21.1 Velocity and acceleration of points on a rigid object in 3-D: $\vec{\omega}$ and $\vec{\alpha}$

Since the assumption that objects move as if they were rigid is so common in mechanics, it is important to know how points on a rigid body move. Understanding the motion of points on a rigid body is useful in two different ways. First, one needs to know something about the motion in order to apply the momentum balance equations. Second, formulas involving the motion of points on a rigid body are useful to understand mechanisms, machines where rigid bodies are attached in various ways to each other.

Angular velocity

When any rigid body moves in any way in two or three dimensions it always has an angular velocity. If we call the body by the name $\mathcal{B}$ (script B), then we call the angular velocity of the body $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. The angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is a vector that may change with time. But what is the angular velocity of a rigid body?

Angular velocity in two dimensions

Let's first review $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ carefully for the case of a two-dimensional body moving in the plane (see chapter 4). We study 2-D rotation by keeping track of straight lines that are drawn on the body. For each line that we draw on the body we keep track of the angle that the line makes with either the positive x or y axis. Let's assume that all angles are measured as positive in the counter-clockwise direction. The x and y axes are fixed but the lines on the body rotate with the body. These angles θ_1, θ_2, ... all change with time. But, though each angle is different from the other, all the angles change at the same rate. That is:

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots$$

Since all lines on a body rotate at the same rate as each other (at a given instant in time) the rotation rate is a single number for the body. We call this number $\omega_{\mathcal{B}}$ for body $\mathcal{B}$. In order to make various formulas work out we define a vector which is perpendicular to the xy plane. We make the magnitude of this vector $\omega_{\mathcal{B}}$. So in two dimensions the angular velocity of a rigid body is

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \underbrace{\omega_{\mathcal{B}}}_{\dot{\theta}} \hat{\boldsymbol{k}}$$

where $\dot{\theta}$ is the rate of change of the angle of *any* line marked on the body.

Angular velocity in three dimensions

In three dimensions there is not such a simple geometric description of the angular velocity vector of a rigid body. At any instant in time, the relative velocities of points on a rigid body can be described by thinking of the body as spinning about an axis in space, although this axis may change with time. The direction of this axis is the instantaneous direction of the angular velocity vector. The spinning rate is the instantaneous magnitude of the angular velocity vector.

Although it is a little unpalatable for those who need initial motivation, one way of defining the angular velocity vector is as follows. The angular velocity vector $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ of a body $\mathcal{B}$ is that vector which makes the formula 17.5 below for relative velocity true. It is a fact that for every rigid body there is, at any instant in time, a unique vector that serves this purpose. That is, for the rigid body $\mathcal{B}$ there is one angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ that can be used correctly to describe the relative velocities of all pairs of points on the body.

Relative velocity of two points on a rigid body

For any two points A and B glued to a rigid body $\mathcal{B}$ the relative velocity of the points ('the velocity of B relative to A') is

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} \tag{21.1}$$

where $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is the angular velocity of body $\mathcal{B}$. In words, the relative velocity of two points on a rigid body is given by the cross product of the angular velocity of the body with the relative position of the two points. This expression should not look mysterious. It says that the relative velocity of two points on a rigid body is the same as would be predicted for one of the points if the other were stationary.

Absolute velocity of a point on a rigid body

If one knows the velocity of one point on a rigid body and one also knows the angular velocity of the body, then one can find the velocity of any other point. How?

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + (\vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A) \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \underbrace{\vec{\boldsymbol{r}}_{B/A}}_{\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A}
\end{aligned}$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A. Because B and A are on the same rigid body, their relative velocity is given by the formula 21.1 given above. The equation is a valid relation no matter what values are known and what are not known. It can be used various ways depending on what is known about the motion of the body being studied and what is known about the motion of the chosen points A and B.

Acceleration of points on a rigid body

Angular acceleration

Because both the magnitude and the direction of the angular velocity vector $\vec{\boldsymbol{\omega}}$ can change with time, we can define the angular acceleration $\vec{\boldsymbol{\alpha}}$ of a rigid body as the rate of change of angular velocity, $\vec{\boldsymbol{\alpha}} = \dot{\vec{\boldsymbol{\omega}}}$. The angular acceleration of a body $\mathcal{B}$ is called $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$. The relative acceleration of two points on a rigid body depends on the angular acceleration.

Angular acceleration in two dimensions

The concept of angular acceleration is most intuitive for two-dimensional bodies moving in the plane. In this case both the angular velocity and the angular acceleration are always perpendicular to the plane. That is $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{k}}$. In this case the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e., $\alpha = \dot{\omega}$. There is no change in the direction of the angular velocity vector.

Relative acceleration of two points on a rigid body

For any two points A and B glued to a rigid body $\mathcal{B}$ the relative acceleration of the points ('the acceleration of B relative to A') is

$$\vec{\boldsymbol{a}}_{B/A} \;=\; \vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A \;=\; \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}) \tag{21.2}$$

If point A has no acceleration, this formula is the same as that for the acceleration of a point going in circles at variable rate in chapter 5. The formula above has a little more generality, however. In three-dimensional motion $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ can change in both magnitude *and direction* so the relative motion of B to A is not circular. Nonetheless, the formula above is correct and valid.

Absolute acceleration of a point on a rigid body

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How?

$$\vec{\boldsymbol{a}}_B \;=\; \vec{\boldsymbol{a}}_A + (\vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A) = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} \tag{21.3}$$

$$\;=\; \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}) + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} \tag{21.4}$$

Again, this equation has various uses depending on what is known and what one is trying to find. Equation 21.4 is often called the three term acceleration formula. The acceleration of a point B on a rigid body is the sum of three terms. The first, $\vec{\boldsymbol{a}}_A$, is the acceleration of some point A on the body. The second term, $\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A})$, is the centripetal acceleration. Its direction is from B towards the line which goes through A and is parallel to the angular velocity. The third term, $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}$, is due to the change of angular velocity.

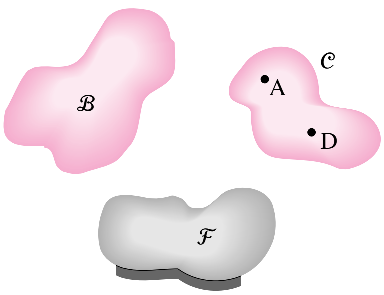


Figure 21.1: Bodies $\mathcal{B}$ and $\mathcal{C}$ move relative to each other.

Angular velocity

We want to determine the absolute angular velocity of a moving body $\mathcal{C}$, $\vec{\boldsymbol{\omega}}_{\mathcal{C}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}}$. We will calculate it using data from another moving frame $\mathcal{B}$. For example, imagine you drive a car $\mathcal{B}$ and rotate the steering wheel $\mathcal{C}$ to make a turn. What is the angular velocity of the steering wheel relative to the ground? Once we have found this angular velocity, we use it as *the* angular velocity of $\mathcal{C}$ for use in any of the formulae earlier in this chapter or in Chapter 7.

Angular velocity relative to a body or frame

We start by considering the angular velocity of a body relative to a body or frame. Not only can an arbitrary point P be viewed from a moving frame $\mathcal{B}$, but so also can an entire rigid body $\mathcal{C}$ with all of its points. The velocities of each of the points on $\mathcal{C}$ can be calculated relative to the moving frame $\mathcal{B}$. So, referring to fig. 21.1, the relative velocities of two points A and D on the body $\mathcal{C}$ as calculated in the moving frame $\mathcal{B}$ can also be calculated.

$$\vec{\boldsymbol{v}}_{D/\mathcal{B}} - \vec{\boldsymbol{v}}_{A/\mathcal{B}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{D/A}.$$

The quantity $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}$ is the angular velocity of body $\mathcal{C}$ relative to body $\mathcal{B}$. This formula is easy enough to understand for relative rotation about a fixed axis but turns out to be true for the general motion of a rigid body.

Absolute angular velocity using data from a moving frame

The equation

$$\overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{C}}}^{\text{absolute angular velocity of } \mathcal{C}} = \underbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}}}_{\text{absolute angular velocity of } \mathcal{B}} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}}^{\text{angular velocity of } \mathcal{C} \text{ relative to } \mathcal{B}} \tag{21.5}$$

is not only plausible but also true. It is often referred to with the deceptively simple words 'angular velocities add.' However plausible, this *addition formula for angular velocities* is not obvious for three-dimensional angular velocity. It is important, however, and is one of the cornerstones of three-dimensional rigid body kinematics. One might develop a feel for the formula by looking at the boxes on rotations on the following pages. Let's illustrate the calculation of absolute angular velocity with the crane of fig. 21.4.

Example: Angular velocity (3-D): A crane, again
Reconsider the crane in fig. 21.3. We have the following: the angular velocity of the cab is $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{\jmath}}'$ and the angular velocity of the boom relative to the cab is $\vec{\omega}_{\mathcal{A}/\mathcal{B}} = \dot{\phi}\hat{\boldsymbol{k}}'$.

So, equation 21.5 on page 4 tells us that the angular velocity of the crane boom relative to the fixed frame is

$$\vec{\omega}_{\mathcal{A}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{A}/\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{\jmath}}' + \dot{\phi}\hat{\boldsymbol{k}}'.$$

21.1 Some comments on the motions of rigid bodies

The kinematics theory we develop is based on two key facts.

