cdot \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} = \frac{mr^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -\omega_1\omega_2 \\ 0 \\ 0 \end{Bmatrix} = -\frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}},$$

and similarly,

$$[\boldsymbol{I}^{\text{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}} = \frac{mr^2}{4}(2\omega_2\hat{\boldsymbol{j}} + \omega_1\hat{\boldsymbol{k}}).$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}} \times ([\boldsymbol{I}^{\text{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}}) &= (\omega_1\hat{\boldsymbol{k}} + \omega_2\hat{\boldsymbol{j}}) \times \frac{mr^2}{4}(2\omega_2\hat{\boldsymbol{j}} + \omega_1\hat{\boldsymbol{k}}) \\
&= \frac{mr^2}{4}(\omega_1\omega_2\hat{\boldsymbol{\imath}} - 2\omega_1\omega_2\hat{\boldsymbol{\imath}}) \\
&= -\frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}}.
\end{aligned}$$

Now substituting these expressions in Eqn. (21.18) we get

$$\begin{aligned}
\vec{\boldsymbol{M}} &= -\frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}} - \frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}} \\
&= -\frac{1}{2}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}} \\
&= -\frac{1}{2}(2\,\text{kg}) \cdot 0.25\,\text{m}^2 \cdot 15\,\text{rad/s}^2 \cdot 3\,\text{rad/s} \\
&= -11.25\,\text{N·m}.
\end{aligned}$$

$$\vec{\boldsymbol{M}} = -11.25\,\text{N·m}$$

21.2 Instantaneous dynamics and "inverse dynamics" in 3-D

21.2.1 Under what circumstances is the linear momentum of a system conserved (that is, does not change with time)?

21.2.2 For a continuous system, where at a given instant in time velocity depends on position, $\vec{v} = \vec{v}(\vec{x})$, (a) how is linear momentum defined? (b) How is rate of change of linear momentum defined?

21.2.3 For what points C, with positions $\vec{r}_C$, are the balance of angular momentum equations $\sum \vec{M}_C = \dot{\vec{H}}_{/C}$, correct:

a) the origin,

b) the center of mass, and

c) any point anywhere?

21.2.4 Bead on a stationary 3-D wire. A bead with mass m slides on a frictionless wire that is fixed in space. The wire is twisted and curved in complicated ways. There is no gravity. The only force on the bead comes from the wire. The initial speed of the bead is v_0. Justify your answers to the following questions with text and/or equations. Add any extra assumptions if you feel they are required.

a) Is it true that $\vec{F} = m\vec{a}$ (where $\vec{F}$ is the total force on the bead and $\boldsymbol{a}$ its acceleration?)?

b) Is the momentum of the bead, $m\vec{v}$, constant?

c) Is the kinetic energy of the bead, $\frac{1}{2}mv^2$, constant?

d) What is the speed of the bead at $t = 3\,\mathrm{s}$?

e) Is the acceleration always such that $\vec{a}\cdot\hat{\boldsymbol{e}}_t = 0$ (where $\hat{\boldsymbol{e}}_t$ is the unit tangent to the wire)?

f) Is the acceleration always such that $\vec{a}\cdot\hat{\boldsymbol{e}}_n = 0$ (where $\hat{\boldsymbol{e}}_n$ is the unit normal to the wire)?

g) Using any notation you like to describe the wire shape, write an expression for the force $\vec{F}$ on the bead from the wire.

h) For a circular wire, what is the direction of the force from the bead on the wire?

i) For a helical wire, (say, $\vec{r} = r_0\cos\theta\hat{\imath} + r_0\sin\theta\hat{\jmath} + C\theta\hat{k}$), what is the direction of the force from the bead on the wire? *

21.2.5 A 500 kg roller coaster car travels along a track defined by the conical spiral $r = z$, $\theta = -2z$. If $\dot{\theta} = 2\,\mathrm{rad/s}$ is constant, determine the force exerted on the car by the track when $z = 5$ meters. Assume that the gravitational force acts in the negative z direction. Remember to express the force in vector form.

