SAMPLE 21.1 Velocity and acceleration of a point on a spinning and precessing body. A three-mass symmetric system is supported by a fan-like structure (Fig. 21.7). The three masses spin about the horizontal shaft at a constant rate $\omega = 10\,\text{rad/s}$. The horizontal shaft is rigidly attached to a vertical shaft which rotates about the vertical axis at a constant rate $\Omega = 2\,\text{rad/s}$. At the instant shown the three masses are in the xz-plane and mass P is aligned with the x-axis.

1. Find the absolute velocity of mass P.
2. Find the absolute acceleration of mass P.

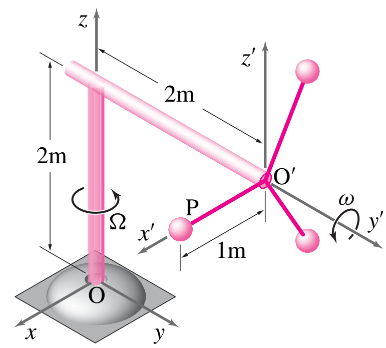

Figure 21.7:

Solution Let us attach a frame $\mathcal{B}$ to the horizontal shaft so that this frame rotates with the horizontal shaft with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \Omega\hat{\boldsymbol{k}}$. For calculations in the rotating frame, let us attach a coordinate system $x'y'z'$ to the shaft at point O'. Since this coordinate system is attached to the rotating frame, its basis vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$ $\hat{\boldsymbol{k}}'$ also rotate with $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$, but at the instant of interest

$$\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}, \quad \hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}.$$

1. **Velocity of P:** The absolute velocity of point P is given by

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \underbrace{\vec{\boldsymbol{v}}_{\text{rel}}}_{\vec{\boldsymbol{v}}_{P/\mathcal{B}}}$$

where P' is a point fixed in the rotating frame and coincides with point P at the instant of interest. To visualize the motion of P', we draw a rigid arm of any shape (usually taking advantage of the given geometry) to the point P' (which is at point P). This imaginary rigid arm rotates with the same angular velocity as the rotating frame $\mathcal{B}$, since P' is a point fixed in the rotating frame. From Fig. 21.8 we see that point P' goes in circles of radius AP' at constant rate Ω. Considering the rigid arm OAP' we can find the velocity of P' as

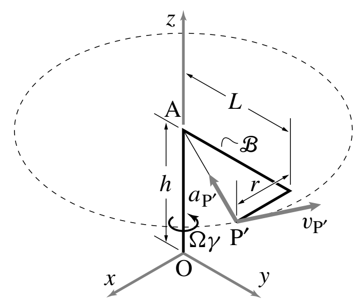

Figure 21.8:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{P'} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'} = \Omega\hat{\boldsymbol{k}} \times (h\hat{\boldsymbol{k}} + L\hat{\boldsymbol{\jmath}}' + r\hat{\boldsymbol{\imath}}') \\
&= -\Omega L\hat{\boldsymbol{\imath}}' + \Omega r\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

To find $\vec{\boldsymbol{v}}_{\text{rel}}$, the velocity of point P relative to the rotating frame, we imagine ourselves sitting anywhere in the rotating frame and observe the motion of point P. That is, we forget about the rotation of the frame (if we are sitting in the rotating frame we cannot see the rotation of the frame) and watch the motion of point P as if the horizontal and the vertical shafts were stationary. It is easy to see that with respect to $\mathcal{B}$ the three masses execute circular motion at a constant rate ω. From Fig. 21.9 we see that

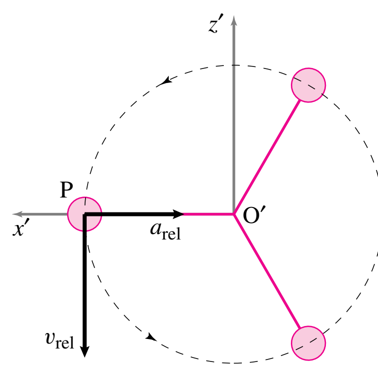

Figure 21.9

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\text{rel}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{P/O'} = \omega\hat{\boldsymbol{\jmath}}' \times r\hat{\boldsymbol{\imath}}' \\
&= -\omega r\hat{\boldsymbol{k}}' = -\omega r\hat{\boldsymbol{k}}. \qquad (\text{since } \hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}})
\end{aligned}$$