1. The absolute rate of change of any vector $\vec{Q}$ glued to a rotating body $\mathcal{C}$ is
$$\dot{\vec{Q}} = \vec{\omega}_{\mathcal{C}} \times \vec{Q}. \tag{21.6}$$
More generally, the rate of change of any vector that is fixed in a frame $\mathcal{C}$ calculated relative to a frame $\mathcal{B}$ is given by
$${}^{\mathcal{B}}\dot{\vec{Q}} = \vec{\omega}_{\mathcal{C}/\mathcal{B}} \times \vec{Q}. \tag{21.7}$$
If for frame $\mathcal{B}$ we use the fixed frame $\mathcal{F}$, then the formula 21.7 above reduces to formula 21.6.
2. The second key fact is: if body $\mathcal{C}$ has angular velocity $\vec{\omega}_{\mathcal{C}/\mathcal{B}}$ with respect to body $\mathcal{B}$ and if body $\mathcal{B}$ has the angular velocity $\vec{\omega}_{\mathcal{B}}$, then the absolute angular velocity of $\mathcal{C}$ is
$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}. \tag{21.8}$$
More generally, we can write
$$\vec{\omega}_{\mathcal{C}/\mathcal{D}} = \vec{\omega}_{\mathcal{B}/\mathcal{D}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}. \tag{21.9}$$
with no need to refer to any fixed frame. If $\mathcal{D}$ is the fixed frame, then formula 21.9 reduces to formula 21.8.

Equations 21.6-21.9 seem natural enough once one is used to the terms and notation. However, to derive formulas 21.6-21.9 from more basic notions involves several steps of geometric and/or algebraic reasoning. Here is a list of some of the high points in that reasoning.

I. The net motion of any rigid body from time t_1 to time t_2 can be represented as a displacement of a point A on the body and a rotation θ about an axis $\hat{\boldsymbol{n}}$ through A. If a different point, say B, is chosen, then the same net motion is a displacement of the point B and a rotation about a fixed axis through B.

II. For a given net motion, the displacement depends on which point A or B is chosen. The direction of the axis $\hat{\boldsymbol{n}}$ of rotation and the amount of rotation about that axis, θ, does not depend on which point is chosen.

III. The net finite rotation after two successive finite rotations of a body depends on the order of the rotations. For example, a 90° rotation about the y-axis followed by a 90° rotation about the x-axis is *not* equivalent to a 90° rotation about the x-axis followed by a 90° rotation about the y-axis. For a more detailed explanation of this idea, see the box 21.2 on page 6.

IV. For infinitesimal rotations, however, the net infinitesimal rotation does *not* depend on the order of successive rotations. In fact, the net rotation is given by vector addition:
$$\Delta\theta\hat{\boldsymbol{n}} = (\Delta\theta_1)\hat{\boldsymbol{n}}_1 + (\Delta\theta_2)\hat{\boldsymbol{n}}_2.$$
For a more detailed explanation of this fact, see the box 21.3 on page 7.

Facts I and II imply that the relative motion of any two points is that of rotation of one point about a fixed axis through the other. For small rotations, this geometric result leads to relation

$$\Delta\vec{\boldsymbol{r}}_{B/A} = (\Delta\theta\hat{\boldsymbol{n}}) \times \vec{\boldsymbol{r}}_{B/A}.$$

Dividing by Δt and taking the limit as $\Delta t \to 0$ gives

$$\dot{\vec{\boldsymbol{r}}}_{B/A} = (\dot{\theta}\hat{\boldsymbol{n}}) \times \vec{\boldsymbol{r}}_{B/A}.$$

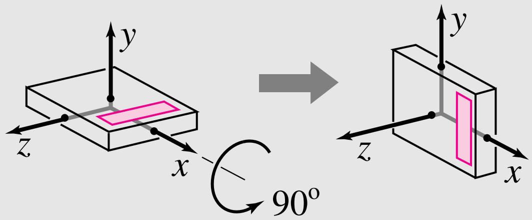
$\vec{\omega}$ is *defined* as the vector $\dot{\theta}\hat{\boldsymbol{n}}$ from this formula. The equation of fact IV, when divided by Δt, leads to the formula for addition of angular velocities, equation 21.8:

$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}. \tag{21.10}$$

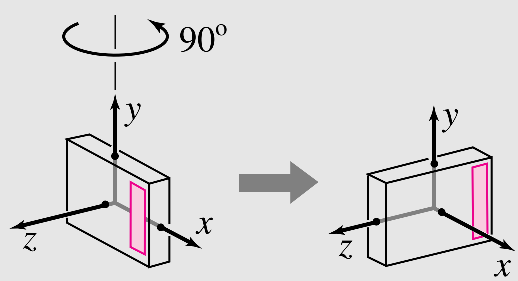
21.2 Finite Rotations

The net finite rotation after two successive finite rotations about a fixed axis depends on the order of the rotations.

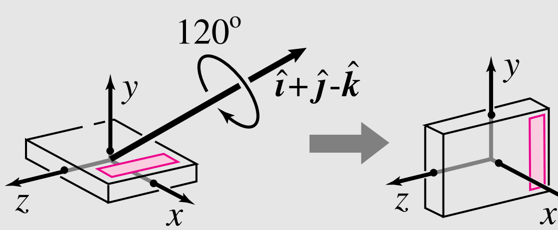
For example, a 90° rotation about the x-axis



followed by a 90° rotation about the y-axis

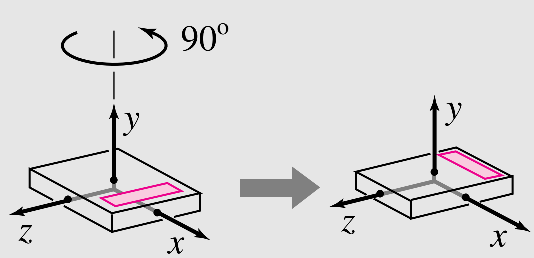


is equivalent to a single 120° rotation about an axis in the direction $\hat{\imath} + \hat{\jmath} - \hat{k}$

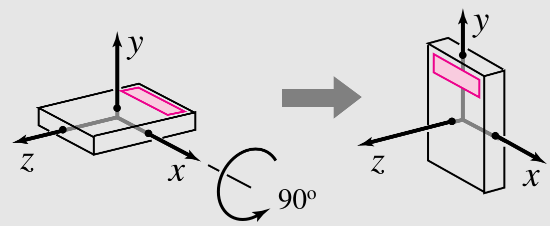


which is, agreeably, hard to picture (this rotation corresponds to a cube rotating about one of its main diagonals).

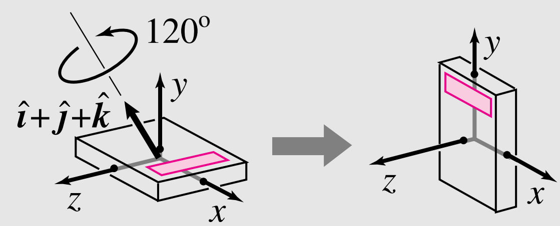
On the other hand, a 90° rotation about the y-axis



followed by a 90° rotation about the x-axis



is equivalent to a single 120° rotation about an axis in the direction $\hat{\imath} + \hat{\jmath} + \hat{k}$



which is also hard to see (this rotation corresponds to a 120° rotation about a different main diagonal of a cube).

Because the order in which the rotations occur affects the resulting net rotation, the result of successive large rotations about axes fixed in space can't be found by vector addition.

Is rotation a vector?

Rotation can be represented by an arrow with magnitude and direction. The direction is the axis of rotation. The magnitude is the angle of rotation. The net rotation after two rotations is also a vector in that it can be represented with a magnitude and direction. But the net rotation cannot be found by addition of the previous vectors. Vector addition is commutative (the order of addition doesn't matter) but the previous example shows that rotation is not commutative. So, in the sense that addition of two finite rotation vectors does not have a physical meaning, one can say 'rotation is not a vector,' meaning that the rules of vector arithmetic are not physically interpretable.

Small (infinitesimal) rotations and rotation rates do add, however.

An irony

Rotations are not vectors in that vector addition does not have meaning for the composition of two 3-D rotation 'vectors'. But an odd twist of history is that the very first use of the word 'vector' to represent an object with magnitude and direction was to describe rotations. This first use of the word 'vector' was by the mathematician and mechanician William Hamilton in 1846 in the context of his invention/discovery: 'quaternions'. One and a half centuries later, despite this history, some people are emphatic in saying 'rotation is not a vector.'

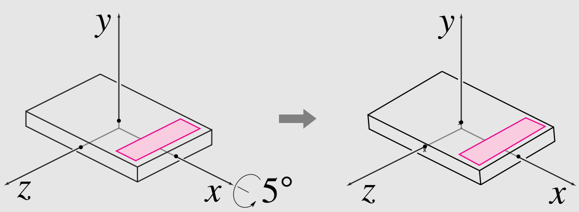
Summary: Rotations and angular velocity

- Finite rotations 'don't add.'
- Small rotations 'do add.'

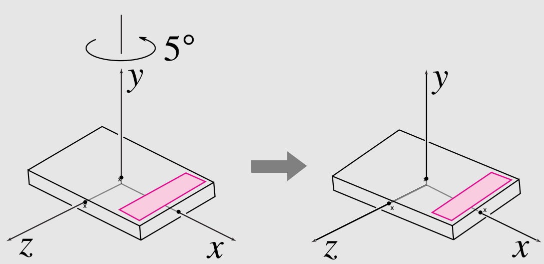
21.3 Small (Infinitesimal) Rotations

Unlike finite rotations, the net infinitesimal rotation does *not* depend on the order of successive rotations.

For example, let's now look at successive 5° rotations about the x and y axes. A 5° rotation about the x-axis

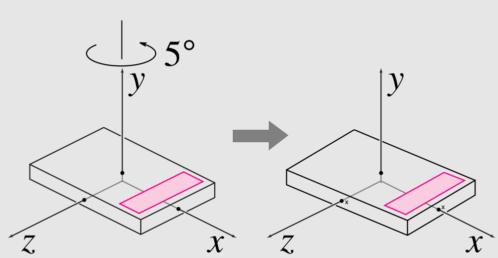


followed by a 5° rotation about the y-axis

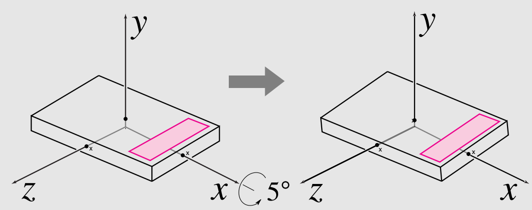


is nearly equal to a single rotation of $\sqrt{5^2+5^2}^{\circ} = \sqrt{50}^{\circ} = 7.0711°$ about the $\hat{\imath}+\hat{\jmath}$ axis. More exactly, the net rotation is 7.0699° about the $\hat{\imath}+\hat{\jmath}-(0.031)\hat{k}$ axis. Thus, the component of the axis of rotation in the z direction is negligible compared to the components in the x and y directions as we have supposed.

