21.1 Velocity and acceleration of points on a rigid object in 3-D: $\vec{\omega}$ and $\vec{\alpha}$

Since the assumption that objects move as if they were rigid is so common in mechanics, it is important to know how points on a rigid body move. Understanding the motion of points on a rigid body is useful in two different ways. First, one needs to know something about the motion in order to apply the momentum balance equations. Second, formulas involving the motion of points on a rigid body are useful to understand mechanisms, machines where rigid bodies are attached in various ways to each other.

Angular velocity

When any rigid body moves in any way in two or three dimensions it always has an angular velocity. If we call the body by the name $\mathcal{B}$ (script B), then we call the angular velocity of the body $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. The angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is a vector that may change with time. But what is the angular velocity of a rigid body?

Angular velocity in two dimensions

Let's first review $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ carefully for the case of a two-dimensional body moving in the plane (see chapter 4). We study 2-D rotation by keeping track of straight lines that are drawn on the body. For each line that we draw on the body we keep track of the angle that the line makes with either the positive x or y axis. Let's assume that all angles are measured as positive in the counter-clockwise direction. The x and y axes are fixed but the lines on the body rotate with the body. These angles θ_1, θ_2, ... all change with time. But, though each angle is different from the other, all the angles change at the same rate. That is:

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots$$

Since all lines on a body rotate at the same rate as each other (at a given instant in time) the rotation rate is a single number for the body. We call this number $\omega_{\mathcal{B}}$ for body $\mathcal{B}$. In order to make various formulas work out we define a vector which is perpendicular to the xy plane. We make the magnitude of this vector $\omega_{\mathcal{B}}$. So in two dimensions the angular velocity of a rigid body is

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \underbrace{\omega_{\mathcal{B}}}_{\dot{\theta}} \hat{\boldsymbol{k}}$$

where $\dot{\theta}$ is the rate of change of the angle of *any* line marked on the body.

Angular velocity in three dimensions

In three dimensions there is not such a simple geometric description of the angular velocity vector of a rigid body. At any instant in time, the relative velocities of points on a rigid body can be described by thinking of the body as spinning about an axis in space, although this axis may change with time. The direction of this axis is the instantaneous direction of the angular velocity vector. The spinning rate is the instantaneous magnitude of the angular velocity vector.

Although it is a little unpalatable for those who need initial motivation, one way of defining the angular velocity vector is as follows. The angular velocity vector $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ of a body $\mathcal{B}$ is that vector which makes the formula 17.5 below for relative velocity true. It is a fact that for every rigid body there is, at any instant in time, a unique vector that serves this purpose. That is, for the rigid body $\mathcal{B}$ there is one angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ that can be used correctly to describe the relative velocities of all pairs of points on the body.

Relative velocity of two points on a rigid body

For any two points A and B glued to a rigid body $\mathcal{B}$ the relative velocity of the points ('the velocity of B relative to A') is

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} \tag{21.1}$$

where $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is the angular velocity of body $\mathcal{B}$. In words, the relative velocity of two points on a rigid body is given by the cross product of the angular velocity of the body with the relative position of the two points. This expression should not look mysterious. It says that the relative velocity of two points on a rigid body is the same as would be predicted for one of the points if the other were stationary.

Absolute velocity of a point on a rigid body

If one knows the velocity of one point on a rigid body and one also knows the angular velocity of the body, then one can find the velocity of any other point. How?

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + (\vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A) \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \underbrace{\vec{\boldsymbol{r}}_{B/A}}_{\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A}
\end{aligned}$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A. Because B and A are on the same rigid body, their relative velocity is given by the

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

formula 21.1 given above. The equation is a valid relation no matter what values are known and what are not known. It can be used various ways depending on what is known about the motion of the body being studied and what is known about the motion of the chosen points A and B.