Thus, the absolute velocity of point P is

$$\begin{aligned}
\vec{\boldsymbol{v}}_P &= \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}} = -\Omega L\hat{\boldsymbol{\imath}} + \Omega r\hat{\boldsymbol{\jmath}} - \omega r\hat{\boldsymbol{k}} \\
&= -2\,\text{rad/s}\cdot 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{rad/s}\cdot 1\,\text{m}\hat{\boldsymbol{\jmath}} - 10\,\text{rad/s}\cdot 1\,\text{m}\hat{\boldsymbol{k}} \\
&= (-4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}})\,\text{m/s}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_P = (-4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}})\,\text{m/s}}$$

2. **Acceleration of point P:** The absolute acceleration of point P is given by

$$\vec{\boldsymbol{a}}_P = \underbrace{\vec{\boldsymbol{a}}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'}) + \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'}}_{\vec{\boldsymbol{a}}_{P'}} + \underbrace{\vec{\boldsymbol{a}}_{P/\mathcal{B}}}_{\vec{\boldsymbol{a}}_{\text{rel}}} + \underbrace{2\omega_{\mathcal{D}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}}}_{\vec{\boldsymbol{a}}_{\text{cor}}}$$

where $\vec{\boldsymbol{a}}_{P'}$ is the acceleration of point P' discussed above, $\vec{\boldsymbol{a}}_{\text{cor}}$ is the Coriolis acceleration, and $\vec{\boldsymbol{a}}_{\text{rel}} \equiv \vec{\boldsymbol{a}}_{P/\mathcal{B}}$ is the relative acceleration of P with respect to the rotating frame. Now we calculate each term separately. Considering the motion of point P' in Fig. 21.8, we see that

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= \underbrace{\dot{\vec{\omega}}_{\mathcal{B}}}_{\vec{0}} \times \vec{\boldsymbol{r}}_{P'} + \vec{\omega}_{\mathcal{B}} \times \overbrace{(\vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'})}^{\vec{\boldsymbol{v}}_{P'}} \\
&= \Omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{P'} \\
&= \Omega\hat{\boldsymbol{k}} \times (-\Omega L\hat{\boldsymbol{\imath}} + \Omega r\,\hat{\boldsymbol{\jmath}}) \\
&= -\Omega^2(L\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{\imath}}).
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{cor}} &= 2\vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\text{rel}} \\
&= 2\Omega\hat{\boldsymbol{k}} \times (-\omega r\hat{\boldsymbol{k}}) \\
&= \vec{\boldsymbol{0}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{rel}} &= \underbrace{\dot{\vec{\omega}}}_{\vec{0}} \times \vec{\boldsymbol{r}}_{P/O'} + \vec{\omega} \times \overbrace{(\vec{\omega} \times \vec{\boldsymbol{r}}_{P/O'})}^{-\omega^2\vec{\boldsymbol{r}}_{P/O'}} \\
&= -\omega^2 r\hat{\boldsymbol{\imath}}' \\
&= -\omega^2 r\hat{\boldsymbol{\imath}} \qquad (\text{since } \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}).
\end{aligned}$$

Thus, the absolute acceleration of point P is

$$\begin{aligned}
\vec{\boldsymbol{a}}_P &= \vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{\text{cor}} + \vec{\boldsymbol{a}}_{\text{rel}} \\
&= -\Omega^2(L\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{\imath}}) + \vec{\boldsymbol{0}} - \omega^2 r\hat{\boldsymbol{\imath}} \\
&= -(\Omega^2 + \omega^2)r\hat{\boldsymbol{\imath}} - \Omega^2 L\hat{\boldsymbol{\jmath}} \\
&= -(4 + 100)(\text{ rad/s})^2 \cdot 1\,\text{m}\hat{\boldsymbol{\imath}} - 4(\text{ rad/s})^2 \cdot 2\,\text{m}\,\hat{\boldsymbol{\jmath}} \\
&= -(104\hat{\boldsymbol{\imath}} + 8\,\hat{\boldsymbol{\jmath}})\,\text{m/s}^2.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_P = -(104\hat{\boldsymbol{\imath}} + 8\,\hat{\boldsymbol{\jmath}})\,\text{m/s}^2}$$