Similarly, a rotation of 5° about the y-axis



followed by a 5° rotation about the x-axis



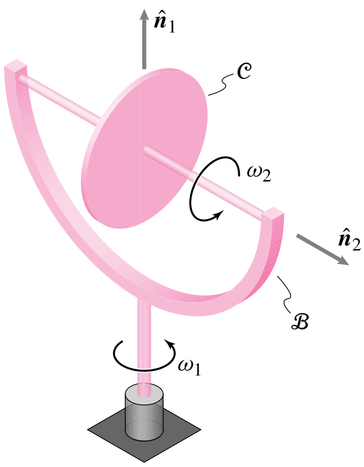
is nearly equal to a $\sqrt{50}^{\circ} = 7.0711°$ rotation about the $\hat{\imath}+\hat{\jmath}$ axis. More exactly, the net rotation is 7.0699° about the $\hat{\imath}+\hat{\jmath}+(0.031)\hat{k}$ axis. Thus, again, the component of the axis of rotation in the z direction is negligible compared to the components in the x and y directions as we have supposed. The final configuration after the two different sequences of rotations is nearly the same.

So, the order of small rotations doesn't matter and the result can be found by vector addition. In the example shown, the small rotation vector is

$$\overbrace{5^\circ\hat{\imath}}^{\text{5° rotation about } x\text{-axis}}\}\} + \varepsilon\ \underbrace{5^\circ\hat{\jmath}}_{\text{5° rotation about } y\text{-axis}} \approx \underbrace{5^\circ(\hat{\imath}+\hat{\jmath})}_{\text{the net rotation}}$$

This reasoning, mathematized, is what justifies the formula for adding angular velocities

$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}.$$



$$\vec{\omega}_{\mathcal{B}} = \omega_1 \hat{\boldsymbol{n}}_1$$

$$\vec{\omega}_{\mathcal{C}/\mathcal{B}} = \omega_2 \hat{\boldsymbol{n}}_2$$

$$\begin{aligned}\vec{\omega}_{\mathcal{C}} &= \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}} \\ &= \omega_1 \hat{\boldsymbol{n}}_1 + \omega_2 \hat{\boldsymbol{n}}_2\end{aligned}$$

Figure 21.2: Example of adding angular velocities to get the absolute angular velocity: rotor on gimbals. The unit vector $\hat{\boldsymbol{n}}_2$ rotates with the gimbals.

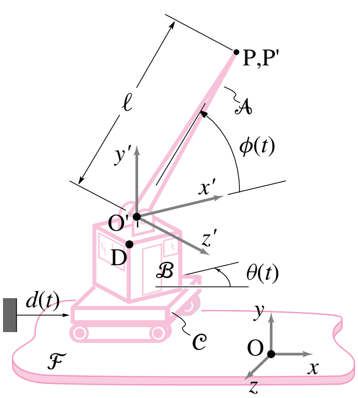


Figure 21.3

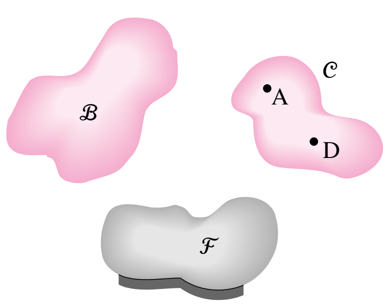


Figure 21.4: Bodies $\mathcal{B}$ and $\mathcal{C}$ move relative to each other.

- Angular velocity 'does add',

$$\vec{\omega}_{\mathcal{C}} = \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}.$$

- If D and A are fixed on $\mathcal{C}$, then

$$\vec{\boldsymbol{v}}_{D/\mathcal{B}} - \vec{\boldsymbol{v}}_{A/\mathcal{B}} = \vec{\omega}_{\mathcal{C}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{D/A}$$

- For any two points O' and D fixed in a body or frame $\mathcal{B}$, $\dot{\vec{\boldsymbol{r}}}_{D/O'} = {}^{\mathcal{F}}\dot{\vec{\boldsymbol{r}}}_{D/O} - {}^{\mathcal{F}}\dot{\vec{\boldsymbol{r}}}_{O'/O} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{D/O'}$.

Absolute angular acceleration using data from a moving frame

We want to determine the absolute angular acceleration of a moving body $\mathcal{C}$, $\vec{\alpha}_{\mathcal{C}} = \vec{\alpha}_{\mathcal{C}/\mathcal{F}}$. We will calculate it using data from another moving frame $\mathcal{B}$. See fig. 21.4. For example, imagine again you drive a car $\mathcal{B}$ and rotate the steering wheel $\mathcal{C}$ to make a turn. What is the absolute angular acceleration of the steering wheel relative to the ground?

Taking the time derivative of our result for the absolute angular velocity of a rigid body, and making use of the 'Q-dot' formula, we get for the bodies $\mathcal{B}$ and $\mathcal{C}$ in fig. 21.4:

$$\begin{aligned}
\vec{\omega}_{\mathcal{C}} &= \vec{\omega}_{\mathcal{B}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}} \\
\dot{\vec{\omega}}_{\mathcal{C}} &= \dot{\vec{\omega}}_{\mathcal{B}} + \dot{\vec{\omega}}_{\mathcal{C}/\mathcal{B}} && \text{(differentiating)} \\
&= \dot{\vec{\omega}}_{\mathcal{B}} + \overbrace{{}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{C}/\mathcal{B}}}^{\dot{\vec{\omega}}_{\mathcal{C}/\mathcal{B}}} && (\dot{\vec{\boldsymbol{Q}}}\text{ formula}) \\
\vec{\alpha}_{\mathcal{C}} &= \vec{\alpha}_{\mathcal{B}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{C}/\mathcal{B}}. && \text{(change of notation)}
\end{aligned}$$

Thus, the absolute angular acceleration of body $\mathcal{C}$ with respect to body $\mathcal{B}$ is

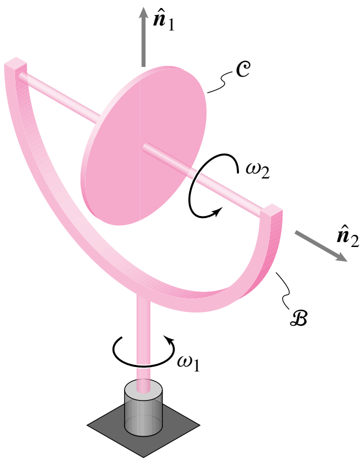
$$\vec{\alpha}_{\mathcal{C}} = \vec{\alpha}_{\mathcal{B}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{C}/\mathcal{B}}$$

or, equivalently,

$$\vec{\alpha}_{\mathcal{C}} = \vec{\alpha}_{\mathcal{B}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{C}}, \tag{21.11}$$

because $\vec{\omega}_{\mathcal{C}/\mathcal{F}} = \vec{\omega}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}}$ and $\vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{B}} = \vec{\boldsymbol{0}}$.

Note: angular accelerations *do not* add unless the angular velocities are parallel.



$$\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \dot{\omega}_1 \hat{\boldsymbol{n}}_1$$

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} = \dot{\omega}_2 \hat{\boldsymbol{n}}_2$$

$$\begin{aligned}\vec{\boldsymbol{\alpha}}_{\mathcal{C}} &= \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}} \\ &= \dot{\omega}_1 \hat{\boldsymbol{n}}_1 + \dot{\omega}_2 \hat{\boldsymbol{n}}_2 + \omega_1 \omega_2 (\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2)\end{aligned}$$

Figure 21.5: Example of total angular acceleration: rotor on gimbals. The unit vector $\hat{\boldsymbol{n}}_2$ rotates with the gimbals.

We now have all the kinematics we need to calculate the right hand sides of the momentum balance and energy equations for a variety of complex problems.

Let's illustrate the calculation of absolute angular acceleration with the crane of fig. 21.4.

Example: Angular acceleration (3-D): A crane, again

Reconsider the crane in fig. 21.6. The angular acceleration of the cab is $\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \ddot{\theta}\,\hat{\boldsymbol{\jmath}}'$, and the angular acceleration of the boom relative to the cab is $\vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{B}} = \ddot{\phi}\hat{\boldsymbol{k}}'$.

Equation 21.11 on page 8 tells us that the angular acceleration of the boom relative to the ground is

$$\begin{aligned}\vec{\boldsymbol{\alpha}}_{\mathcal{A}} &= \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} \\ &= \ddot{\theta}\,\hat{\boldsymbol{\jmath}}' + \ddot{\phi}\hat{\boldsymbol{k}}' + \dot{\theta}\,\hat{\boldsymbol{\jmath}}' \times \dot{\phi}\hat{\boldsymbol{k}}' \\ &= \dot{\theta}\dot{\phi}\hat{\boldsymbol{\imath}}' + \ddot{\theta}\,\hat{\boldsymbol{\jmath}}' + \ddot{\phi}\hat{\boldsymbol{k}}'.\end{aligned}$$

In this example, the absolute acceleration of the crane boom is due to:

1. the change in magnitude of the angular velocity of the cab,
2. the change in the magnitude of the angular velocity of the boom relative to the cab,
3. the change in direction of the angular velocity of the boom relative to the cab caused by the rotation of the cab. This contribution points in the $\hat{\boldsymbol{\imath}}'$ direction.

