20.5 Dynamic balance

Sometimes when something spins it can shake the structure that holds it. A familiar example is an 'unbalanced' car tire which makes a car shake. But rotor balance is important in all kinds of machines. Most often one seeks to eliminate or minimize the imbalance. For example the strange design of car crank-shafts is due in large part to an attempt to minimize its imbalance. But what does it mean, in the language of mechanics, to say a rotating body is 'unbalanced'? It means that non-zero forces and/or torques are required to hold the axis still while the object spins at constant rate. Going back to the basic momentum balance equations:

$$\text{Linear momentum balance:} \qquad \sum \vec{\boldsymbol{F}} = \dot{\vec{\boldsymbol{L}}}$$

and

$$\text{Angular momentum balance:} \qquad \sum \vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}},$$

we see that forces and torques are required if $\dot{\vec{\boldsymbol{L}}} \neq \vec{\boldsymbol{0}}$ or if $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \neq \vec{\boldsymbol{0}}$.

So imbalance means $\dot{\vec{\boldsymbol{L}}}$ and/or $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ is not zero. We break the concept of balance into the following two concepts:

(1) An object is said to be *statically balanced* with respect to a given axis of rotation if $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{0}}$ when it spins at constant rate about that axis.

(2) An object is said to be *dynamically balanced* with respect to a given axis of rotation if both $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{0}}$ and $\dot{\vec{\boldsymbol{H}}} = \vec{\boldsymbol{0}}$ for constant rate rotation about that axis.

The origin of the words 'static' balance and 'dynamic' balance is in how the imbalance can be measured. Static imbalance can be measured with a static test, dynamic imbalance requires a dynamic test.

Static balance

If the center-of-mass of a rigid body is on the fixed axis of rotation then it will not accelerate. Thus, $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot} = \vec{\boldsymbol{0}}$ and the object is statically balanced. An equivalent definition of static balance is that the *net* force on the spinning body is zero. Whether or not this net force is so can be tested with a statics experiment. Put the axis of rotation on good bearings and see if the mass hangs down in any preferred direction. In a tire shop, this kind of balancing is sometimes called bubble balancing because of the bubble in the level measuring device.

Dynamic balance

The first condition for dynamic balance of an object with respect to spinning about an axis is that the object be statically balanced. The center-of-mass must lie on the axis of rotation. The second condition for dynamic balance, that $\dot{\vec{\boldsymbol{H}}} = \vec{\mathbf{0}}$, is a little more subtle. To make things more specific, let's calculate the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ with respect to a point O that is on the axis of rotation. So, for constant rate rotation about a fixed axis we have:

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\omega} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \tag{20.50}$$

because $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ spins with the body. $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ is evidently zero if the angular momentum, $\vec{\boldsymbol{H}}_{/\mathrm{O}}$, is parallel to the angular velocity, $\vec{\omega}$. However, $\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\omega}$ which means $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is parallel to $\vec{\omega}$ only when $\vec{\omega}$ is an eigenvector of $[\boldsymbol{I}^{\mathrm{O}}]$①.

> *So an object which is spinning about a fixed axis is dynamically balanced if its center-of-mass is on the axis (static balance)* and *the angular velocity vector $\vec{\omega}$ is an eigenvector of the moment of inertia matrix.*

If we restrict ② our attention to cases where the axis of rotation is the z-axis then this condition is easy to recognize. It is when $I_{xz} = I_{yz} = 0$, that is, when the only non-zero element in the third column and in the third row of the $[\boldsymbol{I}^{\mathrm{O}}]$ matrix is I^{O}_{zz} in the lower right corner.

Since these terms cause wobbling if one of the coordinate axis is an axis of rotation, the terms that are off the main diagonal in the moment of inertia matrix are sometimes called the 'imbalance' terms. These terms lead to dynamic imbalance when the object is spun about the x, y, or z-axes. The off diagonal terms are also sometimes called the 'centrifugal' terms. This naming has an intuitive basis. One way to understand dynamic imbalance is to think of it being due to the unbalanced centrifugal pull of bits of mass that are spinning in circles.