SAMPLE 21.2 A student stands on a turntable that rotates about the z-axis at a constant rate $\omega_1 = 2\,\text{rad/s}$. He holds a wheel that rotates with respect to his arm at a constant rate $\omega_2 = 6\,\text{rad/s}$ (see Figure 21.10. At the instant when the student's arm is parallel to the y-axis, find

1. the absolute angular velocity of the wheel,
2. the magnitude of the angular velocity of the wheel, and
3. the axis of instantaneous rotation of the wheel.

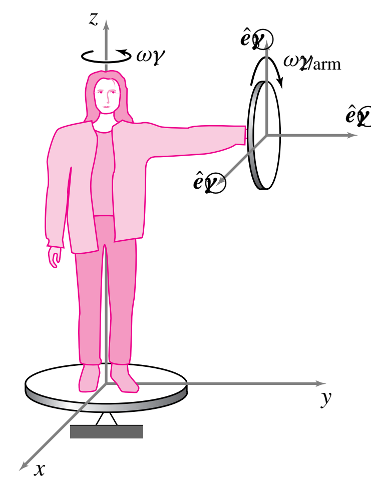

Figure 21.10

Solution Let a coordinate system with basis vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ be attached to the hand of the student. At the instant of interest, these basis vectors are parallel to the fixed basis vectors $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$, respectively, *i.e.*,

$$\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{\jmath}}, \quad \text{and} \quad \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{k}}.$$

1. The absolute angular velocity of the wheel is:

$$\begin{aligned} \vec{\omega} &= \vec{\omega}_{\text{arm}} + \vec{\omega}_{\text{wheel/arm}} \\ &= \omega_1 \hat{\boldsymbol{k}} + \omega_2(-\hat{\boldsymbol{e}}_2) \\ &= \omega_1 \hat{\boldsymbol{k}} - \omega_2 \hat{\boldsymbol{\jmath}} \\ &= (2\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{\jmath}})\,\text{rad/s}. \end{aligned}$$

$$\boxed{\vec{\omega} = (2\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{\jmath}})\,\text{rad/s}}$$

2. The magnitude of the absolute angular velocity of the wheel is:

$$\omega = |\vec{\omega}| = \sqrt{4+36}\,\text{rad/s} = 6.32\,\text{rad/s}.$$

$$\boxed{\omega = 6.32\,\text{rad/s}}$$

3. The instantaneous axis of rotation is the direction of the angular velocity $\vec{\omega}$. Let $\hat{\boldsymbol{\lambda}}$, a unit vector in the direction of $\vec{\omega}$, represent the axis of rotation. Then,

$$\begin{aligned} \hat{\boldsymbol{\lambda}} &= \frac{\vec{\omega}}{\omega} = \frac{(2\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{\jmath}})\,\text{rad/s}}{6.32\,\text{rad/s}} \\ &= 0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}}. \end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} = 0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}}}$$

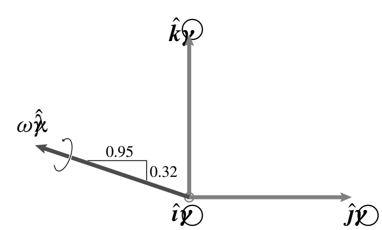

Figure 21.11: The angular velocity of the wheel and the instantaneous axis of rotation.

Comments:

1. After computing any unit vector, you should check that it has a magnitude of 1.
2. $\hat{\boldsymbol{\lambda}} \equiv 0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}}$ is a unit vector along the axis of instantaneous rotation. Therefore, $\vec{\omega}$ can now also be written as

$$\vec{\omega} = \omega\hat{\boldsymbol{\lambda}} = 6.37\,\text{rad/s}(0.32\hat{\boldsymbol{k}} - 0.95\hat{\boldsymbol{\jmath}})$$

.