21.2.6 When do you use which term in $\dot{\vec{H}}_O$ formula for a rigid body? To do the angular momentum balance about an arbitrary point O we write $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$ where the most general formula for $\dot{\vec{H}}_O$ for one rigid body is

$$\dot{\vec{H}}_O = \vec{r}_{cm/o}\times m\vec{a}_{cm} + [\mathbf{I}^{cm}]\cdot\dot{\vec{\omega}} + \vec{\omega}\times([\mathbf{I}^{cm}]\cdot\vec{\omega}).$$

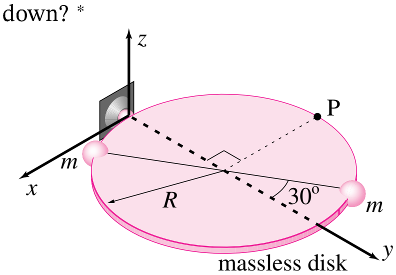
Give examples of motions of a rigid body (you may also use a point mass) in which

a) only one term at a time in the above formula for $\dot{\vec{H}}_O$ survives (three examples) and

b) only two terms at a time in the formula survive (three examples).

21.2.7
(See also problems 21.1.17, 21.1.19, and 21.2.16.) Find the reaction supplied by the environment to the system of bodies $\mathcal{ABD}$ so that each body spins at a constant rate in problem 21.1.11. *

21.2.8 Disk suspended at corner by a ball and socket joint. See also problem 21.2.10. A disk with negligible mass has two point masses glued to it. It is stationary and horizontal when it is allowed to fall, but it is held back by a ball-and-socket joint at one point on its perimeter. Just after the disk is released, what is the acceleration of the point P? Is it up or down? *



Problem 21.2.8

21.2.9 A car driving in circles counterclockwise at constant speed so that its left rear tire has speed v_o (the middle of the tire, that is, or the location of the ground contact point.) The radius of the circle that the tire travels on is R_o. The radius of the tire is r_o. Right next to the tire is the car fender (*i.e.,* the fender overlaps the outer face of the tire). In fact the fender is rubbing on the tire just a little, but not quite enough to disturb any insects that might be in the neighborhood).

a) What is the angular velocity of the car? *

b) What is the angular velocity of the tire relative to the car? *

c) What is the angular velocity of the tire? *

d) What is the total torque applied to the car (not including the tires as part of the car, the rotating engine is also massless) relative to the car center of mass? *

e) What is the total torque on the car relative to the center of the circle around which it is traveling? *

f) What is the angular momentum of the tire relative to its center of mass (use any sensible model of the tire, say a rigid disk.)? *

g) What is the rate of change of angular momentum of the tire (relative to its center of mass)? *

h) What is the total torque of all the forces applied to the tire from the road and the car as calculated at the center of the tire? *

i) A bug is climbing straight up on the car fender named at speed v_g. At the instant of interest she is right next to the center of the wheel. What is her velocity and acceleration? What is the total force acting on her? *

j) Another bug is climbing on a straight line marked on the tire at constant speed v_b. He is in the middle of the tire at the instant of interest. What is his absolute velocity and acceleration? What is the total force acting on him? (Both bugs are at the same place at the instant of interest and walking on lines that are, at the moment, parallel.)

k) Redo the problem, questions (a) -(j), but this time the car speed is increasing with acceleration a_o. Questions (a) - (j) of this problem are a special case of question (k). You should check that your answers from (k) reduce to the answers in questions (a) - (j). Or, if you are really cocky, you should skip the first questions and only do (k). Then, display your answers for the special case $a_o = 0$.

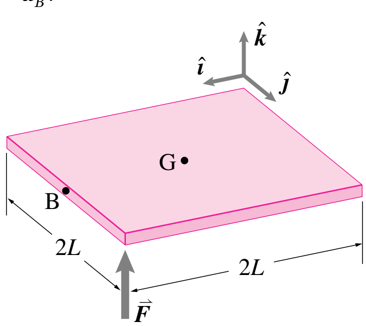
21.2.10 The moment of inertia matrix. Reconsider the system of problem 21.2.8 — a disk with negligible mass that has two point masses glued to it.

a) Find the moment of inertia matrix $[\mathbf{I}^O]$ of the system.
b) Find the angular acceleration of the system when released from rest in the configuration shown. You may use your answer from problem 21.2.8.