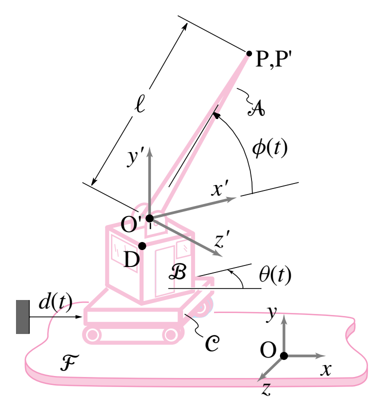


Figure 21.6

Consider the following special case at the instant of interest. Suppose the crane boom *does not* have a rotational acceleration relative to the cab and the cab does not have a rotational acceleration relative to the ground; thus, $\ddot{\phi} = 0$ and $\ddot{\theta} = 0$. Even in this case, the boom *does* have angular acceleration relative to the ground, $\vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{F}} = \dot{\theta}\dot{\phi}\hat{\boldsymbol{\imath}}' + 0\hat{\boldsymbol{\jmath}}' + 0\hat{\boldsymbol{k}}' \neq \vec{\boldsymbol{0}}$. In this special case, the absolute angular acceleration of the boom is due to the changing direction of its angular velocity vector caused by the rotation of the cab at constant rate relative to the ground.

In the 2-D case, if a body $\mathcal{A}$ rotates relative to a fixed frame at constant rate and a frame $\mathcal{B}$ rotates relative to the $\mathcal{A}$ at constant rate, body $\mathcal{B}$ does not have an angular acceleration relative to the fixed frame. That compounding angular velocities does not lead to angular acceleration in planar problems can be explained mathematically as follows: $\vec{\boldsymbol{\omega}}_{\mathcal{A}} = \omega_{\mathcal{A}}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}} = \omega_{\mathcal{B}/\mathcal{A}}\hat{\boldsymbol{k}}$ so that $\vec{\boldsymbol{\omega}}_{\mathcal{A}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}} = \vec{\mathbf{0}}$.

Summary of angular acceleration

Relative angular accelerations do not 'add' (an extra term is needed).

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}}^{\text{extra term}}$$

or, equivalently, because $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}} = \vec{\mathbf{0}}$,

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \underbrace{\vec{\boldsymbol{\omega}}_{\mathcal{C}}}_{\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}}}.$$

SAMPLE 21.1 Velocity and acceleration of a point on a spinning and precessing body. A three-mass symmetric system is supported by a fan-like structure (Fig. 21.7). The three masses spin about the horizontal shaft at a constant rate $\omega = 10\,\text{rad/s}$. The horizontal shaft is rigidly attached to a vertical shaft which rotates about the vertical axis at a constant rate $\Omega = 2\,\text{rad/s}$. At the instant shown the three masses are in the xz-plane and mass P is aligned with the x-axis.

1. Find the absolute velocity of mass P.
2. Find the absolute acceleration of mass P.

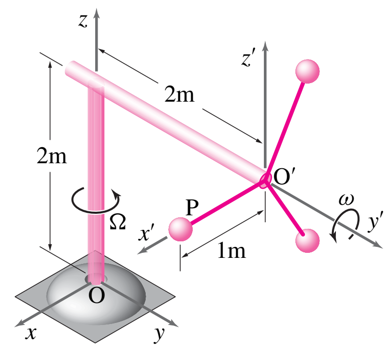


Figure 21.7:

Solution Let us attach a frame $\mathcal{B}$ to the horizontal shaft so that this frame rotates with the horizontal shaft with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \Omega\hat{\boldsymbol{k}}$. For calculations in the rotating frame, let us attach a coordinate system $x'y'z'$ to the shaft at point O'. Since this coordinate system is attached to the rotating frame, its basis vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$ $\hat{\boldsymbol{k}}'$ also rotate with $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$, but at the instant of interest

$$\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}, \quad \hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}.$$

1. **Velocity of P:** The absolute velocity of point P is given by

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \underbrace{\vec{\boldsymbol{v}}_{\text{rel}}}_{\vec{\boldsymbol{v}}_{P/\mathcal{B}}}$$

where P' is a point fixed in the rotating frame and coincides with point P at the instant of interest. To visualize the motion of P', we draw a rigid arm of any shape (usually taking advantage of the given geometry) to the point P' (which is at point P). This imaginary rigid arm rotates with the same angular velocity as the rotating frame $\mathcal{B}$, since P' is a point fixed in the rotating frame. From Fig. 21.8 we see that point P' goes in circles of radius AP' at constant rate Ω. Considering the rigid arm OAP' we can find the velocity of P' as

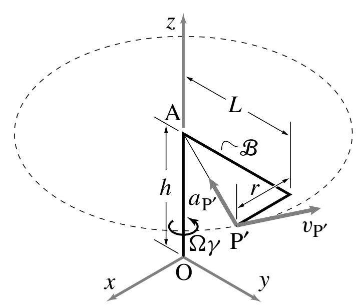


Figure 21.8:

$$\begin{aligned}\vec{\boldsymbol{v}}_{P'} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'} = \Omega\hat{\boldsymbol{k}} \times (h\hat{\boldsymbol{k}} + L\hat{\boldsymbol{\jmath}}' + r\hat{\boldsymbol{\imath}}') \\ &= -\Omega L\hat{\boldsymbol{\imath}}' + \Omega r\hat{\boldsymbol{\jmath}}'.\end{aligned}$$

To find $\vec{\boldsymbol{v}}_{\text{rel}}$, the velocity of point P relative to the rotating frame, we imagine ourselves sitting anywhere in the rotating frame and observe the motion of point P. That is, we forget about the rotation of the frame (if we are sitting in the rotating frame we cannot see the rotation of the frame) and watch the motion of point P as if the horizontal and the vertical shafts were stationary. It is easy to see that with respect to $\mathcal{B}$ the three masses execute circular motion at a constant rate ω. From Fig. 21.9 we see that

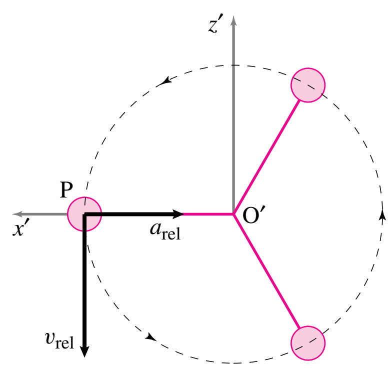


Figure 21.9

$$\begin{aligned}\vec{\boldsymbol{v}}_{\text{rel}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{P/O'} = \omega\hat{\boldsymbol{\jmath}}' \times r\hat{\boldsymbol{\imath}}' \\ &= -\omega r\hat{\boldsymbol{k}}' = -\omega r\hat{\boldsymbol{k}}. \qquad (\text{since } \hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}})\end{aligned}$$

Thus, the absolute velocity of point P is

$$\begin{aligned}\vec{\boldsymbol{v}}_P &= \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}} = -\Omega L\hat{\boldsymbol{\imath}} + \Omega r\hat{\boldsymbol{\jmath}} - \omega r\hat{\boldsymbol{k}} \\ &= -2\,\text{rad/s} \cdot 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{rad/s} \cdot 1\,\text{m}\hat{\boldsymbol{\jmath}} - 10\,\text{rad/s} \cdot 1\,\text{m}\hat{\boldsymbol{k}} \\ &= (-4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}})\,\text{m/s}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_P = (-4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}})\,\text{m/s}}$$

2. **Acceleration of point P:** The absolute acceleration of point P is given by

$$\vec{\boldsymbol{a}}_P = \underbrace{\vec{\boldsymbol{a}}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'}) + \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'}}_{\vec{\boldsymbol{a}}_{P'}} + \underbrace{\vec{\boldsymbol{a}}_{P/\mathcal{B}}}_{\vec{\boldsymbol{a}}_{\text{rel}}} + \underbrace{2\omega_{\mathcal{D}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}}}_{\vec{\boldsymbol{a}}_{\text{cor}}}$$

where $\vec{\boldsymbol{a}}_{P'}$ is the acceleration of point P' discussed above, $\vec{\boldsymbol{a}}_{\text{cor}}$ is the Coriolis acceleration, and $\vec{\boldsymbol{a}}_{\text{rel}} \equiv \vec{\boldsymbol{a}}_{P/\mathcal{B}}$ is the relative acceleration of P with respect to the rotating frame. Now we calculate each term separately. Considering the motion of point P' in Fig. 21.8, we see that

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= \underbrace{\dot{\vec{\omega}}_{\mathcal{B}}}_{\vec{\mathbf{0}}} \times \vec{\boldsymbol{r}}_{P'} + \vec{\omega}_{\mathcal{B}} \times \overbrace{(\vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'})}^{\vec{\boldsymbol{v}}_{P'}} \\
&= \Omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{P'} \\
&= \Omega\hat{\boldsymbol{k}} \times (-\Omega L\hat{\boldsymbol{\imath}} + \Omega r\hat{\boldsymbol{\jmath}}) \\
&= -\Omega^2(L\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{\imath}}).
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{cor}} &= 2\vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\text{rel}} \\
&= 2\Omega\hat{\boldsymbol{k}} \times (-\omega r\hat{\boldsymbol{k}}) \\
&= \vec{\mathbf{0}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{rel}} &= \underbrace{\dot{\vec{\omega}}}_{\vec{\mathbf{0}}} \times \vec{\boldsymbol{r}}_{P/O'} + \vec{\omega} \times \overbrace{(\vec{\omega} \times \vec{\boldsymbol{r}}_{P/O'})}^{-\omega^2\vec{\boldsymbol{r}}_{P/O'}} \\
&= -\omega^2 r\hat{\boldsymbol{\imath}}' \\
&= -\omega^2 r\hat{\boldsymbol{\imath}} \qquad (\text{since } \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}).
\end{aligned}$$

Thus, the absolute acceleration of point P is

$$\begin{aligned}
\vec{\boldsymbol{a}}_P &= \vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{\text{cor}} + \vec{\boldsymbol{a}}_{\text{rel}} \\
&= -\Omega^2(L\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{\imath}}) + \vec{\mathbf{0}} - \omega^2 r\hat{\boldsymbol{\imath}} \\
&= -(\Omega^2 + \omega^2)r\hat{\boldsymbol{\imath}} - \Omega^2 L\hat{\boldsymbol{\jmath}} \\
&= -(4 + 100)(\text{ rad/s})^2 \cdot 1\,\text{m}\hat{\boldsymbol{\imath}} - 4(\text{ rad/s})^2 \cdot 2\,\text{m}\,\hat{\boldsymbol{\jmath}} \\
&= -(104\hat{\boldsymbol{\imath}} + 8\hat{\boldsymbol{\jmath}})\,\text{m/s}^2.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_P = -(104\hat{\boldsymbol{\imath}} + 8\hat{\boldsymbol{\jmath}})\,\text{m/s}^2}$$

SAMPLE 21.2 A student stands on a turntable that rotates about the z-axis at a constant rate $\omega_1 = 2\,\mathrm{rad/s}$. He holds a wheel that rotates with respect to his arm at a constant rate $\omega_2 = 6\,\mathrm{rad/s}$ (see Figure 21.10. At the instant when the student's arm is parallel to the y-axis, find

1. the absolute angular velocity of the wheel,
2. the magnitude of the angular velocity of the wheel, and
3. the axis of instantaneous rotation of the wheel.