Acceleration of points on a rigid body

Angular acceleration

Because both the magnitude and the direction of the angular velocity vector $\vec{\boldsymbol{\omega}}$ can change with time, we can define the angular acceleration $\vec{\boldsymbol{\alpha}}$ of a rigid body as the rate of change of angular velocity, $\vec{\boldsymbol{\alpha}} = \dot{\vec{\boldsymbol{\omega}}}$. The angular acceleration of a body $\mathcal{B}$ is called $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$. The relative acceleration of two points on a rigid body depends on the angular acceleration.

Angular acceleration in two dimensions

The concept of angular acceleration is most intuitive for two-dimensional bodies moving in the plane. In this case both the angular velocity and the angular acceleration are always perpendicular to the plane. That is $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{k}}$. In this case the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e., $\alpha = \dot{\omega}$. There is no change in the direction of the angular velocity vector.

Relative acceleration of two points on a rigid body

For any two points A and B glued to a rigid body $\mathcal{B}$ the relative acceleration of the points ('the acceleration of B relative to A') is

$$\vec{\boldsymbol{a}}_{B/A} = \vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A = \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}) \tag{21.2}$$

If point A has no acceleration, this formula is the same as that for the acceleration of a point going in circles at variable rate in chapter 5. The formula above has a little more generality, however. In three-dimensional motion $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ can change in both magnitude *and direction* so the relative motion of B to A is not circular. Nonetheless, the formula above is correct and valid.

Absolute acceleration of a point on a rigid body

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How?

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A + (\vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A) = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} && (21.3)\\
&= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}) + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} && (21.4)
\end{aligned}$$

Again, this equation has various uses depending on what is known and what one is trying to find. Equation 21.4 is often called the three term acceleration formula. The acceleration of a point B on a rigid body is the sum of three terms. The first, $\vec{\boldsymbol{a}}_A$, is the acceleration of some point A on the body. The second term, $\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A})$, is the centripetal acceleration. Its direction is from B towards the line which goes through A and is parallel to the angular velocity. The third term, $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}$, is due to the change of angular velocity.

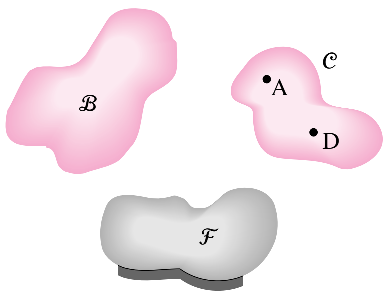


Figure 21.1: Bodies $\mathcal{B}$ and $\mathcal{C}$ move relative to each other.

Angular velocity

We want to determine the absolute angular velocity of a moving body $\mathcal{C}$, $\vec{\boldsymbol{\omega}}_{\mathcal{C}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}}$. We will calculate it using data from another moving frame $\mathcal{B}$. For example, imagine you drive a car $\mathcal{B}$ and rotate the steering wheel $\mathcal{C}$ to make a turn. What is the angular velocity of the steering wheel relative to the ground? Once we have found this angular velocity, we use it as *the* angular velocity of $\mathcal{C}$ for use in any of the formulae earlier in this chapter or in Chapter 7.

Angular velocity relative to a body or frame

We start by considering the angular velocity of a body relative to a body or frame. Not only can an arbitrary point P be viewed from a moving frame $\mathcal{B}$, but so also can an entire rigid body $\mathcal{C}$ with all of its points. The velocities of each of the points on $\mathcal{C}$ can be calculated relative to the moving frame $\mathcal{B}$. So, referring to fig. 21.1, the relative velocities of two points A and D on the body $\mathcal{C}$ as calculated in the moving frame $\mathcal{B}$ can also be calculated.

$$\vec{\boldsymbol{v}}_{D/\mathcal{B}} - \vec{\boldsymbol{v}}_{A/\mathcal{B}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{D/A}.$$

The quantity $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}$ is the angular velocity of body $\mathcal{C}$ relative to body $\mathcal{B}$. This formula is easy enough to understand for relative rotation about a fixed axis but turns out to be true for the general motion of a rigid body.