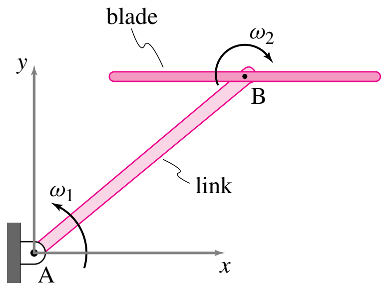

Figure 21.12

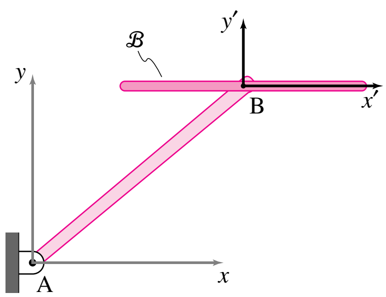

Figure 21.13

SAMPLE 21.3 **Windshield wiper of a bus.** The windshield wipers of big vehicles such as buses and trucks are usually designed such that the blades of the wipers always keep a fixed angle during the motion of the connecting link. This angle is maintained by making the blade rotate with respect to the link about pin B. At any instant, let the angular velocity of the link be $\vec{\boldsymbol{\Omega}} = \omega_1 \hat{\boldsymbol{k}}$. What must be the angular velocity of the blade with respect to the link so that the blade stays horizontal?

Solution Intuitively, the answer should be more or less obvious. If the link rotates by $\Delta\theta$ in some time interval Δt, then the blade must rotate, relative to the link, by the same amount $\Delta\theta$ in the opposite direction so that its net rotation is zero and it stays horizontal. Now let us see how we can get the same answer using angular velocities and rotating frames.

Let $\vec{\boldsymbol{\omega}}_2 = \omega_2 \hat{\boldsymbol{k}}$ be the angular velocity of the blade with respect to the link. Let us attach a frame $\mathcal{B}$ to the blades and fix a coordinate axes $x'y'z'$ in this frame. (see Figure 21.13). Then

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{\mathcal{B}} &= \vec{\boldsymbol{\omega}}_{\text{blade}} = \vec{\boldsymbol{\omega}}_{\text{blade/link}} + \vec{\boldsymbol{\omega}}_{\text{link}} \\
&= \omega_2 \hat{\boldsymbol{k}} + \omega_1 \hat{\boldsymbol{k}} = (\omega_2 + \omega_1)\hat{\boldsymbol{k}}.
\end{aligned}$$

Since, in our example, the blade remains horizontal, the unit vector $\hat{\boldsymbol{\imath}}'$ does not change its direction, *i.e.*,

$$\begin{aligned}
\dot{\hat{\boldsymbol{\imath}}}' &= \vec{\boldsymbol{0}}. \\
\text{But} \quad \dot{\hat{\boldsymbol{\imath}}}' &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}' \\
&= (\omega_1 + \omega_2)\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}' \\
&= (\omega_1 + \omega_2)\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\omega_1 + \omega_2)\hat{\boldsymbol{\jmath}}' &= \vec{\boldsymbol{0}}. \\
\Rightarrow \quad \omega_1 + \omega_2 &= 0 \\
\Rightarrow \quad \omega_2 &= -\omega_1.
\end{aligned}$$

$$\boxed{\omega_2 = -\omega_1}$$

Comments: In general, justification of such simple ideas in such detail would preclude efficient solving of more complex problems. We present it just to show how the formulae *do* ultimately agree with common sense.

SAMPLE 21.4 Relative rotations in 2-D and 3-D. In the figures shown below, the rigid arm AB rotates with constant angular speed $\Omega = 5\,\text{rad/s}$ and the disk $\mathcal{D}$ rotates with respect to the arm at constant angular speed $\omega = 10\,\text{rad/s}$. Is the angular acceleration of the disk the same in each case?