21.2.11 A square plate in space. A uniform square plate with mass m and sides with length $2L$ is floating stationary in space. A force of magnitude F is suddenly applied to it at point A. Using the coordinates shown the moment of inertia matrix for this plate is:

$$[I_{/cm}] = \frac{mL^2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

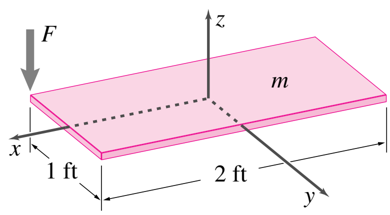
a) What is the acceleration of the center of the plate $\vec{a}_G$? *
b) What is the angular acceleration of the plate $\dot{\vec{\omega}}$ *
c) What is the acceleration of point B $\vec{a}_B$? *



Problem 21.2.11

21.2.12 Force on a rectangular plate in space. A uniform rectangular plate of mass $m = 10\,\text{lbm}$ is floating in space. A force $F = 5\,\text{lbf}$ is suddenly applied at a corner that is perpendicular to the plate.

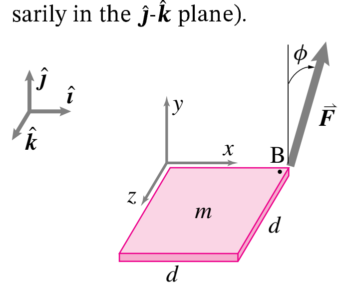
a) Find the acceleration of the center of mass of the plate. $^*\vec{a} = -16\,\text{ft/s}^2\hat{k}$
b) Find the angular acceleration of the plate. *
c) Some of the plate accelerates in the direction of the force and some accelerates back towards the tail of the force vector. What part of the plate accelerates in the opposite direction of the applied force?



Problem 21.2.12

21.2.13 Force on a square plate. A constant force with magnitude F is applied, in the direction shown in the yz plane, to the corner B of a uniform square plate with side length d and mass m. Before the force is applied the plate is at rest in space with no other forces applied to it. Consider the time immediately after the time of the force application.

a) What, in terms of m, F, d, $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, and ϕ is the acceleration of point B?
b) For what angle ϕ is the acceleration of point B in the direction of the force?
c) Find another direction for F so that the force at B and the acceleration at B are parallel (not necessarily in the $\hat{\jmath}$-$\hat{k}$ plane). [Physical reasoning will probably work more easily than mathematical reasoning.]
d) Find yet another direction for F so that the force at B and the acceleration at B are parallel (not necessarily in the $\hat{\jmath}$-$\hat{k}$ plane).



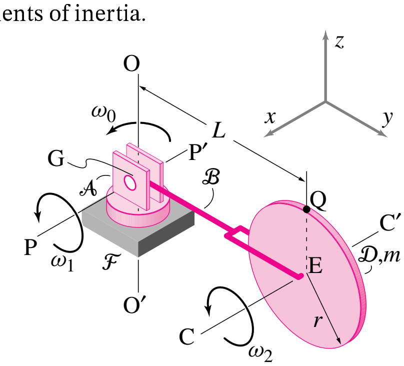
Problem 21.2.13

21.2.14
(See also problems 21.1.16, 21.1.18, and 21.2.15.) Find the reaction supplied by the environment to the system of bodies $\mathcal{ABD}$ so that each body spins at a constant rate in problem 21.1.15.

21.2.15 (See also problems 21.1.16, 21.1.18, and 21.2.14.) Reconsider the system in problem 21.1.15. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass and body $\mathcal{D}$ is a uniform disc of mass m. Find the angular momentum and rate of change of angular momentum of the system of bodies $\mathcal{ABD}$ about:

a) point G
b) point E.

Make reasonable assumptions about moments of inertia.

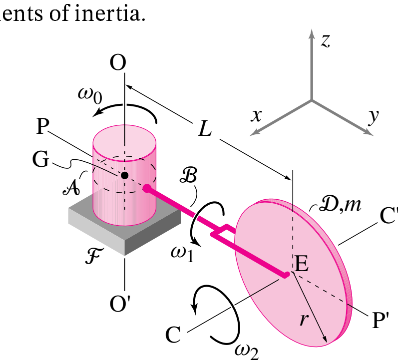


Problem 21.2.15

21.2.16 (See also problems 21.1.17, 21.1.19, and 21.2.7.) Reconsider the system in problem 21.1.11. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass and body $\mathcal{D}$ is a uniform disc of mass m. Find the angular momentum and rate of change of angular momentum of the system of bodies $\mathcal{ABD}$ about:

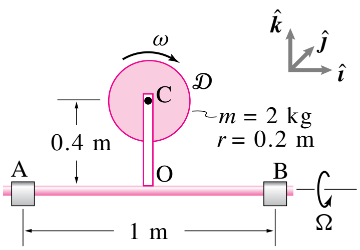
a) point G *
b) point E. *

Make reasonable assumptions about moments of inertia.