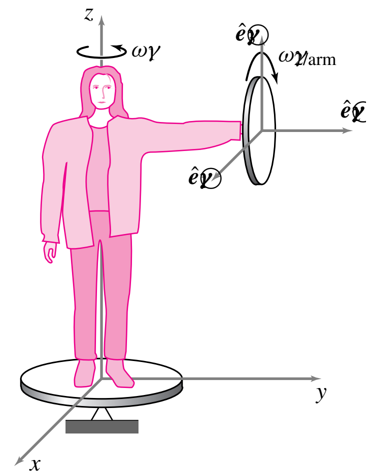


Figure 21.10

Solution Let a coordinate system with basis vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ be attached to the hand of the student. At the instant of interest, these basis vectors are parallel to the fixed basis vectors $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$, respectively, *i.e.,*

$$\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{\jmath}}, \quad \text{and} \quad \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{k}}.$$

1. The absolute angular velocity of the wheel is:

$$\begin{aligned}
\vec{\omega} &= \vec{\omega}_{\mathrm{arm}} + \vec{\omega}_{\mathrm{wheel/arm}} \\
&= \omega_1 \hat{\boldsymbol{k}} + \omega_2(-\hat{\boldsymbol{e}}_2) \\
&= \omega_1 \hat{\boldsymbol{k}} - \omega_2 \hat{\boldsymbol{\jmath}} \\
&= (2\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s}.
\end{aligned}$$

$$\boxed{\vec{\omega} = (2\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s}}$$

2. The magnitude of the absolute angular velocity of the wheel is:

$$\omega = |\vec{\omega}| = \sqrt{4+36}\,\mathrm{rad/s} = 6.32\,\mathrm{rad/s}.$$

$$\boxed{\omega = 6.32\,\mathrm{rad/s}}$$

3. The instantaneous axis of rotation is the direction of the angular velocity $\vec{\omega}$. Let $\hat{\boldsymbol{\lambda}}$, a unit vector in the direction of $\vec{\omega}$, represent the axis of rotation. Then,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} &= \frac{\vec{\omega}}{\omega} = \frac{(2\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s}}{6.32\,\mathrm{rad/s}} \\
&= 0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} = 0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}}}$$

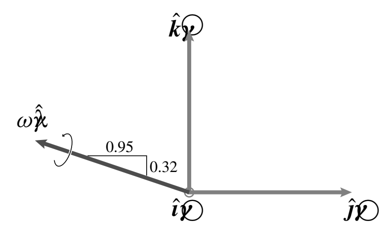


Figure 21.11: The angular velocity of the wheel and the instantaneous axis of rotation.

Comments:

1. After computing any unit vector, you should check that it has a magnitude of 1.
2. $\hat{\boldsymbol{\lambda}} \equiv 0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}}$ is a unit vector along the axis of instantaneous rotation. Therefore, $\vec{\omega}$ can now also be written as

$$\vec{\omega} = \omega\hat{\boldsymbol{\lambda}} = 6.37\,\mathrm{rad/s}(0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}})$$

.

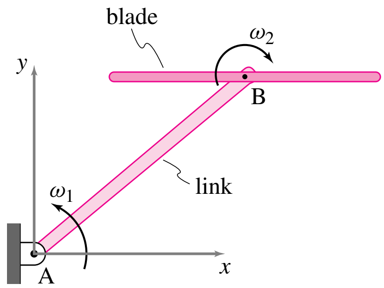


Figure 21.12

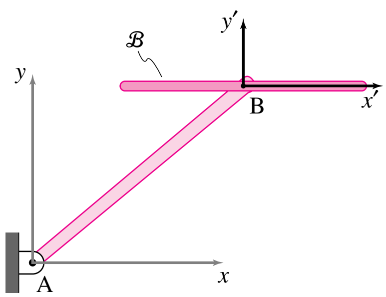


Figure 21.13

SAMPLE 21.3 Windshield wiper of a bus. The windshield wipers of big vehicles such as buses and trucks are usually designed such that the blades of the wipers always keep a fixed angle during the motion of the connecting link. This angle is maintained by making the blade rotate with respect to the link about pin B. At any instant, let the angular velocity of the link be $\vec{\mathbf{\Omega}} = \omega_1 \hat{\boldsymbol{k}}$. What must be the angular velocity of the blade with respect to the link so that the blade stays horizontal?

Solution Intuitively, the answer should be more or less obvious. If the link rotates by $\Delta\theta$ in some time interval Δt, then the blade must rotate, relative to the link, by the same amount $\Delta\theta$ in the opposite direction so that its net rotation is zero and it stays horizontal. Now let us see how we can get the same answer using angular velocities and rotating frames.

Let $\vec{\boldsymbol{\omega}}_2 = \omega_2 \hat{\boldsymbol{k}}$ be the angular velocity of the blade with respect to the link. Let us attach a frame $\mathcal{B}$ to the blades and fix a coordinate axes $x'y'z'$ in this frame. (see Figure 21.13). Then

$$
\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{B}} &= \vec{\boldsymbol{\omega}}_{\text{blade}} = \vec{\boldsymbol{\omega}}_{\text{blade/link}} + \vec{\boldsymbol{\omega}}_{\text{link}} \\
&= \omega_2 \hat{\boldsymbol{k}} + \omega_1 \hat{\boldsymbol{k}} = (\omega_2 + \omega_1)\hat{\boldsymbol{k}}.
\end{aligned}
$$

Since, in our example, the blade remains horizontal, the unit vector $\hat{\boldsymbol{\imath}}'$ does not change its direction, *i.e.*,

$$
\begin{aligned}
\dot{\hat{\boldsymbol{\imath}}}' &= \vec{\mathbf{0}}. \\
\text{But} \quad \dot{\hat{\boldsymbol{\imath}}}' &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}' \\
&= (\omega_1 + \omega_2)\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}' \\
&= (\omega_1 + \omega_2)\hat{\boldsymbol{\jmath}}'.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\omega_1 + \omega_2)\hat{\boldsymbol{\jmath}}' &= \vec{\mathbf{0}}. \\
\Rightarrow \quad \omega_1 + \omega_2 &= 0 \\
\Rightarrow \quad \omega_2 &= -\omega_1.
\end{aligned}
$$

$$\omega_2 = -\omega_1$$

Comments: In general, justification of such simple ideas in such detail would preclude efficient solving of more complex problems. We present it just to show how the formulae *do* ultimately agree with common sense.

SAMPLE 21.4 **Relative rotations in 2-D and 3-D.** In the figures shown below, the rigid arm AB rotates with constant angular speed $\Omega = 5\,\mathrm{rad/s}$ and the disk $\mathcal{D}$ rotates with respect to the arm at constant angular speed $\omega = 10\,\mathrm{rad/s}$. Is the angular acceleration of the disk the same in each case?

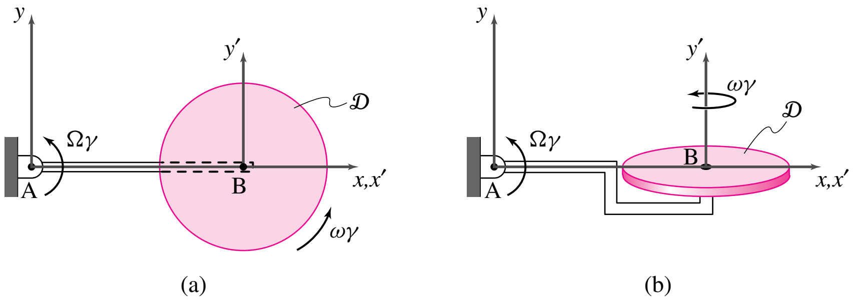


Figure 21.14:

Solution In each case, let us attach a frame $\mathcal{B}$ to the rod and fix coordinate axes $x'y'z'$ in $\mathcal{B}$ with the origin at the center of the disk. Let the primed coordinate axes be parallel to the inertial coordinate axes at the moment of interest.

In case (a):

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \Omega\hat{\boldsymbol{k}} \quad \text{and} \quad \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} = \omega\hat{\boldsymbol{k}}' = \omega\hat{\boldsymbol{k}}.$$

Therefore, the angular acceleration of the disk is

$$\begin{aligned}\vec{\boldsymbol{\alpha}}_{\mathcal{D}} &= \overbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{B}}}^{\vec{\mathbf{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\ &= \Omega\hat{\boldsymbol{k}} \times \omega\hat{\boldsymbol{k}} = \vec{\mathbf{0}}.\end{aligned}$$

In case (b):

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \Omega\hat{\boldsymbol{k}} \quad \text{and} \quad \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} = \omega\hat{\boldsymbol{\jmath}}' = \omega\hat{\boldsymbol{\jmath}}.$$

Therefore, the angular acceleration of the disk is

$$\begin{aligned}\vec{\boldsymbol{\alpha}}_{\mathcal{D}} &= \overbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{B}}}^{\vec{\mathbf{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\ &= \Omega\hat{\boldsymbol{k}} \times \omega\hat{\boldsymbol{\jmath}} = -\Omega\omega\hat{\boldsymbol{\imath}} \\ &= -50\,\mathrm{rad/s^2}\hat{\boldsymbol{\imath}}.\end{aligned}$$

Thus, the angular acceleration of the disk is not the same in each case.