Absolute angular velocity using data from a moving frame

The equation

$$\overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{C}}}^{\text{absolute angular velocity of } \mathcal{C}} = \underbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}}}_{\text{absolute angular velocity of } \mathcal{B}} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}}^{\text{angular velocity of } \mathcal{C} \text{ relative to } \mathcal{B}} \qquad (21.5)$$

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

is not only plausible but also true. It is often referred to with the deceptively simple words 'angular velocities add.' However plausible, this *addition formula for angular velocities* is not obvious for three-dimensional angular velocity. It is important, however, and is one of the cornerstones of three-dimensional rigid body kinematics. One might develop a feel for the formula by looking at the boxes on rotations on the following pages. Let's illustrate the calculation of absolute angular velocity with the crane of fig. 21.4.

Example: Angular velocity (3-D): A crane, again
Reconsider the crane in fig. 21.3. We have the following: the angular velocity of the cab is $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{j}}'$ and the angular velocity of the boom relative to the cab is $\vec{\omega}_{\mathcal{A}/\mathcal{B}} = \dot{\phi}\hat{\boldsymbol{k}}'$.

So, equation 21.5 on page 4 tells us that the angular velocity of the crane boom relative to the fixed frame is

$$\vec{\omega}_{\mathcal{A}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{A}/\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{j}}' + \dot{\phi}\hat{\boldsymbol{k}}'.$$

21.1 Some comments on the motions of rigid bodies

The kinematics theory we develop is based on two key facts.

1. The absolute rate of change of any vector $\vec{Q}$ glued to a rotating body $\mathcal{C}$ is
$$\dot{\vec{Q}} = \vec{\omega}_{\mathcal{C}} \times \vec{Q}. \tag{21.6}$$
More generally, the rate of change of any vector that is fixed in a frame $\mathcal{C}$ calculated relative to a frame $\mathcal{B}$ is given by
$${}^{\mathcal{B}}\dot{\vec{Q}} = \vec{\omega}_{\mathcal{C}/\mathcal{B}} \times \vec{Q}. \tag{21.7}$$
If for frame $\mathcal{B}$ we use the fixed frame $\mathcal{F}$, then the formula 21.7 above reduces to formula 21.6.
2. The second key fact is: if body $\mathcal{C}$ has angular velocity $\vec{\omega}_{\mathcal{C}/\mathcal{B}}$ with respect to body $\mathcal{B}$ and if body $\mathcal{B}$ has the angular velocity $\vec{\omega}_{\mathcal{B}}$, then the absolute angular velocity of $\mathcal{C}$ is
$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}. \tag{21.8}$$
More generally, we can write
$$\vec{\omega}_{\mathcal{C}/\mathcal{D}} = \vec{\omega}_{\mathcal{B}/\mathcal{D}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}. \tag{21.9}$$
with no need to refer to any fixed frame. If $\mathcal{D}$ is the fixed frame, then formula 21.9 reduces to formula 21.8.

Equations 21.6-21.9 seem natural enough once one is used to the terms and notation. However, to derive formulas 21.6-21.9 from more basic notions involves several steps of geometric and/or algebraic reasoning. Here is a list of some of the high points in that reasoning.

I. The net motion of any rigid body from time t_1 to time t_2 can be represented as a displacement of a point A on the body and a rotation θ about an axis $\hat{\boldsymbol{n}}$ through A. If a different point, say B, is chosen, then the same net motion is a displacement of the point B and a rotation about a fixed axis through B.

II. For a given net motion, the displacement depends on which point A or B is chosen. The direction of the axis $\hat{\boldsymbol{n}}$ of rotation and the amount of rotation about that axis, θ, does not depend on which point is chosen.

