21.1 Velocity and acceleration of points on a rigid body: $\vec{\omega}$ and $\vec{\alpha}$

21.1.1 A particle moves on a helix. Say you know that a particle moves according to the equation

$$\vec{r} = R\cos\theta\hat{\imath} + R\sin\theta\,\hat{\jmath} + z\hat{k}$$

where R is a constant, $\theta = c_1 t$, and $z = c_2 t$.

a) Find $\vec{v}$ at general time t.

b) Find $\vec{a}$ at general time t.

c) Find $\hat{e}_t$ at general time t.

d) Find $\hat{e}_n$ at general time t.

e) Write $\vec{a}$ in terms of $\hat{e}_t$ and $\hat{e}_n$.

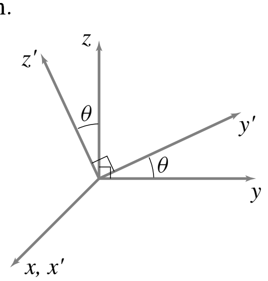
f) What is the radius of the osculating circle? Check your answer to see if it makes sense in the special case when $c_2 = 0$.

21.1.2 A curious chef tosses a potato in the air. Being a part-time dynamicist, she has just the right instruments in her kitchen and measures the absolute angular velocity of the potato to be $\vec{\omega} = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}(\,\mathrm{rad/s})$ and the absolute center of mass velocity to be $\vec{v}_G = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}(\,\mathrm{in/s})$, at a particular instant in time. At the same instant in time, she notes the position of an eye on the potato relative to its center of mass, point G, is $\vec{r}_{E/G} = 1 \in \hat{\imath}$. (The high quality potato came from the market with the center of mass already marked inside it.) $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are basis vectors in the kitchen frame.

a) What is the velocity of the potato eye relative to G, $\vec{v}_{E/G}$?

b) What is the absolute velocity of the potato eye, $\vec{v}_E$?

21.1.3 Find the rotation matrix $[\mathbf{R}]$ for $\theta = 30^\circ$ such that $\{\hat{e}'\} = [\mathbf{R}]\{\hat{e}\}$, where $\hat{e} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. Using the rotation matrix, find the components of $\vec{v} = 2\,\mathrm{m/s}\hat{\imath} - 3\,\mathrm{m/s}\hat{\jmath} + 1\,\mathrm{m/s}\hat{k}$ in the rotated coordinate system.



Problem 21.1.3

21.1.4 A circular plate rotates at a constant angular velocity $\vec{\omega} = 5\,\mathrm{rad/s}\hat{k}$. A set of coordinate axes $x'y'z'$ is glued to the plate at its center and thus rotates with $\vec{\omega}$. Find the rate of change of basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$ using the $\dot{\vec{Q}}$ formula.

21.1.5 A rigid body $\mathcal{B}$ rotates in space with angular velocity $\vec{\omega} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k})\,\mathrm{rad/s}$. A set of local coordinate axes $x'y'z'$ is fixed to the body at its center of mass. The position vector $\vec{r}_P$ of a point P is given in local coordinates: $\vec{r}_P = (5\hat{\imath}' + \hat{\jmath}')\,\mathrm{m}$. Find the velocity ($\dot{\vec{r}}_P$) of point P using the $\dot{\vec{Q}}$ formula.

21.1.6 The rotation matrix between two coordinate systems with basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is $\mathbf{Q}$ such that $\{\hat{e}\} = [\mathbf{Q}]\{\boldsymbol{E}\}$. Find the components of $\vec{a} = (-2\hat{e}_1 + 3\hat{e}_3)\,\mathrm{m/s^2}$ in the $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ basis if $\vec{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

21.1.7 Let $\mathcal{B}$ and $\mathcal{F}$ be two moving rigid bodies. Show that ${}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} = {}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}}$.