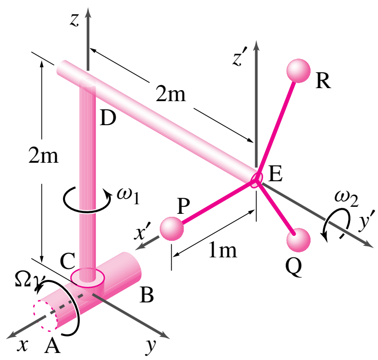


Figure 21.15:

SAMPLE 21.5 Relative and absolute angular velocity and acceleration. A three-mass body is made up of three point masses P, Q, and R, connected by three identical rigid rods at 120^o. The three-mass system is attached to shaft CDE and spins with respect to the shaft at a non-constant rate ω_2. Shaft CDE rotates with a constant rate ω_1 with respect to the base sleeve which, in turn, rotates with the base shaft AB with constant angular speed Ω. At the instant shown, $\omega_2 = 10\,\text{rad/s}$ and is changing at the rate of $\dot{\omega}_2 = 2\,\text{rad/s}^2$, $\omega_1 = 5\,\text{rad/s}$ (constant), and $\Omega = 2\,\text{rad/s}$ (constant).

1. Find the angular velocity of the three-mass body PQR with respect to the base shaft AB.
2. Find the absolute angular velocity of the body PQR.
3. Find the angular acceleration of the body PQR relative to the base shaft AB.
4. Find the absolute angular acceleration of the body PQR.

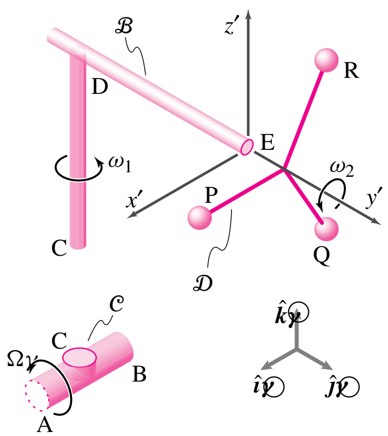


Figure 21.16: Frame $\mathcal{B}$ is attached to the shaft CDE and frame $\mathcal{C}$ to the base shaft AB. $\mathcal{D}$ represents the three mass system. $\mathcal{D}$ rotates with respect to $\mathcal{B}$, $\mathcal{B}$ rotates with respect to $\mathcal{C}$, and $\mathcal{C}$ rotates with respect to the fixed frame.

Solution Let us attach a frame $\mathcal{B}$ to the shaft CDE and a frame $\mathcal{C}$ to the base shaft AB. Thus frame $\mathcal{B}$ rotates with respect to frame $\mathcal{C}$ with constant speed ω_1 or at the instant shown,

$$
\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} &= \omega_1 \hat{\boldsymbol{k}} \quad \text{and} \\
\vec{\boldsymbol{\omega}}_{\mathcal{C}} &= \Omega \hat{\boldsymbol{\imath}}.
\end{aligned}
$$

Let the symbol $\mathcal{D}$ denote the three-mass rigid body PQR. Also, let the coordinate axes $x'y'z'$ be attached to the frame $\mathcal{D}$ at point E.

1. The angular velocity of $\mathcal{D}$ with respect to frame $\mathcal{C}$ is

$$
\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \\
&= \omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}} \\
&= (10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}})\,\text{rad/s}
\end{aligned}
$$

where the last line follows from the fact that at the instant of interest the primed coordinate axes x', y', and z', glued to the frame $\mathcal{B}$, are parallel to the fixed coordinates x, y, and z.

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} = (10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}})\,\text{rad/s}$$

2. The absolute angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{D}} \equiv \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}}$ where $\mathcal{F}$ denotes the fixed frame, can be written as

$$
\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{D}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}} \\
&= \omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}} + \Omega \hat{\boldsymbol{\imath}} \\
&= (10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} + 2\hat{\boldsymbol{\imath}})\,\text{rad/s}.
\end{aligned}
$$

$$\vec{\boldsymbol{\omega}}_{\mathcal{D}} = (2\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}})\,\text{rad/s}$$

3. Now we calculate the angular acceleration. We have to be extra careful in calculating angular accelerations relative to the intermediate frames. Explicit notations for the time derivatives taken in different frames usually help.

The angular acceleration of $\mathcal{D}$ relative to $\mathcal{C}$ may be calculated as follows.

$$
\begin{aligned}
\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}} &= \overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{C}} \\
&= \overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{B}/\mathcal{C}} \\
&= \underbrace{\overset{\mathcal{B}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}}}_{\overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}}} \quad + \quad \vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{C}} \\
&= \overset{\mathcal{B}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \omega_1 \hat{\boldsymbol{k}} \times \omega_2 \hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{C}} \\
&= \dot{\omega}_2 \hat{\boldsymbol{\jmath}} - \omega_1 \omega_2 \hat{\boldsymbol{\imath}} + \underbrace{\dot{\omega}_1}_{0} \hat{\boldsymbol{k}} \\
&= 2\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\jmath}} - 10\,\mathrm{rad/s} \cdot 5\,\mathrm{rad/s}\,\hat{\boldsymbol{\imath}} \\
&= (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s^2}.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}} = (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s^2}}$$

4. The absolute angular acceleration $\vec{\boldsymbol{\alpha}}_{\mathcal{D}}$ of the body can be found in a similar way by carrying the time derivative of the absolute angular velocity in the fixed frame. In the following calculations, all angular quantities without explicit reference to a frame stand for quantities with respect to the fixed frame $\mathcal{F}$.

$$
\begin{aligned}
\vec{\boldsymbol{\alpha}}_{\mathcal{D}} &= \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}} \\
&= \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{B}/\mathcal{C}} + \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{C}} \\
&= (\overset{\mathcal{B}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}}) + (\overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{B}/\mathcal{C}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}}) + \underbrace{\dot{\Omega}}_{0} \hat{\boldsymbol{\imath}} \\
&= (\dot{\omega}_2 \hat{\boldsymbol{\jmath}} + (\omega_1 \hat{\boldsymbol{k}} + \Omega \hat{\boldsymbol{\imath}}) \times \omega_2 \hat{\boldsymbol{\jmath}}) + (\underbrace{\dot{\omega}_1}_{0} \hat{\boldsymbol{k}} + \Omega \hat{\boldsymbol{\imath}} \times \omega_1 \hat{\boldsymbol{k}}) \\
&= (\dot{\omega}_2 \hat{\boldsymbol{\jmath}} - \omega_1 \omega_2 \hat{\boldsymbol{\imath}} + \Omega \omega_2 \hat{\boldsymbol{k}}) + (-\Omega \omega_1 \hat{\boldsymbol{\jmath}}) \\
&= (2\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\jmath}} - 50\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\imath}} + 20\,\mathrm{rad/s^2}\,\hat{\boldsymbol{k}}) - 10\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\jmath}} \\
&= (-50\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 20\hat{\boldsymbol{k}})\,\mathrm{rad/s^2}.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{\alpha}}_{\mathcal{D}} = (-50\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 20\hat{\boldsymbol{k}})\,\mathrm{rad/s^2}}$$

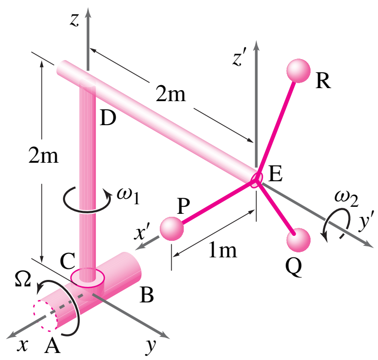


Figure 21.17:

SAMPLE 21.6 Relative and absolute angular velocity and acceleration. Consider Sample 21.5 again: A three-mass body consists of three point masses P, Q, and R, connected by three identical rigid rods 120º apart. The three-mass system is attached to shaft CDE and spins with respect to the shaft at a non-constant rate ω_2. Shaft CDE rotates with a constant rate ω_1 with respect to the base sleeve which, in turn, rotates with the base shaft AB with constant angular speed Ω. At the instant shown, $\omega_2 = 10\,\text{rad/s}$ and is changing at the rate of $\dot{\omega}_2 = 2\,\text{rad/s}^2$; $\omega_1 = 5\,\text{rad/s}$ (constant), and $\Omega = 2\,\text{rad/s}$ (constant).

1. Find the angular acceleration of the body relative to the base shaft AB.
2. Find the absolute angular acceleration of the body.

Solution Let $\mathcal{D}$ represent the rotating mass system. Let us attach frame $\mathcal{B}$ to the shaft CDE and frame $\mathcal{C}$ to the shaft AB. Coordinate axes $x'y'z'$ are fixed in $\mathcal{B}$ and rotate with shaft CDE.