III. The net finite rotation after two successive finite rotations of a body depends on the order of the rotations. For example, a 90° rotation about the y-axis followed by a 90° rotation about the x-axis is *not* equivalent to a 90° rotation about the x-axis followed by a 90° rotation about the y-axis. For a more detailed explanation of this idea, see the box 21.2 on page 6.

IV. For infinitesimal rotations, however, the net infinitesimal rotation does *not* depend on the order of successive rotations. In fact, the net rotation is given by vector addition:
$$\Delta\theta\hat{\boldsymbol{n}} = (\Delta\theta_1)\hat{\boldsymbol{n}}_1 + (\Delta\theta_2)\hat{\boldsymbol{n}}_2.$$
For a more detailed explanation of this fact, see the box 21.3 on page 7.

Facts I and II imply that the relative motion of any two points is that of rotation of one point about a fixed axis through the other. For small rotations, this geometric result leads to relation

$$\Delta\vec{\boldsymbol{r}}_{B/A} = (\Delta\theta\hat{\boldsymbol{n}}) \times \vec{\boldsymbol{r}}_{B/A}.$$

Dividing by Δt and taking the limit as $\Delta t \to 0$ gives

$$\dot{\vec{\boldsymbol{r}}}_{B/A} = (\dot{\theta}\hat{\boldsymbol{n}}) \times \vec{\boldsymbol{r}}_{B/A}.$$

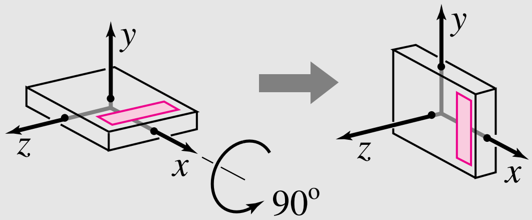
$\vec{\omega}$ is *defined* as the vector $\dot{\theta}\hat{\boldsymbol{n}}$ from this formula. The equation of fact IV, when divided by Δt, leads to the formula for addition of angular velocities, equation 21.8:

$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}. \tag{21.10}$$

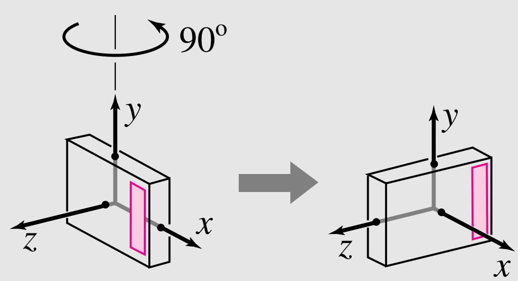
21.2 Finite Rotations

The net finite rotation after two successive finite rotations about a fixed axis depends on the order of the rotations.

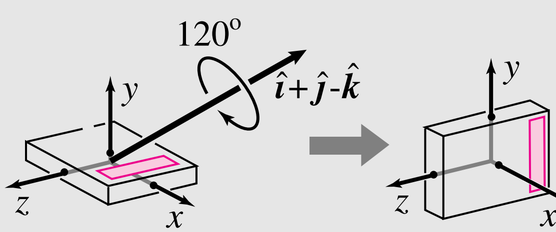
For example, a 90° rotation about the x-axis



followed by a 90° rotation about the y-axis

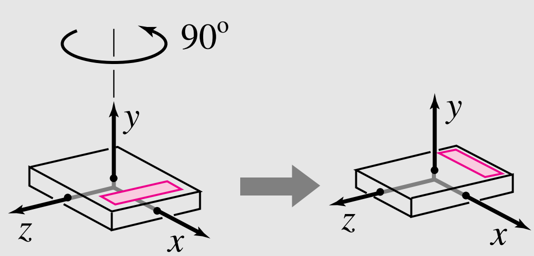


is equivalent to a single 120° rotation about an axis in the direction $\hat{\imath} + \hat{\jmath} - \hat{k}$

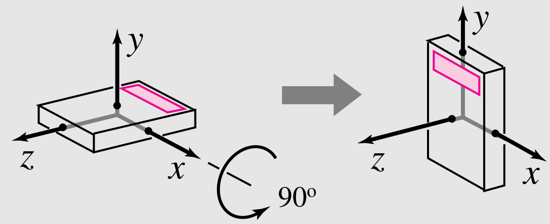


which is, agreeably, hard to picture (this rotation corresponds to a cube rotating about one of its main diagonals).