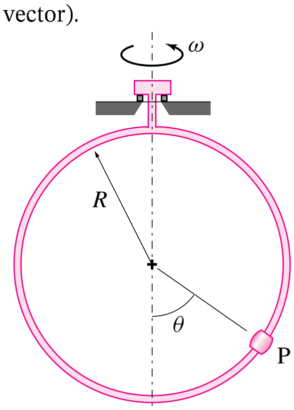
Problem 21.2.16

21.2.17 Reconsider problem 21.2.26. A thin uniform disk $\mathcal{D}$ of mass $m = 2\,\text{kg}$ and radius $r = 0.2\,\text{m}$ is mounted on a massless shaft AB as shown in the figure. Rod CO is welded to the shaft and has negligible mass. The shaft rotates at a constant angular speed $\Omega = 120\,\text{rpm}$ about the positive x-axis. The disk spins about its own axis at a constant rate $\omega = 60\,\text{rpm}$ with respect to the shaft (or bar CO). Find the dynamic reactions on the bearings at A and B at the instant shown. *



Problem 21.2.17

21.2.18 A bead P rides on a frictionless thin circular hoop of radius R. The hoop rotates about its diameter at a constant angular speed ω in the vertical plane. Find, in terms of $\theta, \dot{\theta}$ and ω and base vectors you define (a) the velocity of the bead (a vector) and (b) the acceleration of the bead (a vector).



Problem 21.2.18

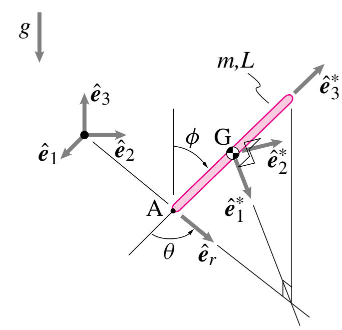
21.2.19 A thin rod of mass m and length L moves while making contact with the frictionless surface (the $\hat{\boldsymbol{e}}_1 - \hat{\boldsymbol{e}}_2$ plane) at its tip A, as shown in the figure. θ is assumed to be constant and the rod always remains in the $\hat{\boldsymbol{e}}_3, \hat{\boldsymbol{e}}_r$ plane.

a) What is the kinematical relation between the velocity of the center of mass, $\vec{\boldsymbol{v}}_G$, and the angular velocity of the rod, $\vec{\boldsymbol{\omega}}$?

b) Obtain the equations of motion of the rod in terms of the speed of point A and the angle ϕ.

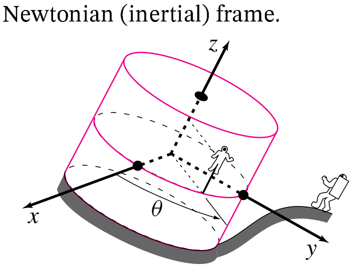
The unit vectors $\hat{\boldsymbol{e}}_1^*$ and $\hat{\boldsymbol{e}}_3^*$ lie in the $(\hat{\boldsymbol{e}}_3 - \boldsymbol{e}_r)$ plane and $\hat{\boldsymbol{e}}_2^*$ is normal to the $(\hat{\boldsymbol{e}}_3 - \hat{\boldsymbol{e}}_r)$ plane. The starred basis vectors are written in terms of the standard basis as follows:

$$
\begin{aligned}
\hat{\boldsymbol{e}}_r &= \cos\theta\hat{\boldsymbol{e}}_1 + \sin\theta\hat{\boldsymbol{e}}_2, \\
\hat{\boldsymbol{e}}_1^* &= \sin\phi\hat{\boldsymbol{e}}_3 + \cos\phi\hat{\boldsymbol{e}}_r, \\
\hat{\boldsymbol{e}}_3^* &= \cos\phi\hat{\boldsymbol{e}}_3 + \sin\phi\hat{\boldsymbol{e}}_r, \qquad \text{and} \\
\hat{\boldsymbol{e}}_2^* &= -\sin\theta\hat{\boldsymbol{e}}_1 + \cos\theta\hat{\boldsymbol{e}}_1.
\end{aligned}
$$