1. The angular velocity of $\mathcal{D}$ with respect to shaft AB (or frame $\mathcal{C}$) is:

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \\ &= \omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}}'.\end{aligned}$$

Therefore, the angular acceleration of $\mathcal{D}$ with respect to shaft AB is:

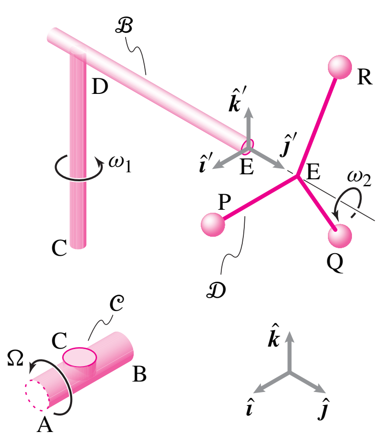


Figure 21.18:

$$\begin{aligned}{}^{\mathcal{C}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{C}} &= \text{time derivative of } \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} \text{ in frame } \mathcal{C}. \\ &= \frac{{}^{\mathcal{C}}d}{dt}[\omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}}'] \\ &= \dot{\omega}_2 \hat{\boldsymbol{\jmath}}' + \omega_2 \dot{\hat{\boldsymbol{\jmath}}}' + \underbrace{\dot{\omega}_1}_{0} \hat{\boldsymbol{k}}' + \omega_1 \dot{\hat{\boldsymbol{k}}} \\ &= \dot{\omega}_2 \hat{\boldsymbol{\jmath}}' + \omega_2(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') + \omega_1(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \times \hat{\boldsymbol{k}}') \\ &= \dot{\omega}_2 \hat{\boldsymbol{\jmath}}' + \omega_2(\omega_1 \underbrace{\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}'}_{-\hat{\boldsymbol{\imath}}'}) + \omega_1(\omega_1 \underbrace{\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}'}_{0}) \\ &= -\omega_1\omega_2 \hat{\boldsymbol{\imath}}' + \dot{\omega}_2 \hat{\boldsymbol{\jmath}}' \\ &= (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{rad/s}^2\end{aligned}$$

since $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ at the instant of interest.

$$\boxed{{}^{\mathcal{C}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{C}} = (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{rad/s}^2}$$

2. The absolute angular velocity of $\mathcal{D}$ is

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} \\ &= \underbrace{\omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}}'}_{\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}}} + \Omega\hat{\boldsymbol{\imath}}.\end{aligned}$$

Therefore, the absolute angular acceleration of $\mathcal{D}$ is

$$\begin{aligned}{}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{F}} &= {}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{C}} + {}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}/\mathcal{F}} \\ &= \frac{{}^{\mathcal{F}}d}{dt}[\omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}}'] + \frac{{}^{\mathcal{F}}d}{dt}[\Omega\hat{\boldsymbol{\imath}}] \\ &= \dot{\omega}_2 \hat{\boldsymbol{\jmath}}' + \omega_2 \dot{\hat{\boldsymbol{\jmath}}}' + \underbrace{\dot{\omega}_1}_{0} \hat{\boldsymbol{k}}' + \omega_1 \dot{\hat{\boldsymbol{k}}}' + \underbrace{\dot{\Omega}}_{0}\,\hat{\boldsymbol{\imath}} \\ &= \dot{\omega}_2 \hat{\boldsymbol{\jmath}}' + \omega_2(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \hat{\boldsymbol{\jmath}}') + \omega_1(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \hat{\boldsymbol{k}}').\end{aligned}$$

But

$$\begin{aligned}\vec{\omega}_{\mathcal{B}/\mathcal{F}} &= \vec{\omega}_{\mathcal{C}/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{C}} \\ &= \Omega\hat{\boldsymbol{\imath}} + \omega_1\hat{\boldsymbol{k}}.\end{aligned}$$

Therefore,

$$\begin{aligned}{}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{D}/\mathcal{F}} &= \dot{\omega}_2\hat{\boldsymbol{\jmath}}' + \omega_2([\Omega\hat{\boldsymbol{\imath}} + \omega_1\hat{\boldsymbol{k}}] \times \hat{\boldsymbol{\jmath}}') + \omega_1([\Omega_1 + \omega_1\hat{\boldsymbol{k}}] \times \hat{\boldsymbol{k}}') \\ &= \dot{\omega}_2\hat{\boldsymbol{\jmath}} + \omega_2\Omega\hat{\boldsymbol{k}} - \omega_1\omega_2\hat{\boldsymbol{\imath}} - \omega_1\Omega\hat{\boldsymbol{\jmath}} \\ &= -\omega_1\omega_2\hat{\boldsymbol{\imath}} + (\dot{\omega}_2 - \omega_1\Omega)\hat{\boldsymbol{\jmath}} + \omega_2\Omega\hat{\boldsymbol{k}} \\ &= (-50\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 20\hat{\boldsymbol{k}})\,\mathrm{rad/s^2}\end{aligned}$$

$$\boxed{\vec{\omega}_{\mathcal{D}/\mathcal{F}} = (-50\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 20\hat{\boldsymbol{k}})\,\mathrm{rad/s^2}}$$

Comments: Again, this approach based on direct differentiation, agrees with the result obtained by quoting the formula for general motion.

21.1 Velocity and acceleration of points on a rigid body: $\vec{\omega}$ and $\vec{\alpha}$

21.1.1 A particle moves on a helix. Say you know that a particle moves according to the equation

$$\vec{r} = R\cos\theta\hat{\imath} + R\sin\theta\hat{\jmath} + z\hat{k}$$

where R is a constant, $\theta = c_1 t$, and $z = c_2 t$.

a) Find $\vec{v}$ at general time t.

b) Find $\vec{a}$ at general time t.

c) Find $\hat{e}_t$ at general time t.

d) Find $\hat{e}_n$ at general time t.

e) Write $\vec{a}$ in terms of $\hat{e}_t$ and $\hat{e}_n$.

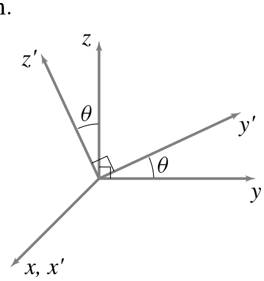
f) What is the radius of the osculating circle? Check your answer to see if it makes sense in the special case when $c_2 = 0$.

21.1.2 A curious chef tosses a potato in the air. Being a part-time dynamicist, she has just the right instruments in her kitchen and measures the absolute angular velocity of the potato to be $\vec{\omega} = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}(\text{rad/s})$ and the absolute center of mass velocity to be $\vec{v}_G = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}(\text{in/s})$, at a particular instant in time. At the same instant in time, she notes the position of an eye on the potato relative to its center of mass, point G, is $\vec{r}_{E/G} = 1 \in \hat{\imath}$. (The high quality potato came from the market with the center of mass already marked inside it.) $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are basis vectors in the kitchen frame.

a) What is the velocity of the potato eye relative to G, $\vec{v}_{E/G}$?

b) What is the absolute velocity of the potato eye, $\vec{v}_E$?

21.1.3 Find the rotation matrix $[\mathbf{R}]$ for $\theta = 30^\circ$ such that $\{\hat{e}'\} = [\mathbf{R}]\{\hat{e}\}$, where $\hat{e} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. Using the rotation matrix, find the components of $\vec{v} = 2\,\text{m/s}\hat{\imath} - 3\,\text{m/s}\hat{\jmath} + 1\,\text{m/s}\hat{k}$ in the rotated coordinate system.



Problem 21.1.3

21.1.4 A circular plate rotates at a constant angular velocity $\vec{\omega} = 5\,\text{rad/s}\hat{k}$. A set of coordinate axes $x'y'z'$ is glued to the plate at its center and thus rotates with $\vec{\omega}$. Find the rate of change of basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$ using the $\dot{\vec{Q}}$ formula.

21.1.5 A rigid body $\mathcal{B}$ rotates in space with angular velocity $\vec{\omega} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k})\,\text{rad/s}$. A set of local coordinate axes $x'y'z'$ is fixed to the body at its center of mass. The position vector $\vec{r}_P$ of a point P is given in local coordinates: $\vec{r}_P = (5\hat{\imath}' + \hat{\jmath}')\,\text{m}$. Find the velocity ($\dot{\vec{r}}_P$) of point P using the $\dot{\vec{Q}}$ formula.

21.1.6 The rotation matrix between two coordinate systems with basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is $\mathbf{Q}$ such that $\{\hat{e}\} = [\mathbf{Q}]\{E\}$. Find the components of $\vec{a} = (-2\hat{e}_1 + 3\hat{e}_3)\,\text{m/s}^2$ in the $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ basis if $\vec{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

21.1.7 Let $\mathcal{B}$ and $\mathcal{F}$ be two moving rigid bodies. Show that ${}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} = {}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}}$.

21.1.8 George is driving his car on the highway on a rainy day. His windshield wipers are on. He notices that a drop of water on the wiper blade is moving along the blade at approximately 3 in/s. Although the wipers rotate at variable angular speed, he approximates that when the wiper is parallel to his nose, its angular speed is $\pi/2$ rad/s. In George's coordinate system (attached to his frame of reference and moving with him), the wiper is rotating in the positive $\hat{\jmath}$ direction and the water-drop is moving in the positive $\hat{k}$ direction at the instant of interest. Find the velocity of the water-drop in George's frame of reference.

21.1.9 A person sits on a bar stool which spins at angular speed Ω. Simultaneously, (s)he hoists a beer glass of mass m up at angular speed ω relative to the bar stool with a straight, rigid arm of length L. θ measures the angle of her/his arm with respect to the XY plane. XYZ is an inertial coordinate system; YZ lies in the plane of the paper. If you use moving axes to do this problem, specify clearly his/her orientation and motion.

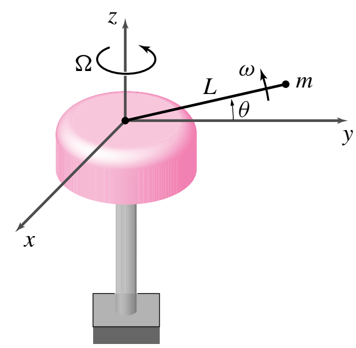
a) What is the angular velocity of the person's arm?

b) What is the angular acceleration of the person's arm?

c) What is the linear velocity of the beer glass relative to inertial space?

d) What is the linear acceleration of the beer glass relative to inertial space?

e) Describe how you would point an absolutely full glass such that the beer will not spill out.