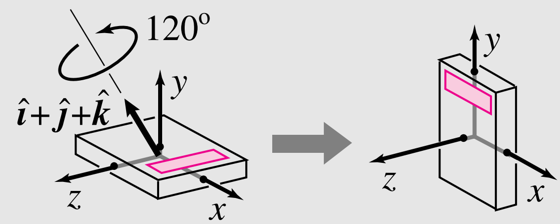
On the other hand, a 90° rotation about the y-axis



followed by a 90° rotation about the x-axis



is equivalent to a single 120° rotation about an axis in the direction $\hat{\imath} + \hat{\jmath} + \hat{k}$



which is also hard to see (this rotation corresponds to a 120° rotation about a different main diagonal of a cube).

Because the order in which the rotations occur affects the resulting net rotation, the result of successive large rotations about axes fixed in space can't be found by vector addition.

Is rotation a vector?

Rotation can be represented by an arrow with magnitude and direction. The direction is the axis of rotation. The magnitude is the angle of rotation. The net rotation after two rotations is also a vector in that it can be represented with a magnitude and direction. But the net rotation cannot be found by addition of the previous vectors. Vector addition is commutative (the order of addition doesn't matter) but the previous example shows that rotation is not commutative. So, in the sense that addition of two finite rotation vectors does not have a physical meaning, one can say 'rotation is not a vector,' meaning that the rules of vector arithmetic are not physically interpretable.

Small (infinitesimal) rotations and rotation rates do add, however.

An irony

Rotations are not vectors in that vector addition does not have meaning for the composition of two 3-D rotation 'vectors'. But an odd twist of history is that the very first use of the word 'vector' to represent an object with magnitude and direction was to describe rotations. This first use of the word 'vector' was by the mathematician and mechanician William Hamilton in 1846 in the context of his invention/discovery: 'quaternions'. One and a half centuries later, despite this history, some people are emphatic in saying 'rotation is not a vector.'

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

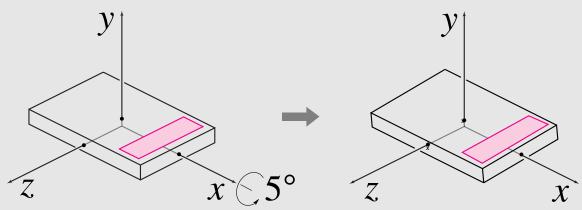
Summary: Rotations and angular velocity

- Finite rotations 'don't add.'
- Small rotations 'do add.'

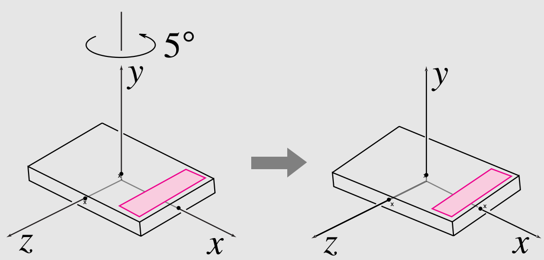
21.3 Small (Infinitesimal) Rotations

Unlike finite rotations, the net infinitesimal rotation does *not* depend on the order of successive rotations.