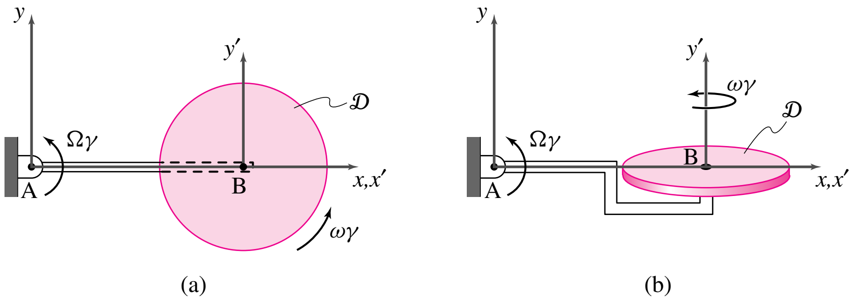

Figure 21.14:

Solution In each case, let us attach a frame $\mathcal{B}$ to the rod and fix coordinate axes $x'y'z'$ in $\mathcal{B}$ with the origin at the center of the disk. Let the primed coordinate axes be parallel to the inertial coordinate axes at the moment of interest.
In case (a):

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \Omega\hat{\boldsymbol{k}} \quad \text{and} \quad \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} = \omega\hat{\boldsymbol{k}}' = \omega\hat{\boldsymbol{k}}.$$

Therefore, the angular acceleration of the disk is

$$\begin{aligned}\vec{\boldsymbol{\alpha}}_{\mathcal{D}} &= \overbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{B}}}^{\vec{\mathbf{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\ &= \Omega\hat{\boldsymbol{k}} \times \omega\hat{\boldsymbol{k}} = \vec{\mathbf{0}}.\end{aligned}$$

In case (b):

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \Omega\hat{\boldsymbol{k}} \quad \text{and} \quad \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} = \omega\hat{\boldsymbol{\jmath}}' = \omega\hat{\boldsymbol{\jmath}}.$$

Therefore, the angular acceleration of the disk is

$$\begin{aligned}\vec{\boldsymbol{\alpha}}_{\mathcal{D}} &= \overbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{B}}}^{\vec{\mathbf{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} \\ &= \Omega\hat{\boldsymbol{k}} \times \omega\hat{\boldsymbol{\jmath}} = -\Omega\omega\hat{\boldsymbol{\imath}} \\ &= -50\,\text{rad/s}^2\hat{\boldsymbol{\imath}}.\end{aligned}$$

Thus, the angular acceleration of the disk is not the same in each case.

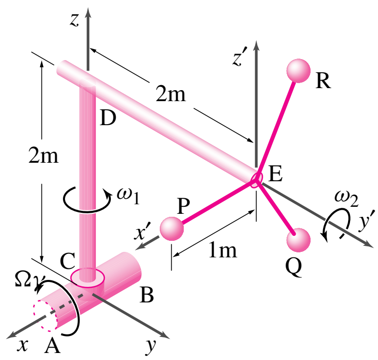

Figure 21.15:

SAMPLE 21.5 Relative and absolute angular velocity and acceleration. A three-mass body is made up of three point masses P, Q, and R, connected by three identical rigid rods at 120^o. The three-mass system is attached to shaft CDE and spins with respect to the shaft at a non-constant rate ω_2. Shaft CDE rotates with a constant rate ω_1 with respect to the base sleeve which, in turn, rotates with the base shaft AB with constant angular speed Ω. At the instant shown, $\omega_2 = 10\,\text{rad/s}$ and is changing at the rate of $\dot{\omega}_2 = 2\,\text{rad/s}^2$, $\omega_1 = 5\,\text{rad/s}$ (constant), and $\Omega = 2\,\text{rad/s}$ (constant).

1. Find the angular velocity of the three-mass body PQR with respect to the base shaft AB.
2. Find the absolute angular velocity of the body PQR.
3. Find the angular acceleration of the body PQR relative to the base shaft AB.
4. Find the absolute angular acceleration of the body PQR.

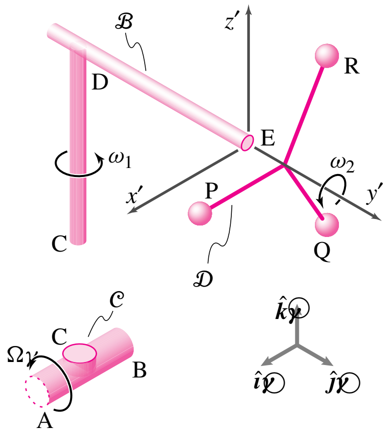

Figure 21.16: Frame $\mathcal{B}$ is attached to the shaft CDE and frame $\mathcal{C}$ to the base shaft AB. $\mathcal{D}$ represents the three mass system. $\mathcal{D}$ rotates with respect to $\mathcal{B}$, $\mathcal{B}$ rotates with respect to $\mathcal{C}$, and $\mathcal{C}$ rotates with respect to the fixed frame.