Problem 21.1.9

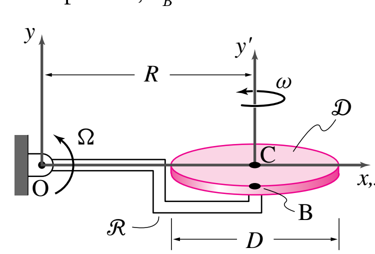
21.1.10 Kinematics of a disk spinning on a spinning rod. The rigid rod $\mathcal{R}$ rotates at constant rate Ω about the z axis. Attached to this rod, a distance R from the origin, is the center C of a disk $\mathcal{C}$ with diameter D. The disk is rotating at constant rate ω about an axis y' that is attached to, and rotates with, the rod. A point B is on the outer edge of the disk. At the instant of interest, the system is in the configuration shown with $\vec{r}_{CB} = (D/2)\hat{k}$.

a) What is the absolute angular velocity of the disk, $\vec{\omega}_{\mathcal{D}}$? *

b) What is the absolute angular acceleration of the disk $\vec{\alpha}_{\mathcal{D}}$? *

c) What is the absolute velocity of point B, $\vec{v}_B$? *

d) What is the absolute acceleration of point B, $\vec{a}_B$? *



Problem 21.1.10

21.1.11
See also problems 21.1.17, 21.1.19, and 21.2.16. Body $\mathcal{A}$ rotates with respect to a Newtonian frame $\mathcal{F}$ at a constant angular rate ω_0 about axis OO'. Body $\mathcal{B}$, a 'fork', rotates with respect to body $\mathcal{A}$ at a constant angular rate ω_1 about axis PP'. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass. Body $\mathcal{D}$, a uniform disc of mass m, rotates with respect to body $\mathcal{B}$, at a constant angular rate ω_2 about axis CC'. Each of the bodies is turned at constant rate by motors(not shown.) At the instant shown, find the absolute angular velocity of body $\mathcal{D}$, $\vec{\omega}_{\mathcal{D}/\mathcal{F}}$. *

Problem 21.1.11

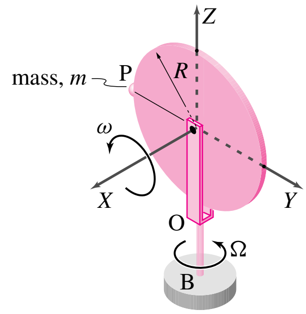
21.1.12 A particle P of mass m is attached to the edge of the disk of radius R, as shown in the figure. The disk spins about its center line with constant angular speed ω rad/s relative to the shaft OB. The shaft OB rotates with angular speed Ω rad/s. For the instant when the disk lies in the YZ plane, and the particle is on the Y axis, as shown:

a) Find the particle's velocity relative to XYZ.

b) Calculate the particle's acceleration relative to XYZ.

c) Calculate the force $\vec{F}$ acting on the particle.

If you choose to use another coordinate system, define it explicitly.



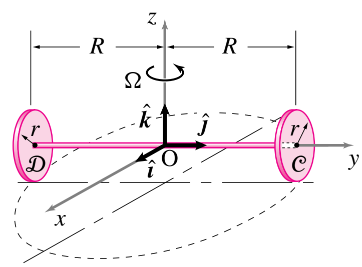
xyz: fixed frame

Problem 21.1.12

21.1.13 See also problem 21.2.31. Two identical thin disks, bodies $\mathcal{C}$ and $\mathcal{D}$, of radius r are connected (perpendicularly) to opposite ends of a thin axle, as depicted below. The axle rotates about the fixed Z axis at a constant angular speed Ω, and the disks roll without slipping. Determine:

1. The total angular velocity of each disk, $\vec{\boldsymbol{\omega}}_{\mathcal{C}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$, and
2. The total angular acceleration of each disk, $\boldsymbol{\alpha}_{\mathcal{C}}$ and $\boldsymbol{\alpha}_{\mathcal{D}}$.

(Express your answers relative to the rotating basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$.)



Problem 21.1.13

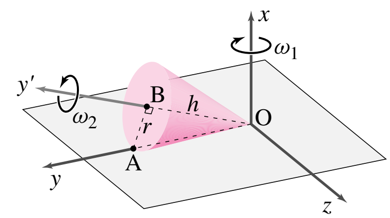
21.1.14 Kinematics of a rolling cone. See also problem 20.1.20. A solid cone with circular-base radius r and height h moves as follows. Point O is fixed in space. The cone surface always contacts the plane but may roll and/or slide. The axis of the cone OB rotates about the vertical axis with constant angular velocity $-\omega_1\hat{\boldsymbol{\imath}}$. The cone spins about this moving axis $\hat{\boldsymbol{\jmath}}'$ with constant angular rate ω_2.

a) At the instant shown what is the velocity of point B? (in terms of some or all of ω_1, ω_2, r, h, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$) *

b) At the instant shown what is the angular acceleration of the cone? (in terms of some or all of ω_1, ω_2, r, h, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$) *

c) If the disk rolls with no slip, find ω_1 in terms of ω_2, r and h. *

d) If the disk rolls with no slip and point B has constant speed v find the magnitude of the angular acceleration of the cone in terms of v, r and h. *

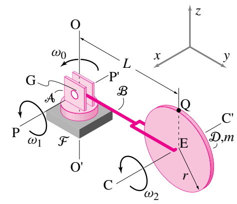


Problem 21.1.14: **Cone rolling on a plane**

21.1.15 See also problems 21.1.16, 21.1.18, 21.2.15, and 21.2.14. Body $\mathcal{A}$ rotates with respect to a Newtonian frame $\mathcal{F}$ at a constant angular rate ω_0 about axis OO'. Body $\mathcal{B}$, a 'fork', rotates with respect to body $\mathcal{A}$ at a constant angular rate ω_1 about axis PP'. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass. Body $\mathcal{D}$, a uniform disc of mass m, rotates with respect to body $\mathcal{B}$, at a constant angular rate ω_2 about axis CC'. Each of the bodies is turned at constant rate by motors(not shown.) At the instant shown, find the absolute velocity and acceleration of point Q on the disk.

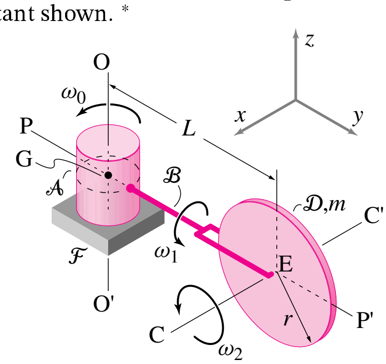
Problem 21.1.15

21.1.16 (See also problems 21.1.18, 21.2.15, and 21.2.14.) For the system in problem 21.1.15, find the absolute angular acceleration of frame $\mathcal{D}$, ${}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{D}}$, at the instant shown.



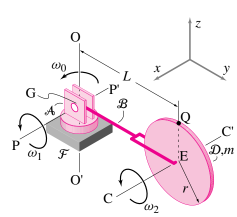
Problem 21.1.16

21.1.17 (See also problems 21.1.19, 21.2.16, and 21.2.7.) For the system in problem 21.1.11, find the absolute angular acceleration of frame $\mathcal{D}$, ${}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{D}}$, at the instant shown. *



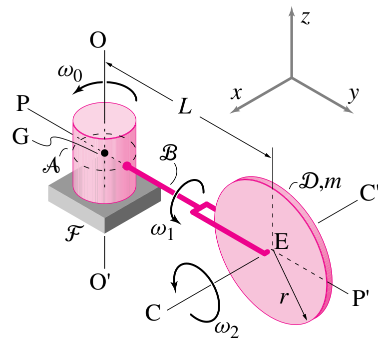
Problem 21.1.17

21.1.18 (See also problems 21.1.16, 21.2.15, and 21.2.14.) For the configuration in problem 21.1.15, find the velocity and acceleration of point Q on the disk using the alternative method.



Problem 21.1.18

21.1.19 (See also problems 21.1.17, 21.2.16, and 21.2.7.) For the configuration in problem 21.1.11, find the velocity and acceleration of the point on the edge of the disk directly above its center point E using the alternative method.



Problem 21.1.19

21.1.20 To simulate the flight conditions of a space vehicle, engineers have developed the centrifuge, shown diagrammatically in the figure. A main truss arm of length $\ell = 15\,\text{m}$, rotates about the AA axis. The pilot sits in the capsule which may rotate about axis CC. The seat for the pilot may rotate inside the capsule about an axis perpendicular to the page and going through the point B. These rotations are controlled by a computer that is set to simulate certain maneuvers corresponding to the entry and exit from the earth's atmosphere, malfunctions of the control system, etc. When a pilot sits in the capsule, his head, particularly his ears, has the position shown in the enlarged figure, a distance $r = 1\,\text{m}$ from point B.

The main arm is rotating at $\Omega = 10\,\text{rpm}$ and accelerating at $\dot{\Omega} = 5\,\text{rev}/\text{min}^2$, the capsule is rotating at a constant speed about CC at $\omega_c = 10\,\text{rpm}$, and the seat rotates at a speed of $\omega_s = 5\,\text{rpm}$ inside the capsule.

a) Determine the acceleration, in terms of how many g's, that the pilot's head is subjected to by choosing a moving coordinates system xyz centered at B and fixed to the capsule.

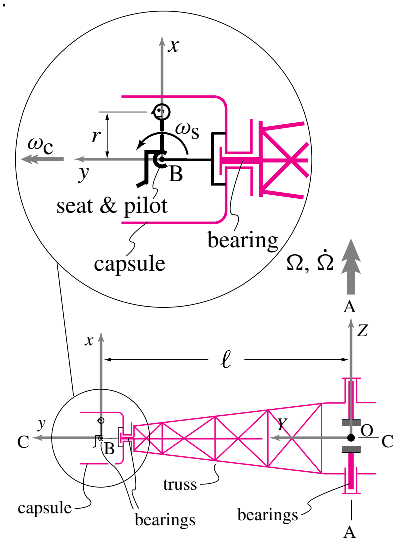
b) Repeat the calculations of (A) by letting xyz be fixed to the seat. *

c) Choose another moving coordinates system to solve the same problem.

d) A pilot has different tolerances for acceleration components. They are roughly:

   1. In a vertical direction, i.e., toe to head – 5 g's. (Here the pilot experiences blackout or may pass out completely.)
   2. Front to rear – 1.5 g's. (Here vision becomes greatly distorted.)

Arrange a test program on the centrifuge so that each acceleration is reached separately while the other accelerations are kept below one- half their tolerance levels.



Problem 21.1.20