21.1.8 George is driving his car on the highway on a rainy day. His windshield wipers are on. He notices that a drop of water on the wiper blade is moving along the blade at approximately $3\,\mathrm{in/s}$. Although the wipers rotate at variable angular speed, he approximates that when the wiper is parallel to his nose, its angular speed is $\pi/2\,\mathrm{rad/s}$. In George's coordinate system (attached to his frame of reference and moving with him), the wiper is rotating in the positive $\hat{\jmath}$ direction and the water-drop is moving in the positive $\hat{k}$ direction at the instant of interest. Find the velocity of the water-drop in George's frame of reference.

21.1.9 A person sits on a bar stool which spins at angular speed Ω. Simultaneously, (s)he hoists a beer glass of mass m up at angular speed ω relative to the bar stool with a straight, rigid arm of length L. θ measures the angle of her/his arm with respect to the XY plane. XYZ is an inertial coordinate system; YZ lies in the plane of the paper. If you use moving axes to do this problem, specify clearly his/her orientation and motion.

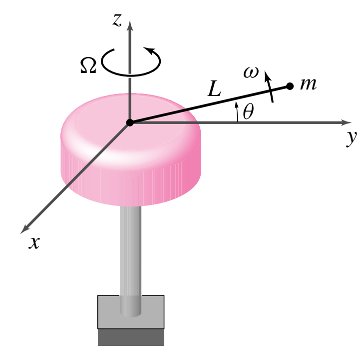
a) What is the angular velocity of the person's arm?

b) What is the angular acceleration of the person's arm?

c) What is the linear velocity of the beer glass relative to inertial space?

d) What is the linear acceleration of the beer glass relative to inertial space?

e) Describe how you would point an absolutely full glass such that the beer will not spill out.



Problem 21.1.9

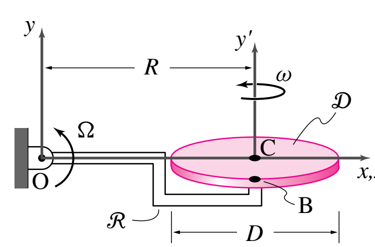
21.1.10 Kinematics of a disk spinning on a spinning rod. The rigid rod $\mathcal{R}$ rotates at constant rate Ω about the z axis. Attached to this rod, a distance R from the origin, is the center C of a disk $\mathcal{C}$ with diameter D. The disk is rotating at constant rate ω about an axis y' that is attached to, and rotates with, the rod. A point B is on the outer edge of the disk. At the instant of interest, the system is in the configuration shown with $\vec{r}_{CB} = (D/2)\hat{k}$.

a) What is the absolute angular velocity of the disk, $\vec{\omega}_{\mathcal{D}}$? *

b) What is the absolute angular acceleration of the disk $\vec{\alpha}_{\mathcal{D}}$? *

c) What is the absolute velocity of point B, $\vec{v}_B$? *

d) What is the absolute acceleration of point B, $\vec{a}_B$? *



Problem 21.1.10

21.1.11
See also problems 21.1.17, 21.1.19, and 21.2.16. Body $\mathcal{A}$ rotates with respect to a Newtonian frame $\mathcal{F}$ at a constant angular rate ω_0 about axis OO'. Body $\mathcal{B}$, a 'fork', rotates with respect to body $\mathcal{A}$ at a constant angular rate ω_1 about axis PP'. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass. Body $\mathcal{D}$, a uniform disc of mass m, rotates with respect to body $\mathcal{B}$, at a constant angular rate ω_2 about axis CC'. Each of the bodies is turned at constant rate by motors(not shown.) At the instant shown, find the absolute angular velocity of body $\mathcal{D}$, $\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}}$. *

Problem 21.1.11

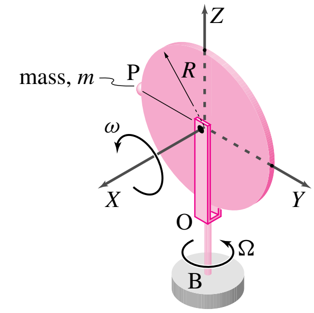
21.1.12 A particle P of mass m is attached to the edge of the disk of radius R, as shown in the figure. The disk spins about its center line with constant angular speed ω rad/s relative to the shaft OB. The shaft OB rotates with angular speed Ω rad/s. For the instant when the disk lies in the YZ plane, and the particle is on the Y axis, as shown:

a) Find the particle's velocity relative to XYZ.

b) Calculate the particle's acceleration relative to XYZ.

c) Calculate the force $\vec{\boldsymbol{F}}$ acting on the particle.

If you choose to use another coordinate system, define it explicitly.



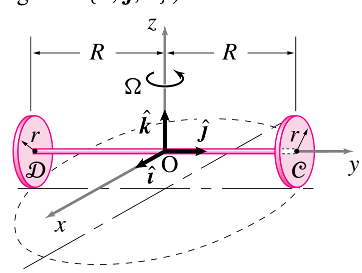
xyz: fixed frame

Problem 21.1.12

21.1.13 See also problem 21.2.31. Two identical thin disks, bodies $\mathcal{C}$ and $\mathcal{D}$, of radius r are connected (perpendicularly) to opposite ends of a thin axle, as depicted below. The axle rotates about the fixed Z axis at a constant angular speed Ω, and the disks roll without slipping. Determine:

1. The total angular velocity of each disk, $\vec{\boldsymbol{\omega}}_{\mathcal{C}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$, and
2. The total angular acceleration of each disk, $\boldsymbol{\alpha}_{\mathcal{C}}$ and $\boldsymbol{\alpha}_{\mathcal{D}}$.

(Express your answers relative to the rotating basis { $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$}.)



Problem 21.1.13

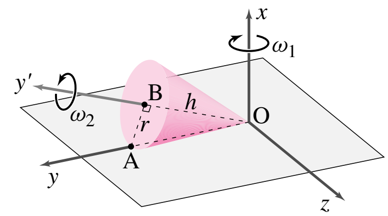
21.1.14 Kinematics of a rolling cone. See also problem 20.1.20. A solid cone with circular-base radius r and height h moves as follows. Point O is fixed in space. The cone surface always contacts the plane but may roll and/or slide. The axis of the cone OB rotates about the vertical axis with constant angular velocity $-\omega_1 \hat{\boldsymbol{\imath}}$. The cone spins about this moving axis $\hat{\boldsymbol{\jmath}}'$ with constant angular rate ω_2.

a) At the instant shown what is the velocity of point B? (in terms of some or all of ω_1, ω_2, r, h, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$) *

b) At the instant shown what is the angular acceleration of the cone? (in terms of some or all of ω_1, ω_2, r, h, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$) *

c) If the disk rolls with no slip, find ω_1 in terms of ω_2, r and h. *

d) If the disk rolls with no slip and point B has constant speed v find the magnitude of the angular acceleration of the cone in terms of v, r and h. *

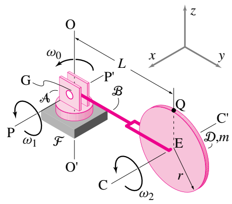


Problem 21.1.14: **Cone rolling on a plane**

21.1.15 See also problems 21.1.16, 21.1.18, 21.2.15, and 21.2.14. Body $\mathcal{A}$ rotates with respect to a Newtonian frame $\mathcal{F}$ at a constant angular rate ω_0 about axis OO'. Body $\mathcal{B}$, a 'fork', rotates with respect to body $\mathcal{A}$ at a constant angular rate ω_1 about axis PP'. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass. Body $\mathcal{D}$, a uniform disc of mass m, rotates with respect to body $\mathcal{B}$, at a constant angular rate ω_2 about axis CC'. Each of the bodies is turned at constant rate by motors(not shown.) At the instant shown, find the absolute velocity and acceleration of point Q on the disk.

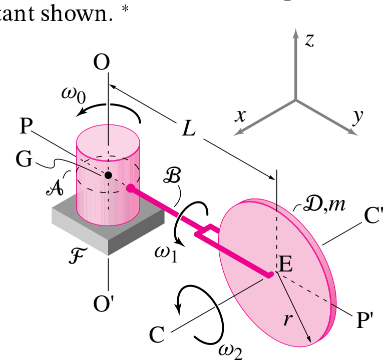
Problem 21.1.15

21.1.16 (See also problems 21.1.18, 21.2.15, and 21.2.14.) For the system in problem 21.1.15, find the absolute angular acceleration of frame $\mathcal{D}$, ${}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}}$, at the instant shown.



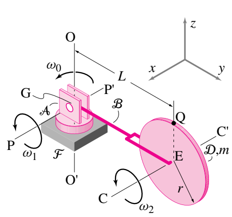
Problem 21.1.16

21.1.17 (See also problems 21.1.19, 21.2.16, and 21.2.7.) For the system in problem 21.1.11, find the absolute angular acceleration of frame $\mathcal{D}$, ${}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{D}}$, at the instant shown. *



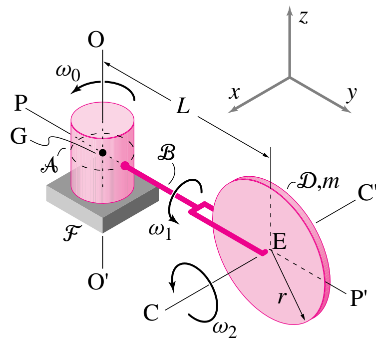
Problem 21.1.17

21.1.18 (See also problems 21.1.16, 21.2.15, and 21.2.14.) For the configuration in problem 21.1.15, find the velocity and acceleration of point Q on the disk using the alternative method.



Problem 21.1.18

21.1.19 (See also problems 21.1.17, 21.2.16, and 21.2.7.) For the configuration in problem 21.1.11, find the velocity and acceleration of the point on the edge of the disk directly above its center point E using the alternative method.



Problem 21.1.19

21.1.20 To simulate the flight conditions of a space vehicle, engineers have developed the centrifuge, shown diagrammatically in the figure. A main truss arm of length $\ell = 15\,\text{m}$, rotates about the AA axis. The pilot sits in the capsule which may rotate about axis CC. The seat for the pilot may rotate inside the capsule about an axis perpendicular to the page and going through the point B. These rotations are controlled by a computer that is set to simulate certain maneuvers corresponding to the entry and exit from the earth's atmosphere, malfunctions of the control system, etc. When a pilot sits in the capsule, his head, particularly his ears, has the position shown in the enlarged figure, a distance $r = 1\,\text{m}$ from point B.

The main arm is rotating at $\Omega = 10\,\text{rpm}$ and accelerating at $\dot{\Omega} = 5\,\text{rev}/\text{min}^2$, the capsule is rotating at a constant speed about CC at $\omega_c = 10\,\text{rpm}$, and the seat rotates at a speed of $\omega_s = 5\,\text{rpm}$ inside the capsule.

a) Determine the acceleration, in terms of how many g's, that the pilot's head is subjected to by choosing a moving coordinates system xyz centered at B and fixed to the capsule.

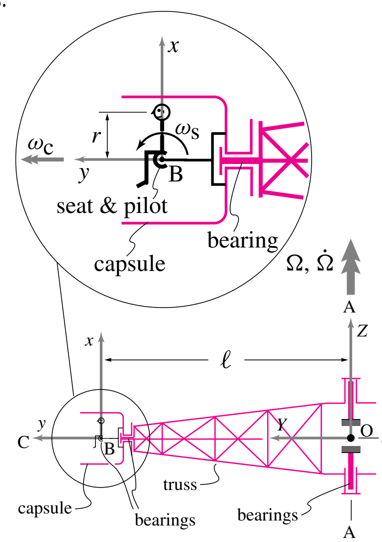
b) Repeat the calculations of (A) by letting xyz be fixed to the seat. *

c) Choose another moving coordinates system to solve the same problem.

d) A pilot has different tolerances for acceleration components. They are roughly:

   1. In a vertical direction, i.e., toe to head – 5 g's. (Here the pilot experiences blackout or may pass out completely.)
   2. Front to rear – 1.5 g's. (Here vision becomes greatly distorted.)

Arrange a test program on the centrifuge so that each acceleration is reached separately while the other accelerations are kept below one- half their tolerance levels.



Problem 21.1.20