Solution Let us attach a frame $\mathcal{B}$ to the shaft CDE and a frame $\mathcal{C}$ to the base shaft AB. Thus frame $\mathcal{B}$ rotates with respect to frame $\mathcal{C}$ with constant speed ω_1 or at the instant shown,

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} &= \omega_1 \hat{\boldsymbol{k}} \quad \text{and}\\ \vec{\boldsymbol{\omega}}_{\mathcal{C}} &= \Omega \hat{\boldsymbol{\imath}}.\end{aligned}$$

Let the symbol $\mathcal{D}$ denote the three-mass rigid body PQR. Also, let the coordinate axes $x'y'z'$ be attached to the frame $\mathcal{D}$ at point E.

1. The angular velocity of $\mathcal{D}$ with respect to frame $\mathcal{C}$ is

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}}\\ &= \omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}}\\ &= (10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}})\,\text{rad/s}\end{aligned}$$

where the last line follows from the fact that at the instant of interest the primed coordinate axes x', y', and z', glued to the frame $\mathcal{B}$, are parallel to the fixed coordinates x, y, and z.

$$\boxed{\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} = (10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}})\,\text{rad/s}}$$

2. The absolute angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{D}} \equiv \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}}$ where $\mathcal{F}$ denotes the fixed frame, can be written as

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{D}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}}\\ &= \omega_2 \hat{\boldsymbol{\jmath}}' + \omega_1 \hat{\boldsymbol{k}} + \Omega\hat{\boldsymbol{\imath}}\\ &= (10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} + 2\hat{\boldsymbol{\imath}})\,\text{rad/s}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{\omega}}_{\mathcal{D}} = (2\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}})\,\text{rad/s}}$$

3. Now we calculate the angular acceleration. We have to be extra careful in calculating angular accelerations relative to the intermediate frames. Explicit notations for the time derivatives taken in different frames usually help.

The angular acceleration of $\mathcal{D}$ relative to $\mathcal{C}$ may be calculated as follows.

$$
\begin{aligned}
\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}} &= \overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{C}} \\
&= \overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{B}/\mathcal{C}} \\
&= \underbrace{\overset{\mathcal{B}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}}}_{\overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}}} \quad + \quad \vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{C}} \\
&= \overset{\mathcal{B}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \omega_1 \hat{\boldsymbol{k}} \times \omega_2 \hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{C}} \\
&= \dot{\omega}_2 \hat{\boldsymbol{\jmath}} - \omega_1 \omega_2 \hat{\boldsymbol{\imath}} + \underbrace{\dot{\omega}_1}_{0} \hat{\boldsymbol{k}} \\
&= 2\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\jmath}} - 10\,\mathrm{rad/s} \cdot 5\,\mathrm{rad/s}\,\hat{\boldsymbol{\imath}} \\
&= (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s^2}.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}} = (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\mathrm{rad/s^2}}$$

4. The absolute angular acceleration $\vec{\boldsymbol{\alpha}}_{\mathcal{D}}$ of the body can be found in a similar way by carrying the time derivative of the absolute angular velocity in the fixed frame. In the following calculations, all angular quantities without explicit reference to a frame stand for quantities with respect to the fixed frame $\mathcal{F}$.