For example, let's now look at successive 5° rotations about the x and y axes. A 5° rotation about the x-axis

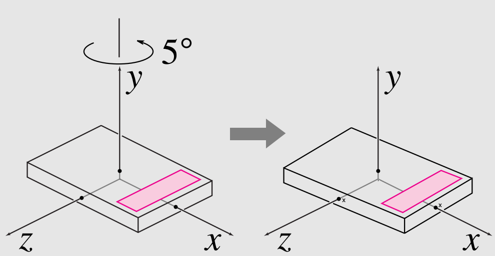


followed by a 5° rotation about the y-axis

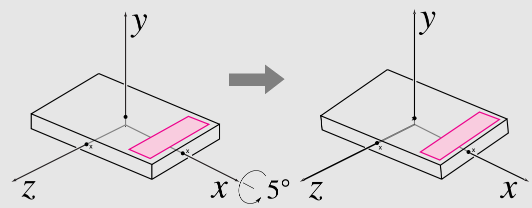


is nearly equal to a single rotation of $\sqrt{5^2+5^2}^{\circ} = \sqrt{50}^{\circ} = 7.0711°$ about the $\hat{\imath}+\hat{\jmath}$ axis. More exactly, the net rotation is 7.0699° about the $\hat{\imath}+\hat{\jmath}-(0.031)\hat{k}$ axis. Thus, the component of the axis of rotation in the z direction is negligible compared to the components in the x and y directions as we have supposed.

Similarly, a rotation of 5° about the y-axis



followed by a 5° rotation about the x-axis



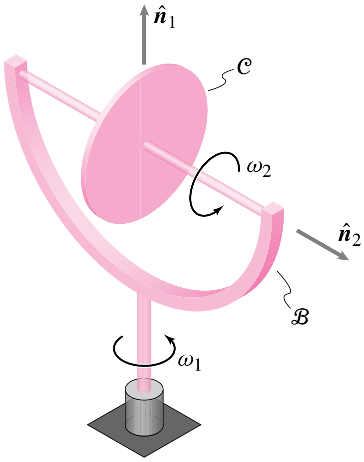
is nearly equal to a $\sqrt{50}^{\circ} = 7.0711°$ rotation about the $\hat{\imath}+\hat{\jmath}$ axis. More exactly, the net rotation is 7.0699° about the $\hat{\imath}+\hat{\jmath}+(0.031)\hat{k}$ axis. Thus, again, the component of the axis of rotation in the z direction is negligible compared to the components in the x and y directions as we have supposed. The final configuration after the two different sequences of rotations is nearly the same.

So, the order of small rotations doesn't matter and the result can be found by vector addition. In the example shown, the small rotation vector is

$$\overbrace{5^\circ\hat{\imath}}^{\text{5° rotation about } x\text{-axis}}\ \}\} + \varepsilon\ \underbrace{5^\circ\hat{\jmath}}_{\text{5° rotation about } y\text{-axis}} \approx \underbrace{5^\circ(\hat{\imath}+\hat{\jmath})}_{\text{the net rotation}}$$

This reasoning, mathematized, is what justifies the formula for adding angular velocities

$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}.$$



$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_1 \hat{\boldsymbol{n}}_1$$
$$\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} = \omega_2 \hat{\boldsymbol{n}}_2$$
$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{C}} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} \\ &= \omega_1 \hat{\boldsymbol{n}}_1 + \omega_2 \hat{\boldsymbol{n}}_2\end{aligned}$$

Figure 21.2: Example of adding angular velocities to get the absolute angular velocity: rotor on gimbals. The unit vector $\hat{\boldsymbol{n}}_2$ rotates with the gimbals.

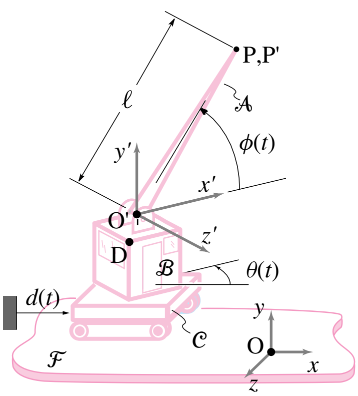


Figure 21.3

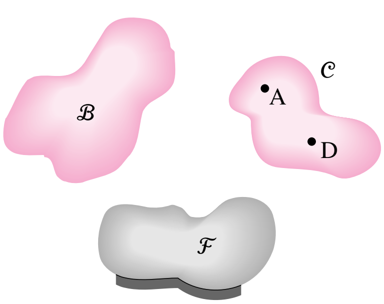


Figure 21.4: Bodies $\mathcal{B}$ and $\mathcal{C}$ move relative to each other.