Often, but not always, you can tell by inspection if something is dynamically balanced for rotation about a certain axis. If, for every bit of mass that is not on the axis there is another equal bit of mass that is exactly opposite, with respect to the axis, then the object is balanced. Some objects that do not meet this symmetry condition are also dynamically balanced, however.

①When you go to ACME garage and get your car tires balanced you can politely ask the mechanic: "Ma'am, could you make sure that one of the eigenvectors of my wheel's moment of inertia matrix is parallel to the car axle? Thanks." The mechanic, if she is a decent person, will make sure that it is the appropriate eigenvector that is parallel to the axle. Otherwise the wheel would point straight sideways with the axle piercing the rubber.

②**Caution:** A common misperception about angular momentum is that it is always parallel to angular velocity. In general, angular momentum is *not* parallel to angular velocity. When is angular momentum about the center-of-mass, say, parallel to angular velocity? When $\vec{\omega}$ is an *eigenvector* of $[\boldsymbol{I}^{\mathrm{cm}}]$. This is always the case for planar objects rotating about an axis perpendicular to the plane, but not generally in 3D.

Example: Some common objects

All of the figures in fig. 2.20 on page 153 are dynamically balanced about all the axes shown and about any axis perpendicular to any pair of these axes.

Example: A cube = a sphere

Both a cube and a sphere have moment of inertia matrices proportional to the identity matrix

$$
\begin{aligned}
[\boldsymbol{I}^{\mathrm{cm}}]_{\mathrm{cube}} &= \frac{m\ell^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
[\boldsymbol{I}^{\mathrm{cm}}]_{\mathrm{sphere}} &= \frac{mD^2}{10}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

So, for any $\vec{\omega}$, we get that $\vec{\boldsymbol{H}}_{\mathrm{cm}}$ is parallel to $\vec{\omega}$ and, thus, both of these objects are dynamically balanced for rotation about any axis through their centers of mass.

Here is an example where the mathematics contradicts intuition; it does not seem like a cube should be balanced for rotation about a random skewed axis through its center-of-mass!

SAMPLE 20.19 Static and dynamic balance in 2-D motion. A system with two equal masses at A and B rotates about point O at a constant rate $\omega = 10\,\text{rpm}$. The system is shown in the figure. The rod connecting the two masses has negligible mass.

1. Is the system statically balanced? If not, suggest a way to balance it.
2. Is the system dynamically balanced? If not, suggest a way to balance it.

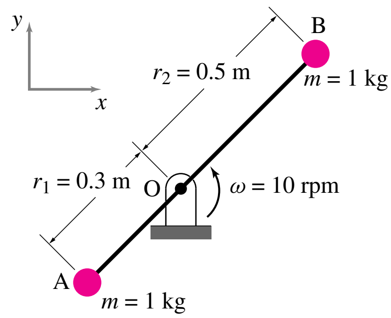


Figure 20.56

Solution

1. Since the two masses are equal, the center-of-mass of the system is at the geometric center of rod AB, i.e., at a point O′, halfway between A and B. Clearly, O′ is not on the axis of rotation which passes through O and is perpendicular to the plane of motion. Therefore, the system is *not* statically balanced.

   To balance the system, we must move the center-of-mass to O or pivot the system at O′. Say we cannot move the pivot point. So, to move the center-of-mass O, we add mass m' to m at A. From the definition of center-of-mass:

$$\begin{aligned} (m+m')r_1 &= mr_2 \\ \Rightarrow \qquad m' &= \frac{m'(r_2-r_1)}{r_1} \\ &= \frac{1\,\text{kg}(0.5\,\text{m}-0.3\,\text{m})}{0.3\,\text{m}} \\ &= 0.67\,\text{kg}. \end{aligned}$$

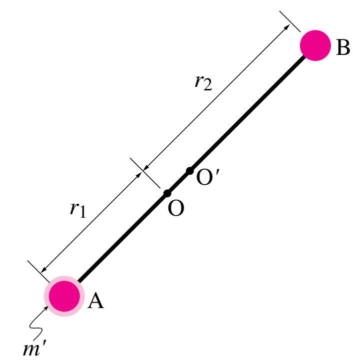


Figure 20.57

Add 0.67 kg to mass at A.