$$
\begin{aligned}
\vec{\boldsymbol{\alpha}}_{\mathcal{D}} &= \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}} \\
&= \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{B}/\mathcal{C}} + \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{C}} \\
&= (\overset{\mathcal{B}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}}) + (\overset{\mathcal{C}}{\dot{\vec{\boldsymbol{\omega}}}}_{\mathcal{B}/\mathcal{C}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}}) + \underbrace{\dot{\Omega}}_{0} \hat{\boldsymbol{\imath}} \\
&= (\dot{\omega}_2 \hat{\boldsymbol{\jmath}} + (\omega_1 \hat{\boldsymbol{k}} + \Omega \hat{\boldsymbol{\imath}}) \times \omega_2 \hat{\boldsymbol{\jmath}}) + (\underbrace{\dot{\omega}_1}_{0} \hat{\boldsymbol{k}} + \Omega \hat{\boldsymbol{\imath}} \times \omega_1 \hat{\boldsymbol{k}}) \\
&= (\dot{\omega}_2 \hat{\boldsymbol{\jmath}} - \omega_1 \omega_2 \hat{\boldsymbol{\imath}} + \Omega \omega_2 \hat{\boldsymbol{k}}) + (-\Omega \omega_1 \hat{\boldsymbol{\jmath}}) \\
&= (2\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\jmath}} - 50\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\imath}} + 20\,\mathrm{rad/s^2}\,\hat{\boldsymbol{k}}) - 10\,\mathrm{rad/s^2}\,\hat{\boldsymbol{\jmath}} \\
&= (-50\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 20\hat{\boldsymbol{k}})\,\mathrm{rad/s^2}.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{\alpha}}_{\mathcal{D}} = (-50\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 20\hat{\boldsymbol{k}})\,\mathrm{rad/s^2}}$$

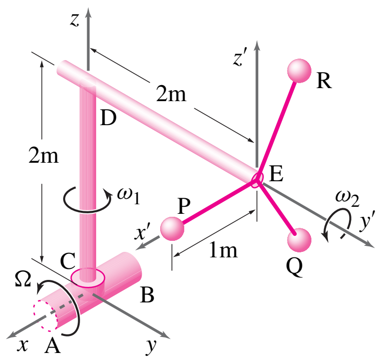

Figure 21.17:

SAMPLE 21.6 Relative and absolute angular velocity and acceleration. Consider Sample 21.5 again: A three-mass body consists of three point masses P, Q, and R, connected by three identical rigid rods 120º apart. The three-mass system is attached to shaft CDE and spins with respect to the shaft at a non-constant rate ω_2. Shaft CDE rotates with a constant rate ω_1 with respect to the base sleeve which, in turn, rotates with the base shaft AB with constant angular speed Ω. At the instant shown, $\omega_2 = 10\,\text{rad/s}$ and is changing at the rate of $\dot{\omega}_2 = 2\,\text{rad/s}^2$; $\omega_1 = 5\,\text{rad/s}$ (constant), and $\Omega = 2\,\text{rad/s}$ (constant).

1. Find the angular acceleration of the body relative to the base shaft AB.
2. Find the absolute angular acceleration of the body.

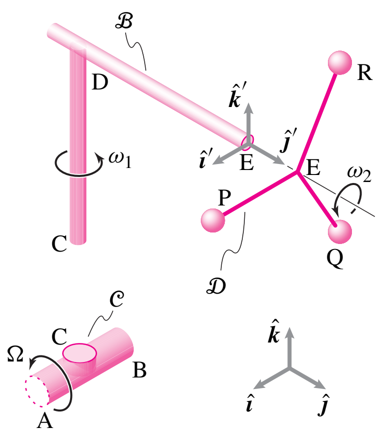

Figure 21.18:

Solution Let $\mathcal{D}$ represent the rotating mass system. Let us attach frame $\mathcal{B}$ to the shaft CDE and frame $\mathcal{C}$ to the shaft AB. Coordinate axes $x'y'z'$ are fixed in $\mathcal{B}$ and rotate with shaft CDE.

1. The angular velocity of $\mathcal{D}$ with respect to shaft AB (or frame $\mathcal{C}$) is:

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \\ &= \omega_2\hat{\boldsymbol{\jmath}}' + \omega_1\hat{\boldsymbol{k}}'.\end{aligned}$$