- Angular velocity 'does add',

$$\vec{\boldsymbol{\omega}}_{\mathcal{C}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}.$$

- If D and A are fixed on $\mathcal{C}$, then

$$\vec{\boldsymbol{v}}_{D/\mathcal{B}} - \vec{\boldsymbol{v}}_{A/\mathcal{B}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{D/A}$$

- For any two points O' and D fixed in a body or frame $\mathcal{B}$, $\dot{\vec{\boldsymbol{r}}}_{D/O'} = {}^{\mathcal{F}}\dot{\vec{\boldsymbol{r}}}_{D/O} - {}^{\mathcal{F}}\dot{\vec{\boldsymbol{r}}}_{O'/O} = \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{D/O'}$.

Absolute angular acceleration using data from a moving frame

We want to determine the absolute angular acceleration of a moving body $\mathcal{C}$, $\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{F}}$. We will calculate it using data from another moving frame $\mathcal{B}$. See fig. 21.4. For example, imagine again you drive a car $\mathcal{B}$ and rotate the steering wheel $\mathcal{C}$ to make a turn. What is the absolute angular acceleration of the steering wheel relative to the ground?

Taking the time derivative of our result for the absolute angular velocity of a rigid body, and making use of the 'Q-dot' formula, we get for the bodies $\mathcal{B}$ and $\mathcal{C}$ in fig. 21.4:

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{C}} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} & \\
\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}} &= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}/\mathcal{B}} & \text{(differentiating)} \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} + \overbrace{{}^{\mathcal{B}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}}^{\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}/\mathcal{B}}} & (\dot{\vec{\boldsymbol{Q}}}\text{ formula}) \\
\vec{\boldsymbol{\alpha}}_{\mathcal{C}} &= \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}. & \text{(change of notation)}
\end{aligned}$$

Thus, the absolute angular acceleration of body $\mathcal{C}$ with respect to body $\mathcal{B}$ is

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}$$

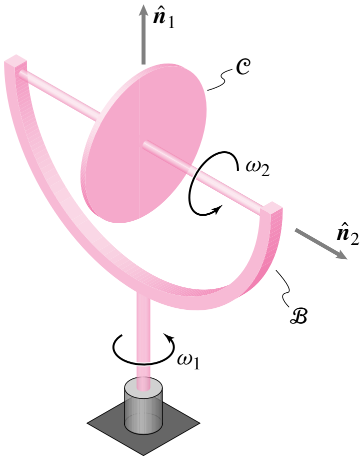
or, equivalently,

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}}, \qquad (21.11)$$

because $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}} = \vec{\boldsymbol{0}}$.

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

Note: angular accelerations *do not* add unless the angular velocities are parallel.



$$\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \dot{\omega}_1 \hat{\boldsymbol{n}}_1$$

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} = \dot{\omega}_2 \hat{\boldsymbol{n}}_2$$

$$\begin{aligned} \vec{\boldsymbol{\alpha}}_{\mathcal{C}} &= \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}} \\ &= \dot{\omega}_1 \hat{\boldsymbol{n}}_1 + \dot{\omega}_2 \hat{\boldsymbol{n}}_2 + \omega_1 \omega_2 (\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2) \end{aligned}$$

Figure 21.5: Example of total angular acceleration: rotor on gimbals. The unit vector $\hat{\boldsymbol{n}}_2$ rotates with the gimbals.

We now have all the kinematics we need to calculate the right hand sides of the momentum balance and energy equations for a variety of complex problems.