Problem 21.2.19

21.2.20 A cosmonaut who is otherwise free to move in 3-dimensional space finds himself stuck to the surface of a giant stationary cylinder with slippery magnets. There is no friction and (net) gravity is negligible. θ and z are cylindrical coordinates that are aligned with the cylinder and mark the position of the cosmonaut relative to the cylinder. The cylinder has radius a. A nearby astronaut notes that at time $t = 0$: $\theta = \theta_0$, $\dot{\theta} = \dot{\theta}_0$, $z = z_0$, and $\dot{z} = \dot{z}_0$. What does she predict for the location of the cosmonaut (the values of θ and z) at an arbitrary later time t? You can assume that both the cylinder and the astronaut were stationary in the same Newtonian (inertial) frame.



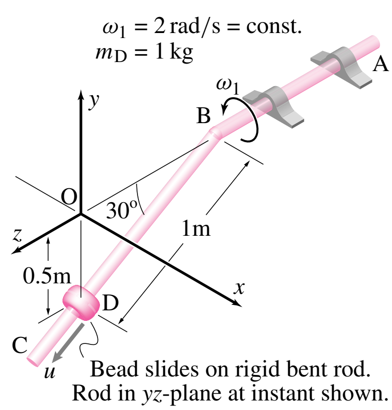
Problem 21.2.20

21.2.21 You are an astronaut in free space and your jet pack is no longer operating. You wish to reorient your body in order to face the central government buildings so as to salute your national leaders before reentering the atmosphere. Discuss whether this maneuver can be done in terms of momentum, energy, or other dynamical concepts. (You are free to move your arms and legs, but you are not allowed to throw anything.) [For aliens and those less patriotic, the same problem arises in the case of a trampoline jumper who decides to change direction in mid-air. Is it possible?]

21.2.22 A collar slides on the rigid rod ABC that has a 30° bend at B as shown in the figure. The collar is currently at point D on the rod. The rod ABC is held by fixed hinges that are aligned with the z-axis, and rotates at the constant rate $\omega_1 = 2\,\text{rad/s}$. The collar has mass $m_D = 1\,\text{kg}$.

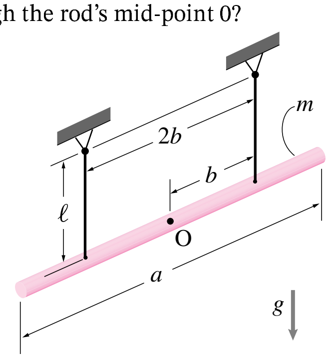
a) Assume a mechanism not shown constrains the collar to slide out along the rod at the constant rate $u = 1\,\text{m/s}$ (u is the distance along the rod traveled per unit time). What is the net force that acts on the collar when it is 1 m from B?

b) Assume that the contact between the collar and the rod is frictionless and that no forces act on the collar besides those due to the rod (e.g. neglect gravity). At the instant shown in the picture, assume that $u = 1\,\text{m/s}$ (because of some initial conditions not discussed here) but is not necessarily constant in time. What is the acceleration of the collar?



Problem 21.2.22

21.2.23 A bifilar pendulum is formed of a rod of length a and mass m, supported by two filaments of length ℓ and separation $2b$. What is the frequency of small 'torsional' oscillations about a vertical axis though the rod's mid-point 0?



Problem 21.2.23

21.2.24 A trifilar pendulum is formed of a thin horizontal disk of radius a and mass m, hung by three wires a distance ℓ below the ceiling. The wire attachments are equally spaced on the perimeter of the disk. The attachment points on the ceiling form an equilateral triangle.

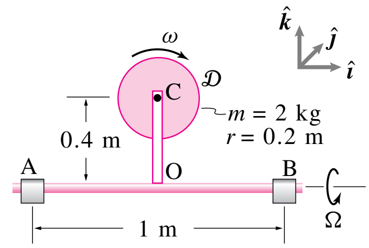
a) If the support wires are vertical what is the period of oscillation?

b) How big should the triangle on the ceiling be to minimize the period of oscillation?

c) To maximize the period of oscillation?

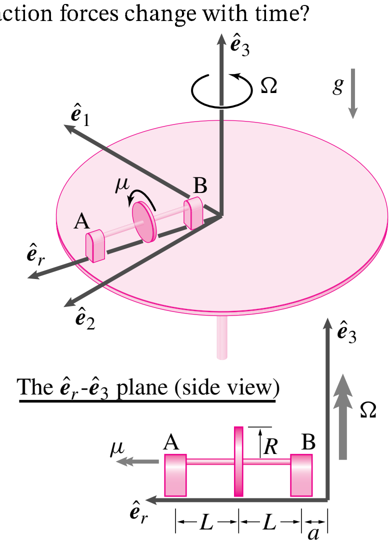
21.2.25 Sometimes thrown footballs (and frisbees too!) wobble and sometimes they spin smoothly. Describe as best you can why these motions differ.

21.2.26 See also problem 21.2.17. A thin uniform disk $\mathcal{D}$ of mass $m = 2\,\mathrm{kg}$ and radius $r = 0.2\,\mathrm{m}$ is mounted on a massless shaft AB as shown in the figure. Rod CO is welded to the shaft and has negligible mass. The shaft rotates at a constant angular speed $\Omega = 120\,\mathrm{rpm}$ about the positive x-axis. The disk spins about its own axis at a constant rate $\omega = 60\,\mathrm{rpm}$ with respect to the shaft (or bar CO). Find the absolute angular velocity and acceleration of the disk. *



Problem 21.2.26

21.2.27 A thin circular disk of mass m and radius R is mounted on a thin light axle, coincident with its axis of symmetry. The axle is supported by bearings A and B attached to a turntable, as shown in the figure. The turntable spins at a constant speed Ω about $\hat{\boldsymbol{e}}_3$, while the disk spins about its axle at a constant speed μ relative to the turntable (via motors not shown). The radial unit vector $\boldsymbol{e}_r = \cos(\Omega t)\hat{\boldsymbol{e}}_1 + \sin(\Omega t)\hat{\boldsymbol{e}}_2$ is fixed to the turntable and aligned with $\hat{\boldsymbol{e}}_1$ at time $t = 0$. Compute the vertical components of the reaction forces at A and B. Will the reaction forces change with time?



Problem 21.2.27

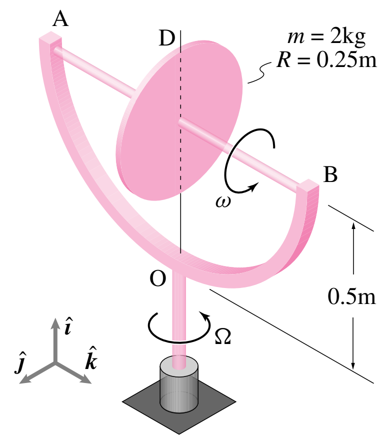
21.2.28 Steady precession of a spinning disk. A uniform disk of mass $m = 2\,\mathrm{kg}$ and radius $R = 0.25\,\mathrm{m}$ is rigidly mounted on a massless axle AB which is supported by frictionless bearings at A and B. The axle, in turn, is mounted on a massless semi-circular gimbal as shown in the figure. The gimbal rotates about the vertical x-axis, which passes through the center of the disk D, at a constant rate $\Omega = 2\,\mathrm{rad/s}$. The axle AB is spinning at a constant rate $\omega = 120\,\mathrm{rpm}$ with respect to the gimbal. At the instant shown, the axle is aligned with the z-axis and the disk D is in the xy-plane. Ignore gravity. At the instant of interest:

a) What is the angular velocity $\vec{\omega}_{\mathcal{D}}$ of the disk? *

b) What is the angular acceleration of the disk? *

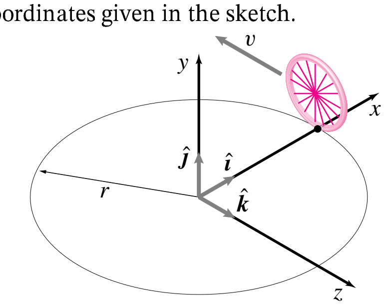
c) What is the net torque required to keep the motion going? *

d) What are the reaction forces at the bearings A and B? *



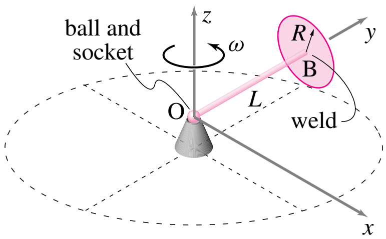
Problem 21.2.28

21.2.29 A cyclist rides a flat, circular track of radius $r = 150\,\mathrm{ft}$ at a constant tangential speed of $v = 22\,\mathrm{ft/s}$ on 27 ∈ diameter wheels. Assuming that the bicycle is vertical, determine the total angular velocity and angular acceleration of the wheel of the bike in the sketch (the rest of the bicycle is not shown). Use the coordinates given in the sketch.



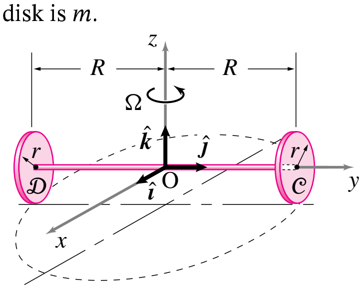
Problem 21.2.29

21.2.30 Rolling disk presses down. A rigid disk (idealized as flat) with radius R and mass m is rigidly attached (welded) to a rigid massless stick with length L which is connected by a ball-and-socket joint to the support at point O. The wheel rolls on a horizontal surface. Point O is at the height of the center of the wheel. The stick revolves around O at a rate of ω. Say gravity is negligible and, even though the wheel is rolling, you may assume there are no frictional forces acting between the wheel and the horizontal surface. What is the vertical reaction on the bottom of the wheel? *



Problem 21.2.30

21.2.31 In problem 21.1.13, find the angular momentum and rate of change of angular momentum of the two disks about point O. Assume that the mass of each disk is m.

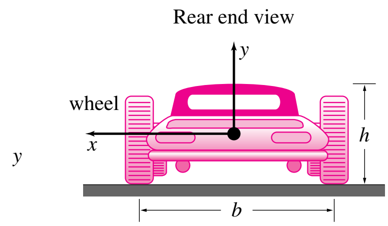


Problem 21.2.31

21.2.32 A vehicle of mass m, width b, and height h rolls without slip along a road which runs into the page(along the z-axis.) The inertial properties about the vehicle center of mass are I_{xx}^{cm},I_{yy}^{cm},I_{zz}^{cm}, and $I_{xy}^{cm} = I_{xz}^{cm} = I_{yz}^{cm} = 0$. Make additional assumptions as needed to get an answer to the problem. Compute the contact forces acting on each wheel normal to the road in the following cases:

a) The vehicle moves at constant speed v along a straight road.

b) The vehicle moves at constant speed v along a circular road (radius ρ) curving to the left. Does the right or left wheel experience the larger force?



Problem 21.2.32

21.2.33 A car drives counter-clockwise in circles at constant speed V. The radius of the circle that the left rear wheel travels on is R. The radius of the wheel is r. Just when the wheel is rolling due north a pebble of mass m, which is stuck in the tread, is passing directly above the wheel axle. (You should assume that the wheel axle is horizontal, that the wheel rolls without slip and that the wheel deformation and gravity are negligible.) At this instant, what is the force that the tread causes on the pebble? Resolve this force into east, north and upwards components. [Hint: several intermediate calculations are probably required.]

21.2.34 A child constructs a spinning top from a solid sphere with radius a and mass m attached to a massless rod of length $2d$. The sphere pivots freely about O. It is set spinning with a constant rate $\dot{\phi}$ and precessing with a constant rate $\dot{\psi}$. The top of the rod is held by a horizontal cord.

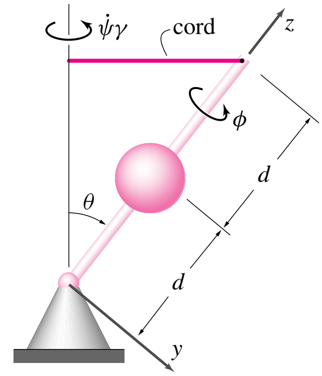
Note: The moments of inertia of a solid sphere of mass m and radius a about any axis through the center is $(2/5)ma^2$. (All products of inertia are zero, due to symmetry.)

a) Using an axis system attached as shown (z lies along the shaft, x is out of the page and y is chosen to produce a right-handed system), write the body's angular velocity $\vec{\omega}$ as well as the angular velocity $\vec{\boldsymbol{\Omega}}$ of the moving coordinate system relative to inertial space.

b) What is the sphere's inertia tensor about O?

c) What is the top's angular momentum about O?

d) Compute the tension in the cord.



Problem 21.2.34

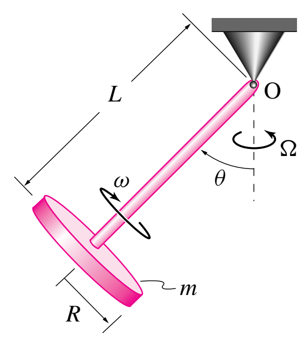
21.2.35 A thin disk of mass m and radius R is connected by a massless shaft of length L to a free swivel joint at O. The disk spins about the shaft at rate ω and proceeds in a steady slow precession (rate Ω) about the vertical.

a) Write the disk's inertia tensor about O. Specify the axes that you have chosen.

b) Write the disk's angular momentum about O, $\bar{\boldsymbol{H}}_{/O}$, in the same coordinate system.

c) Calculate the rate of change of angular momentum about O, $\dot{\bar{\boldsymbol{H}}}_{/O}$.

d) Compute the precession rate Ω for the slow precession case ($\omega \gg \Omega$).



Problem 21.2.35

21.2.36 Spinning and precessing top. (hard question) The picture schematically shows what is going on when a spinning top is in steady precession. The top is, approximately, spinning at constant rate about its symmetry axis. This axis is in turn (so to speak) rotating about the y axis at a constant rate. This rotation about the y axis is called precession. The angular momentum vector thus changes in time as the top precesses and the angular momentum vector also precesses. The picture is a reasonable approximation if the top is rotating much faster than it is precessing. The picture is not accurate if the precession rate is not much smaller than the rotation rate. Here are some questions to help you see why.

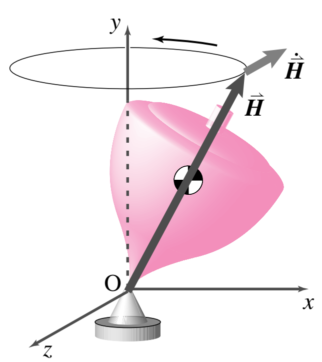
a) If the top is precessing as well as spinning what is the direction of the angular velocity vector?

b) What is the direction of the angular momentum vector of the top? Is it parallel to the top's axis? Is it parallel to the angular velocity vector?

c) How does the angular momentum vector differ from that shown in the figure?

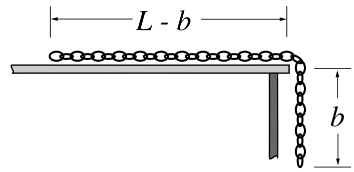
d) Does this difference affect any predictions about the precession rate of a spinning top?

e) Assume the top is on a pedestal and is horizontal as it precesses. In this case the correct precession rate *is* predicted by the naive/approximate theory that the angular momentum is just that due to spin and that this vector precesses with the top axis. Why does this simplification happen to work?



Problem 21.2.36

21.2.37 An amusement park ride consists of two cylinder-shaped cars of length $\ell = 8\,\text{ft}$ and radius $r = 3\,\text{ft}$ mounted at opposite ends of a long arm of length $2h = 20\,\text{ft}$ which is pivoted at its center as shown in the figure. The arm is constrained to rotate about the horizontal axle $O - O$. Each car can also rotate about its own axis. Car A, presently at the top of its flight, is spinning (much to the discomfort of its occupants) on its own axis as shown at a constant rate of $\omega_2 = 5\,\text{rad/s}$ relative to the vertical arm. The vertical arm is rotating about axis $O - O$ at the rate $\omega_1 = 0.5\,\text{rad/s}$ in the direction shown, as seen by a ground observer. The point P, located at $x = a = 2\,\text{ft}$ and $z = b = 1\,\text{ft}$ and representing a point in the inner ear of a passenger, is undergoing absolute velocity and acceleration $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$, respectively. Find the vectors $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$. (If you are using a moving coordinate system to determine $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$, state clearly how it moves).



Problem 21.2.37

21.2.38
buzzing coin If you drop a round coin it will often go round and round progressively faster before buzzing to a stop. In terms of parameters you define, what is the frequency of this buzzing as a function of the tip angle? This problem is more subtle than a straight-forward calculation as there are many solutions for a coin rolling at a given tip angle in circles. What feature picks out the solutions that are observed with real coins?

21.2.39 Estimate the sideways force (Pounds or Newtons) required for a person to walk a straight marked path over the North pole. Radius of earth ≈ 4000 miles ≈ 2×10^7 feet ≈ 6×10^6 meters. Make other simplifying approximations so that you can do the arithmetic without a calculator.