Therefore, the angular acceleration of $\mathcal{D}$ with respect to shaft AB is:

$$\begin{aligned}{}^{\mathcal{C}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{C}} &= \text{time derivative of } \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} \text{ in frame } \mathcal{C}. \\ &= \frac{{}^{\mathcal{C}}d}{dt}[\omega_2\hat{\boldsymbol{\jmath}}' + \omega_1\hat{\boldsymbol{k}}'] \\ &= \dot{\omega}_2\hat{\boldsymbol{\jmath}}' + \omega_2\dot{\hat{\boldsymbol{\jmath}}}' + \underbrace{\dot{\omega}_1}_{0}\hat{\boldsymbol{k}}' + \omega_1\dot{\hat{\boldsymbol{k}}} \\ &= \dot{\omega}_2\hat{\boldsymbol{\jmath}}' + \omega_2(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') + \omega_1(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{C}} \times \hat{\boldsymbol{k}}') \\ &= \dot{\omega}_2\hat{\boldsymbol{\jmath}}' + \omega_2(\omega_1\underbrace{\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}'}_{-\hat{\imath}'}) + \omega_1(\omega_1\underbrace{\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}'}_{0}) \\ &= -\omega_1\omega_2\hat{\boldsymbol{\imath}}' + \dot{\omega}_2\hat{\boldsymbol{\jmath}}' \\ &= (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{rad/s}^2\end{aligned}$$

since $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ at the instant of interest.

$$\boxed{{}^{\mathcal{C}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{C}} = (-50\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{rad/s}^2}$$

2. The absolute angular velocity of $\mathcal{D}$ is

$$\begin{aligned}\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}} &= \vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{C}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} \\ &= \underbrace{\omega_2\hat{\boldsymbol{\jmath}}' + \omega_1\hat{\boldsymbol{k}}'}_{\vec{\omega}_{\mathcal{D}/\mathcal{C}}} + \Omega\hat{\boldsymbol{\imath}}.\end{aligned}$$

Therefore, the absolute angular acceleration of $\mathcal{D}$ is

$$\begin{aligned}{}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{F}} &= {}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{C}} + {}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}/\mathcal{F}} \\ &= \frac{{}^{\mathcal{F}}d}{dt}[\omega_2\hat{\boldsymbol{\jmath}}' + \omega_1\hat{\boldsymbol{k}}'] + \frac{{}^{\mathcal{F}}d}{dt}[\Omega\hat{\boldsymbol{\imath}}] \\ &= \dot{\omega}_2\hat{\boldsymbol{\jmath}}' + \omega_2\dot{\hat{\boldsymbol{\jmath}}}' + \underbrace{\dot{\omega}_1}_{0}\hat{\boldsymbol{k}}' + \omega_1\dot{\hat{\boldsymbol{k}}}' + \underbrace{\dot{\Omega}}_{0}\hat{\boldsymbol{\imath}} \\ &= \dot{\omega}_2\hat{\boldsymbol{\jmath}}' + \omega_2(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \hat{\boldsymbol{\jmath}}') + \omega_1(\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \hat{\boldsymbol{k}}').\end{aligned}$$

But

$$\begin{aligned}\vec{\omega}_{\mathcal{B}/\mathcal{F}} &= \vec{\omega}_{\mathcal{C}/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{C}} \\ &= \Omega\hat{\imath} + \omega_1\hat{k}.\end{aligned}$$

Therefore,

$$\begin{aligned}{}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{D}/\mathcal{F}} &= \dot{\omega}_2\hat{\jmath}' + \omega_2([\Omega\hat{\imath} + \omega_1\hat{k}] \times \hat{\jmath}') + \omega_1([\Omega_1 + \omega_1\hat{k}] \times \hat{k}') \\ &= \dot{\omega}_2\hat{\jmath} + \omega_2\Omega\hat{k} - \omega_1\omega_2\hat{\imath} - \omega_1\Omega\hat{\jmath} \\ &= -\omega_1\omega_2\hat{\imath} + (\dot{\omega}_2 - \omega_1\Omega)\hat{\jmath} + \omega_2\Omega\hat{k} \\ &= (-50\hat{\imath} - 8\hat{\jmath} + 20\hat{k})\,\mathrm{rad/s^2}\end{aligned}$$

$$\boxed{\vec{\omega}_{\mathcal{D}/\mathcal{F}} = (-50\hat{\imath} - 8\hat{\jmath} + 20\hat{k})\,\mathrm{rad/s^2}}$$

Comments: Again, this approach based on direct differentiation, agrees with the result obtained by quoting the formula for general motion.