SAMPLE 21.7 An axisymmetric body with precession and spin. A uniform circular disk of mass $m = 2\,\text{kg}$ is mounted on gimbals which in turn is mounted on a horizontal shaft. The disk spins about its normal axis of symmetry. The spin axis makes an angle $\phi = 15^\circ$ with the horizontal shaft. The shaft rotates at a constant speed $\omega_1 = 60\,\text{rpm}$. The disk spins with respect to the gimbals at a constant speed $\omega_2 = 120\,\text{rpm}$. The moment of inertia of the disk about its diameter is $A = 10\,\text{kg}\cdot\text{m}^2$ and the moment of inertia of the disk about its normal is $B = 20\,\text{kg}\cdot\text{m}^2$. At the instant shown, find

1. the angular momentum $\vec{\boldsymbol{H}}_{/G}$ of the disk,
2. the rate of change of the angular momentum $\dot{\vec{\boldsymbol{H}}}_{/G}$ of the disk.

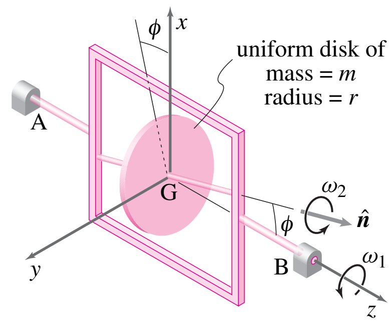
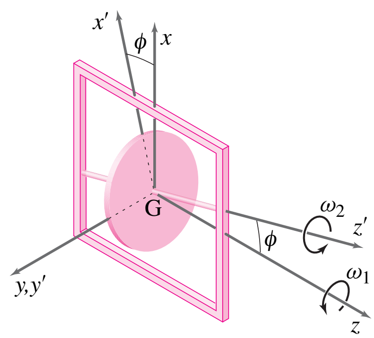


Figure 21.20

Solution Let xyz be the coordinate axes attached to the inertial frame and $x'y'z'$ be the coordinate axes attached to the gimbals $\mathcal{G}$ such that z' is aligned with the spin axis of the disk. Therefore, at the instant shown,

$$\hat{\boldsymbol{\imath}}' = \cos\phi\hat{\boldsymbol{\imath}} - \sin\phi\hat{\boldsymbol{k}}, \quad \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}, \quad \hat{\boldsymbol{k}}' = \sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{k}}. \tag{21.12}$$



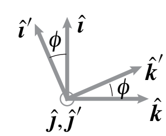


Figure 21.21

1. **Angular momentum:** The angular momentum of the disk is given by

$$\vec{\boldsymbol{H}}_{/G} = [\boldsymbol{I}^{\text{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}}$$

where $[\boldsymbol{I}^{\text{cm}}]$ is the moment of inertia matrix of the disk about its center-of-mass and $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ is the absolute angular velocity of the disk.
Now the absolute angular velocity of the disk

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \vec{\boldsymbol{\omega}}_{\mathcal{G}} + \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{G}}$$

$$\begin{aligned} \text{where} \quad \vec{\boldsymbol{\omega}}_{\mathcal{G}} &= \omega_1\hat{\boldsymbol{k}} = 60\,\text{rpm}\hat{\boldsymbol{k}} = 2\pi\,\text{rad/s}\hat{\boldsymbol{k}}, \\ \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{G}} &= \omega_2\hat{\boldsymbol{k}}' = 120\,\text{rpm}\hat{\boldsymbol{k}}' = 4\pi\,\text{rad/s}\hat{\boldsymbol{k}}'. \end{aligned}$$

The principal moments of inertia for the disk are given. We have chosen the body axes $x'y'z'$ along the principal axes (i.e. $x'y'$ along the two diameters of the disk and z' along the normal of the disk). Therefore in the $x'y'z'$ coordinate system

$$[\boldsymbol{I}^{\text{cm}}]_{x'y'z'} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix}$$