Let's illustrate the calculation of absolute angular acceleration with the crane of fig. 21.4.

Example: Angular acceleration (3-D): A crane, again

Reconsider the crane in fig. 21.6. The angular acceleration of the cab is $\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \ddot{\theta}\hat{\boldsymbol{\jmath}}'$, and the angular acceleration of the boom relative to the cab is $\vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{B}} = \ddot{\phi}\hat{\boldsymbol{k}}'$.

Equation 21.11 on page 8 tells us that the angular acceleration of the boom relative to the ground is

$$\begin{aligned} \vec{\boldsymbol{\alpha}}_{\mathcal{A}} &= \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} \\ &= \ddot{\theta}\hat{\boldsymbol{\jmath}}' + \ddot{\phi}\hat{\boldsymbol{k}}' + \dot{\theta}\hat{\boldsymbol{\jmath}}' \times \dot{\phi}\hat{\boldsymbol{k}}' \\ &= \dot{\theta}\dot{\phi}\hat{\boldsymbol{\imath}}' + \ddot{\theta}\hat{\boldsymbol{\jmath}}' + \ddot{\phi}\hat{\boldsymbol{k}}'. \end{aligned}$$

In this example, the absolute acceleration of the crane boom is due to:

1. the change in magnitude of the angular velocity of the cab,
2. the change in the magnitude of the angular velocity of the boom relative to the cab,
3. the change in direction of the angular velocity of the boom relative to the cab caused by the rotation of the cab. This contribution points in the $\hat{\boldsymbol{\imath}}'$ direction.

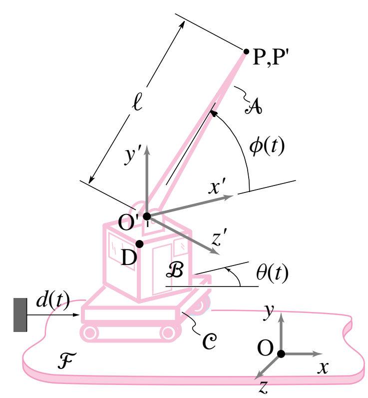


Figure 21.6

Consider the following special case at the instant of interest. Suppose the crane boom *does not* have a rotational acceleration relative to the cab and the cab does not have a rotational acceleration relative to the ground; thus, $\ddot{\phi} = 0$ and $\ddot{\theta} = 0$. Even in this case, the boom *does* have angular acceleration relative to the ground, $\vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{F}} = \dot{\theta}\dot{\phi}\hat{\boldsymbol{\imath}}' + 0\hat{\boldsymbol{\jmath}}' + 0\hat{\boldsymbol{k}}' \neq \vec{\boldsymbol{0}}$. In this special case, the absolute angular acceleration of the boom is due to the changing direction of its angular velocity vector caused by the rotation of the cab at constant rate relative to the ground.

In the 2-D case, if a body $\mathcal{A}$ rotates relative to a fixed frame at constant rate and a frame $\mathcal{B}$ rotates relative to the $\mathcal{A}$ at constant rate, body $\mathcal{B}$ does not have an angular acceleration relative to the fixed frame. That compounding angular velocities does not lead to angular acceleration in planar problems can be explained mathematically as follows: $\vec{\boldsymbol{\omega}}_{\mathcal{A}} = \omega_{\mathcal{A}}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}} = \omega_{\mathcal{B}/\mathcal{A}}\hat{\boldsymbol{k}}$ so that $\vec{\boldsymbol{\omega}}_{\mathcal{A}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}} = \vec{\mathbf{0}}$.

Summary of angular acceleration

Relative angular accelerations do not 'add' (an extra term is needed).

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}}^{\text{extra term}}$$

or, equivalently, because $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}} = \vec{\mathbf{0}}$,

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \underbrace{\vec{\boldsymbol{\omega}}_{\mathcal{C}}}_{\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}}}.$$

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap