20.5 Dynamic balance

Sometimes when something spins it can shake the structure that holds it. A familiar example is an 'unbalanced' car tire which makes a car shake. But rotor balance is important in all kinds of machines. Most often one seeks to eliminate or minimize the imbalance. For example the strange design of car crank-shafts is due in large part to an attempt to minimize its imbalance. But what does it mean, in the language of mechanics, to say a rotating body is 'unbalanced'? It means that non-zero forces and/or torques are required to hold the axis still while the object spins at constant rate. Going back to the basic momentum balance equations:

$$\text{Linear momentum balance:} \qquad \sum \vec{\boldsymbol{F}} = \dot{\vec{\boldsymbol{L}}}$$

and

$$\text{Angular momentum balance:} \qquad \sum \vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}},$$

we see that forces and torques are required if $\dot{\vec{\boldsymbol{L}}} \neq \vec{\mathbf{0}}$ or if $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \neq \vec{\mathbf{0}}$.

So imbalance means $\dot{\vec{\boldsymbol{L}}}$ and/or $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ is not zero. We break the concept of balance into the following two concepts:

(1) An object is said to be *statically balanced* with respect to a given axis of rotation if $\dot{\vec{\boldsymbol{L}}} = \vec{\mathbf{0}}$ when it spins at constant rate about that axis.

(2) An object is said to be *dynamically balanced* with respect to a given axis of rotation if both $\dot{\vec{\boldsymbol{L}}} = \vec{\mathbf{0}}$ and $\dot{\vec{\boldsymbol{H}}} = \vec{\mathbf{0}}$ for constant rate rotation about that axis.

The origin of the words 'static' balance and 'dynamic' balance is in how the imbalance can be measured. Static imbalance can be measured with a static test, dynamic imbalance requires a dynamic test.

Static balance

If the center-of-mass of a rigid body is on the fixed axis of rotation then it will not accelerate. Thus, $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot} = \vec{\mathbf{0}}$ and the object is statically balanced. An equivalent definition of static balance is that the *net* force on the spinning body is zero. Whether or not this net force is so can be tested with a statics experiment. Put the axis of rotation on good bearings and see if the mass hangs down in any preferred direction. In a tire shop, this kind of balancing is sometimes called bubble balancing because of the bubble in the level measuring device.

Dynamic balance

The first condition for dynamic balance of an object with respect to spinning about an axis is that the object be statically balanced. The center-of-mass must lie on the axis of rotation. The second condition for dynamic balance, that $\dot{\vec{\boldsymbol{H}}} = \vec{\mathbf{0}}$, is a little more subtle. To make things more specific, let's calculate the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ with respect to a point O that is on the axis of rotation. So, for constant rate rotation about a fixed axis we have:

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\omega} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \tag{20.50}$$

because $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ spins with the body. $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ is evidently zero if the angular momentum, $\vec{\boldsymbol{H}}_{/\mathrm{O}}$, is parallel to the angular velocity, $\vec{\omega}$. However, $\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\omega}$ which means $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is parallel to $\vec{\omega}$ only when $\vec{\omega}$ is an eigenvector of $[\boldsymbol{I}^{\mathrm{O}}]$①.

So an object which is spinning about a fixed axis is dynamically balanced if its center-of-mass is on the axis (static balance) and *the angular velocity vector $\vec{\omega}$ is an eigenvector of the moment of inertia matrix.*

If we restrict ② our attention to cases where the axis of rotation is the z-axis then this condition is easy to recognize. It is when $I_{xz} = I_{yz} = 0$, that is, when the only non-zero element in the third column and in the third row of the $[\boldsymbol{I}^{\mathrm{O}}]$ matrix is I^{O}_{zz} in the lower right corner.

Since these terms cause wobbling if one of the coordinate axis is an axis of rotation, the terms that are off the main diagonal in the moment of inertia matrix are sometimes called the 'imbalance' terms. These terms lead to dynamic imbalance when the object is spun about the x, y, or z-axes. The off diagonal terms are also sometimes called the 'centrifugal' terms. This naming has an intuitive basis. One way to understand dynamic imbalance is to think of it being due to the unbalanced centrifugal pull of bits of mass that are spinning in circles.

Often, but not always, you can tell by inspection if something is dynamically balanced for rotation about a certain axis. If, for every bit of mass that is not on the axis there is another equal bit of mass that is exactly opposite, with respect to the axis, then the object is balanced. Some objects that do not meet this symmetry condition are also dynamically balanced, however.

①When you go to ACME garage and get your car tires balanced you can politely ask the mechanic: "Ma'am, could you make sure that one of the eigenvectors of my wheel's moment of inertia matrix is parallel to the car axle? Thanks." The mechanic, if she is a decent person, will make sure that it is the appropriate eigenvector that is parallel to the axle. Otherwise the wheel would point straight sideways with the axle piercing the rubber.

②**Caution:** A common misperception about angular momentum is that it is always parallel to angular velocity. In general, angular momentum is *not* parallel to angular velocity. When is angular momentum about the center-of-mass, say, parallel to angular velocity? When $\vec{\omega}$ is an *eigenvector* of $[\boldsymbol{I}^{\mathrm{cm}}]$. This is always the case for planar objects rotating about an axis perpendicular to the plane, but not generally in 3D.

Example: Some common objects

All of the figures in fig. 2.20 on page 153 are dynamically balanced about all the axes shown and about any axis perpendicular to any pair of these axes.

Example: A cube = a sphere

Both a cube and a sphere have moment of inertia matrices proportional to the identity matrix

$$
\begin{aligned}
[\boldsymbol{I}^{\mathrm{cm}}]_{\mathrm{cube}} &= \frac{m\ell^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
[\boldsymbol{I}^{\mathrm{cm}}]_{\mathrm{sphere}} &= \frac{mD^2}{10}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

So, for any $\vec{\omega}$, we get that $\vec{\boldsymbol{H}}_{\mathrm{cm}}$ is parallel to $\vec{\omega}$ and, thus, both of these objects are dynamically balanced for rotation about any axis through their centers of mass.

Here is an example where the mathematics contradicts intuition; it does not seem like a cube should be balanced for rotation about a random skewed axis through its center-of-mass!