Where $A = 10\,\text{kg}\cdot\text{m}^2$ and $B = 20\,\text{kg}\cdot\text{m}^2$. Since the inertia matrix is written in the body coordinate system, we need to express $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ also in the body coordinate system to carry out the product $[\boldsymbol{I}^{\text{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}}$.

$$\text{Now} \quad \vec{\boldsymbol{\omega}}_{\mathcal{D}} = \omega_1\hat{\boldsymbol{k}} + \omega_2\hat{\boldsymbol{k}}',$$

and from Fig. 21.20 we can write $\hat{\boldsymbol{k}} = -\sin\phi\hat{\boldsymbol{\imath}}' + \cos\phi\hat{\boldsymbol{k}}'$, therefore,

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = (-\omega_1\sin\phi)\hat{\boldsymbol{\imath}}' + (\omega_2 + \omega_1\cos\phi)\hat{\boldsymbol{k}}'.$$

So,

$$\begin{aligned} \vec{\boldsymbol{H}}_{/G} &= [\boldsymbol{I}^{\text{cm}}]\cdot\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} \begin{Bmatrix} -\omega_1\sin\phi \\ 0 \\ \omega_2 + \omega_1\cos\phi \end{Bmatrix} \\ &= (-A\omega_1\sin\phi)\hat{\boldsymbol{\imath}}' + B(\omega_2 + \omega_1\cos\phi)\hat{\boldsymbol{k}}'. \end{aligned} \tag{21.13}$$

Substituting the given values of ω_1, ω_2, A, B and ϕ we get

$$\vec{H}_{/G} = [-16.26\hat{\imath}' + 372.71\hat{k}']\,\mathrm{kg\cdot m^2/s}$$

Although we have the answer we need, it is expressed in terms of the body coordinate system's basis vectors. We can easily express $\vec{H}_{/G}$ in the fixed coordinate system using Eq 21.12:

$$\vec{H}_{/G} = [-16.26(\cos\phi\hat{\imath} - \sin\phi\hat{k}) + 372.71(\sin\phi\hat{\imath} + \cos\phi\hat{k})]\,\mathrm{N\cdot m\cdot s}$$

$$\boxed{\vec{H}_{/G} = (80.76\hat{\imath} + 364.22\hat{k})\,\mathrm{N\cdot m\cdot s}}$$

2. **Rate of change of angular momentum:** The rate of change of angular momentum of the disk about point G is given by

$$\dot{\vec{H}}_{/G} = [\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\omega}}_{\mathcal{D}} + \vec{\omega}_{\mathcal{D}} \times [\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}_{\mathcal{D}} \qquad (21.14)$$

Thus, to compute $\dot{\vec{H}}_{/G}$, we need the angular acceleration of the disk $\dot{\vec{\omega}}_{\mathcal{D}}$.

$$\begin{aligned}
\dot{\vec{\omega}}_{\mathcal{D}} &= \dot{\vec{\omega}}_{\mathcal{G}} + \dot{\vec{\omega}}_{\mathcal{D}/\mathcal{G}} \\
&= \underbrace{\dot{\omega}_1}_{0}\hat{k} + \underbrace{\underbrace{\dot{\omega}_2}_{0}\hat{k}' + \vec{\omega}_1\hat{k}\times\vec{\omega}_2\hat{k}'}_{\dot{\vec{\omega}}_{\mathcal{D}/\mathcal{G}}} \\
&= \omega_1\omega_2(\hat{k}\times\hat{k}') = \omega_1\omega_2(-\sin\phi\hat{\imath}' + \cos\phi\hat{k}')\times\hat{k}' = \omega_1\omega_2\sin\phi\,\hat{\jmath}'.
\end{aligned}$$

Here, we calculated $\dot{\vec{\omega}}_{\mathcal{D}}$ in the body coordinate system because we need to calculate $[\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\omega}}_{\mathcal{D}}$ and we know $[\boldsymbol{I}^{\mathrm{cm}}]$ in the body coordinate system. Thus the first term in $\dot{\vec{H}}_{\mathrm{cm}}$ is

$$[\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\omega}}_{\mathcal{D}} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1\omega_2\sin\phi \\ 0 \end{Bmatrix} = A\omega_1\omega_2\sin\phi\,\hat{\jmath}'.$$

The second term in $\dot{\vec{H}}_{/G}$ is obtained by taking the cross product of $\vec{\omega}_{\mathcal{D}}$ expressed in the body coordinate system with the expression for $\vec{H}_{/G}$ given by Eq 21.13:

$$\begin{aligned}
\vec{\omega}_{\mathcal{D}}\times[\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}_{\mathcal{D}} &= \vec{\omega}_{\mathcal{D}}\times\vec{H}_{/G} \\
&= [(-\omega_1\sin\phi)\hat{\imath}' + (\omega_2+\omega_1\cos\phi)\hat{k}']\times[(-A\omega_1\sin\phi)\hat{\imath}' + B(\omega_2+\omega_1\cos\phi \\
&= (B-A)\omega_1\sin\phi[\omega_1\cos\phi+\omega_2]\hat{\jmath}'.
\end{aligned}$$

Adding the two terms together we get

$$\begin{aligned}
\dot{\vec{H}}_{/G} &= A\omega_1\omega_2\sin\phi\,\hat{\jmath}' + (B-A)\omega_1\sin\phi[\omega_1\cos\phi+\omega_2]\hat{\jmath}' \\
&= [(B-A)\omega_1\cos\phi + B\omega_2]\omega_1\sin\phi\,\hat{\jmath}'. \qquad (21.15)
\end{aligned}$$

Substituting $B = 20\,\mathrm{kg\cdot m^2}$, $A = 10\,\mathrm{kg\cdot m^2}$, $\omega_1 = 2\pi\,\mathrm{rad/s}$, $\omega_2 = 4\pi\,\mathrm{rad/s}$ and $\phi = 15^\circ$ we get the answer

$$\boxed{\dot{\vec{H}}_{/G} = 507.41\,\mathrm{N{\cdot}m}\;\hat{\jmath}.}$$

SAMPLE 21.8 Torque free motion of an axisymmetric rigid body Consider the gimbal-mounted disk of sample 21.7 again. Assume that the net torque on the disk is zero. That is, for what motions could the gimbals be eliminated and have the motion proceed?

1. Derive an expression for the ratio of the two angular speeds, $\frac{\omega_1}{\omega_2}$, for the torque free motion of the disk.
2. Assume ϕ to be very small. What is the ratio of the angular speeds for the given disks (or any flat axisymmetric object with $B = 2A$)?

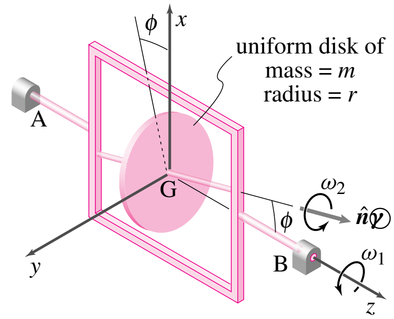


Figure 21.22: The origin G is at the center-of-mass of the disk in this problem.

Solution

1. The angular momentum balance for the disk about point G gives:

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

We are given that the net torque on the disk must be zero, *i.e.*, $\sum \vec{M}_G = \dot{\vec{H}}_G = \vec{0}$. We found an expression for $\dot{\vec{H}}_G$ in the previous sample problem (see Eq 21.15 of Sample 21.7) to be:

$$\dot{\vec{H}}_G = [(B-A)\omega_1\cos\phi + B\omega_2]\omega_1\sin\phi\,\hat{\jmath}.$$

Now under the torque free condition,

$$\begin{aligned} \dot{\vec{H}}_G &= \vec{0} \\ \Rightarrow \quad (B-A)\omega_1\cos\phi &= -B\omega_2 \\ \Rightarrow \quad \frac{\omega_1}{\omega_2} &= \frac{B}{(A-B)\cos\phi} \end{aligned} \tag{21.16}$$

2. for small ϕ, $\cos\phi \approx 1$. Also, for the disk, $B = 2A$. Substituting these values into the above expression for $\frac{\omega_1}{\omega_2}$, we get

$$\frac{\omega_1}{\omega_2} = \frac{2A}{A-2A} = -2$$

That is, the rate of precession is twice the rate of relative spin. Also, if $\vec{\omega}_1 = \omega_1\hat{k}$, then $\vec{\omega}_2 = -\frac{\omega_1}{2}\hat{k}'$.

$$\boxed{\frac{\omega_1}{\omega_2} = -2}$$

Comments: The results obtained here are more general than they seem. In particular, it can be shown that the most general motion of a torque free axisymmetric body satisfies Eq 21.16. If you throw a coin in the air it wobbles according to the motion described as constant rate motion about an axis that itself rotates at a constant rate.

In case (b), the absolute angular velocity is $\omega_1\hat{k} + \omega_2\hat{k}'$ which is $\omega_1\boldsymbol{k} + (-\frac{\omega_1}{2})\boldsymbol{k}'$ which is approximately $\frac{\omega_1}{2}\boldsymbol{k}'$. So, the wobble rate ω_1 is twice the spin rate $\frac{\omega_1}{2}$ for flat round objects in torque free motion.

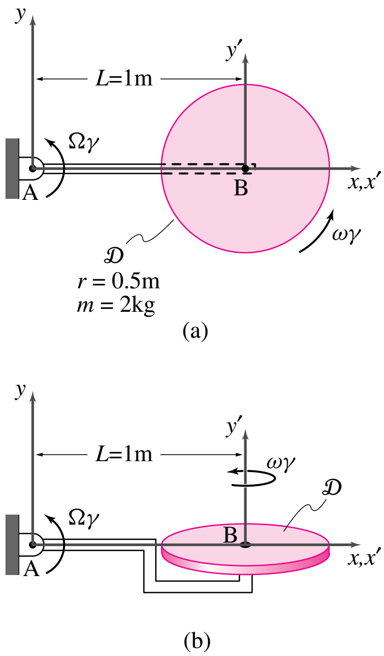


Figure 21.23

SAMPLE 21.9 **Calculation of energy.** In the figures shown, the rigid arm AB rotates with constant angular velocity $\Omega = 5\,\mathrm{rad/s}$ and the disk $\mathcal{D}$ rotates with respect to the arm at constant angular speed $\omega = 10\,\mathrm{rad/s}$. Assume that the rigid arm is massless. The uniform disk has a mass of 2 kg and measures 50 cm along the diameter. Find the kinetic energy of the system in each case.

Solution Case (a): The motion of the disk is in 2-D . Therefore, the kinetic energy of the disk is

$$E_{\mathrm{K}} = \frac{1}{2} m v_{\mathrm{cm}}^2 + \frac{1}{2} I_{zz}^{\mathrm{cm}} \omega^2$$

Where v_{cm} is the speed of the center-of-mass of the disk and ω is the magnitude of the absolute angular velocity of the disk. Since the center-of-mass of the disk executes constant-speed circular motion with radius $L = 1\,\mathrm{m}$,

$$v_{\mathrm{cm}} = \Omega L = 5\,\mathrm{rad/s} \cdot 1\,\mathrm{m} = 5\,\mathrm{m/s}.$$

The absolute angular velocity of the disk is

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{k}} = (\Omega + \omega)\hat{\boldsymbol{k}}.$$

Therefore, the angular speed of the disk is $\omega_{\mathcal{D}} = \Omega + \omega = 15\,\mathrm{rad/s}$. Substituting these values and $m = 2\,\mathrm{kg}$ and $I_{zz}^{\mathrm{cm}} = \frac{1}{2} m r^2 = 0.0625\,\mathrm{kg\,m^2}$, we get

$$E_{\mathrm{K}} = \frac{1}{2} \cdot 2\,\mathrm{kg} \cdot (5\,\mathrm{m/s})^2 + \frac{1}{2} \cdot (0.0625\,\mathrm{kg\,m^2}) \cdot (15\,\mathrm{rad/s})^2 = 32\,\mathrm{N{\cdot}m}$$

$$\boxed{E_{\mathrm{K}} = 32\,\mathrm{N{\cdot}m}}$$

Case (b): The motion of the disk is in 3-D. The kinetic energy is now given by

$$E_{\mathrm{K}} = \frac{1}{2} m v_{\mathrm{cm}}^2 + \frac{1}{2} \vec{\boldsymbol{\omega}}_{\mathcal{D}} \cdot ([\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}})$$

The translational part of the energy is the same as in case (a). For the rotational part, note that $\vec{\boldsymbol{\omega}}_{\mathcal{D}} = \Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{\jmath}}' = \Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{\jmath}}$, and ①

$$[\boldsymbol{I}^{\mathrm{cm}}] = \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore, the rotational kinetic energy is

$$\begin{aligned}
\frac{1}{2} \vec{\boldsymbol{\omega}}_{\mathcal{D}} \cdot ([\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}}) &= \frac{1}{2}(\Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{\jmath}}) \cdot \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ \Omega \end{Bmatrix} \\
&= \frac{1}{2}(\Omega \hat{\boldsymbol{k}} + \omega \hat{\boldsymbol{\jmath}}) \cdot \frac{1}{4} m r^2 (\omega \hat{\boldsymbol{\jmath}} + 2\Omega \hat{\boldsymbol{k}}) \\
&= \frac{1}{8} m r^2 (2\Omega^2 + \omega^2) \\
&= 2.34\,\mathrm{N{\cdot}m}
\end{aligned}$$

Therefore,

$$E_{\mathrm{K}} = \frac{1}{2} \cdot 2\,\mathrm{kg} \cdot (5\,\mathrm{m/s})^2 + 2.34\,\mathrm{N{\cdot}m} \approx 27\,\mathrm{N{\cdot}m}.$$

$$\boxed{E_{\mathrm{K}} = 27\,\mathrm{N{\cdot}m}}$$

① Without calculation, could you have predicted that the kinetic energy in case (a) would be greater than in case (b)?

SAMPLE 21.10 **Dynamic reactions.** A uniform disk of mass $m = 2\,\text{kg}$ and radius $r = 0.5\,\text{m}$ is mounted on frictionless bearings on a 'L' shaped massless shaft OBC. The disk spins about its centroidal axis at a constant rate $\omega_2 = 15\,\text{rad/s}$ while the shaft rotates about O in the vertical plane at a constant rate $\omega_1 = 3\,\text{rad/s}$. Given that $L = 1\,\text{m}$, $R = 0.5\,\text{m}$, find the dynamic reactions at the support O at the instant shown in Fig. 21.24.

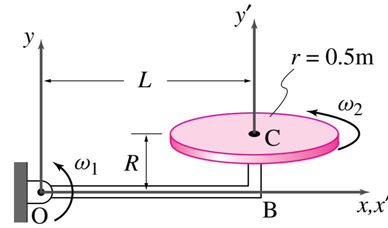


Figure 21.24:

Solution Let us take the disk together with the massless shaft as our system. The free-body diagram of this system is shown in Fig. 21.25. Since we are interested only in dynamic reactions, the force on the system due to gravity, *i.e.,* the weight of the disk is ignored (because this force induces static reactions). Now let us carry out the momentum balance for our system.

From the linear momentum balance for the system,

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \vec{R} &= m\vec{a}_C. \end{aligned} \tag{21.17}$$

Thus, we need to find the acceleration of point C, the center-of-mass of the disk, to evaluate $\vec{R}$. Since point C is at the end of the shaft OBC, it rotates about point O with constant speed ω_1, i.e. it executes constant rate circular motion with radius OC (see Figure 21.26) and its motion lies in the xy-plane. Therefore,

$$\begin{aligned}\vec{a}_C &= \underbrace{\dot{\omega}_1}_{0}\,\hat{k}\times\vec{r}_C - \omega_1^2\vec{r}_C \\ &= -\omega_1^2(L\hat{\imath} + R\hat{\jmath}) \\ &= -9(\,\text{rad/s})^2\cdot(1\,\text{m}\hat{\imath} + 0.5\,\text{m}\hat{\jmath}) \\ &= -(9.0\hat{\imath} + 4.5\hat{\jmath})\,\text{m/s}^2.\end{aligned}$$

Substituting the result in Eqn.(21.17) we get

$$\begin{aligned}\vec{R} &= 2\,\text{kg}\cdot(-9.0\hat{\imath} - 4.5\hat{\jmath})\,\text{m/s}^2 \\ &= -18\,\text{N}\hat{\imath} - 9\,\text{N}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{R} = -18\,\text{N}\hat{\imath} - 9\,\text{N}\hat{\jmath}}$$

To find $\vec{M}$ let us do angular momentum balance for the system about point O:

$$\begin{aligned}\sum \vec{M}_O &= \dot{\vec{H}}_{/O} \\ \vec{M} &= [I^{\text{cm}}]\cdot\dot{\vec{\omega}}_{\mathcal{D}} + \vec{\omega}_{\mathcal{D}}\times([I^{\text{cm}}]\cdot\vec{\omega}_{\mathcal{D}}) + \underbrace{\vec{r}_C\times m\vec{a}_C}_{=\vec{0}}\end{aligned} \tag{21.18}$$

where the last term is zero because $\vec{r}_C || \vec{a}_C$ (see Fig .21.26). Thus, to find $\vec{M}$ we need to find the absolute angular acceleration $\dot{\vec{\omega}}_{\mathcal{D}}$, the absolute angular velocity $\vec{\omega}_{\mathcal{D}}$ and $[I^{\text{cm}}]$ of the disk.

Let us attach a frame $\mathcal{B}$ to the shaft OBC. Thus frame $\mathcal{B}$ rotates with the shaft with angular velocity $\vec{\omega}_{\mathcal{B}} = \omega_1\hat{k}$. For calculations in the rotating frame, we attach a coordinate system $x'y'z'$ with basis vectors $\hat{\imath}', \hat{\jmath}', \hat{k}'$ to the shaft (and *not* to the disk) at point C (see

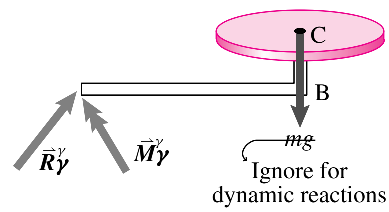


Figure 21.25: Free-body diagram of the shaft and the disk system. $\vec{R}$ and $\vec{M}$ are the unknown dynamic reactions at the support O.

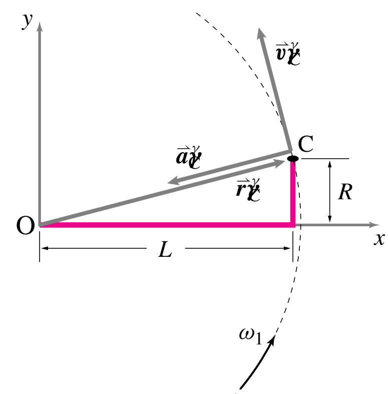


Figure 21.26: The center-of-mass of the disc, point C, executes constant rate circular motion about point O.

Fig .21.27). Note that the moment of inertia matrix $[\boldsymbol{I}^{\text{cm}}]$ of the disk in the $x'y'z'$ coordinate system remains constant due to the symmetry of the disk about these axes and this matrix in the primed coordinate system is written as

$$[\boldsymbol{I}^{\text{cm}}] = \frac{mr^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

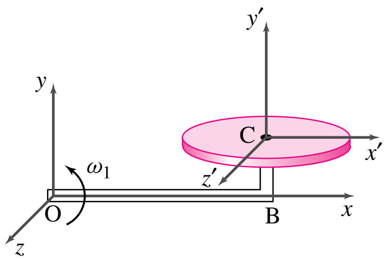


Figure 21.27: The primed coordinate system $x'y'z'$ is glued to the rotating frame $\mathcal{B}$ which rotates with the shaft OBC with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_1\hat{\boldsymbol{k}}$. At the instant shown, the two coordinate axes xyz and $x'y'z'$ are parallel to each other.

Now we find the absolute angular velocity and angular acceleration of the disk:

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\
&= \omega_1\hat{\boldsymbol{k}} + \omega_2\hat{\boldsymbol{\jmath}}. \\
\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} &= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\
&= \underbrace{\dot{\omega}_1}_{0}\hat{\boldsymbol{k}} + \underbrace{\dot{\omega}_2}_{0}\hat{\boldsymbol{\jmath}} + \omega_1\hat{\boldsymbol{k}} \times \omega_2\hat{\boldsymbol{\jmath}} \\
&= -\omega_1\omega_2\hat{\boldsymbol{\imath}}.
\end{aligned}$$

But, for the matrix and vector products in Eqn. (21.18) to be defined both quantities *must* be expressed in the same coordinate system. Since $[\boldsymbol{I}^{\text{cm}}]$ is in the primed coordinate system, we must express $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ and $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$ in the primed coordinate system. Fortunately, at the instant of interest

$$\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}, \quad \hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}.$$

Therefore, the components of any vector in the two coordinate systems are trivially the same. Hence,

$$[\boldsymbol{I}^{\text{cm}}] \cdot \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} = \frac{mr^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -\omega_1\omega_2 \\ 0 \\ 0 \end{Bmatrix} = -\frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}},$$

and similarly,

$$[\boldsymbol{I}^{\text{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}} = \frac{mr^2}{4}(2\omega_2\hat{\boldsymbol{\jmath}} + \omega_1\hat{\boldsymbol{k}}).$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}} \times ([\boldsymbol{I}^{\text{cm}}] \cdot \vec{\boldsymbol{\omega}}_{\mathcal{D}}) &= (\omega_1\hat{\boldsymbol{k}} + \omega_2\hat{\boldsymbol{\jmath}}) \times \frac{mr^2}{4}(2\omega_2\hat{\boldsymbol{\jmath}} + \omega_1\hat{\boldsymbol{k}}) \\
&= \frac{mr^2}{4}(\omega_1\omega_2\hat{\boldsymbol{\imath}} - 2\omega_1\omega_2\hat{\boldsymbol{\imath}}) \\
&= -\frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}}.
\end{aligned}$$

Now substituting these expressions in Eqn. (21.18) we get

$$\begin{aligned}
\vec{\boldsymbol{M}} &= -\frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}} - \frac{1}{4}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}} \\
&= -\frac{1}{2}mr^2\omega_1\omega_2\hat{\boldsymbol{\imath}} \\
&= -\frac{1}{2}(2\,\text{kg}) \cdot 0.25\,\text{m}^2 \cdot 15\,\text{rad/s}^2 \cdot 3\,\text{rad/s} \\
&= -11.25\,\text{N·m}.
\end{aligned}$$

$$\vec{\boldsymbol{M}} = -11.25\,\text{N·m}$$