2. The system, as given, is *not* dynamically balanced since it is not statically balanced. Let us check if it is dynamically balanced after adding m' to A as suggested above.

$$\sum \vec{\boldsymbol{M}}_{\text{O}} = \dot{\vec{\boldsymbol{H}}}_{/\text{O}}.$$

   Let us calculate $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$ to see if it is zero:

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{/\text{O}} &= \vec{\boldsymbol{r}}_1 \times m_1\vec{\boldsymbol{a}}_1 \quad + \quad \vec{\boldsymbol{r}}_2 \times m_2\vec{\boldsymbol{a}}_2 \\ &= (-r_1\hat{\boldsymbol{e}}_R)\times m_1(\omega^2 r_1\hat{\boldsymbol{e}}_R) \quad + \quad (r_2\hat{\boldsymbol{e}}_R)\times(-\omega^2 r_2\hat{\boldsymbol{e}}_R)m_2 \\ &= -m_1\omega^2 r_1^2\underbrace{(\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_R)}_{\vec{\boldsymbol{0}}} \quad - \quad m_2\omega^2 r_2^2\underbrace{(\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_R)}_{\vec{\boldsymbol{0}}} \\ &= \vec{\boldsymbol{0}}. \end{aligned}$$

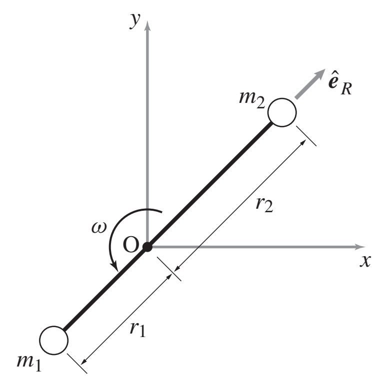


Figure 20.58

   The net torque on the system is zero, therefore it is dynamically balanced. Note that $\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{0}}$ irrespective of the values of m_1, m_2, r_1, and r_2. Thus $\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{0}}$ even for the system as given. But the system, as given, is not dynamically balanced because *static balance is a necessary condition for dynamic balance.*

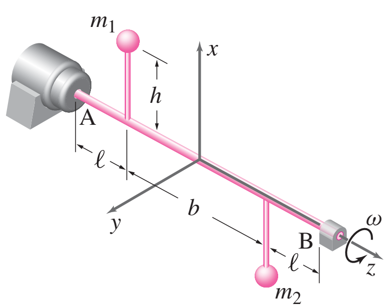


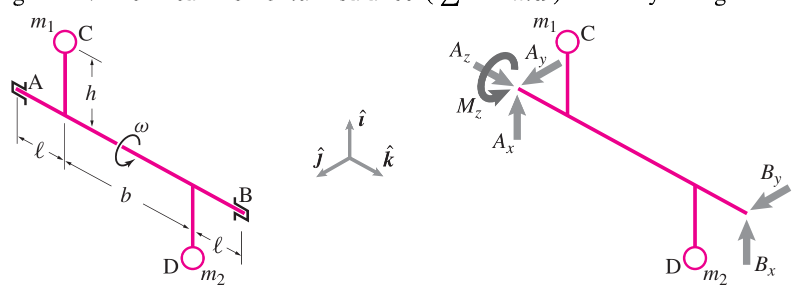
Figure 20.59: Imbalance of a rotor due to point masses spinning at a constant rate.

SAMPLE 20.20 Imbalance of a rotor due to rotating masses. Two masses m_1 and m_2 are attached to a massless shaft AB by massless rigid rods of length h each. The two masses are separated by distance b along the shaft and are in the same plane but on the opposite sides of the shaft. The shaft is rotating with a constant angular speed ω. The shaft is free to move along the z-axis at point B. Ignore gravity.

1. Is the system *statically balanced*?
2. What is the torque required to keep the motion going about the z-axis (i.e., M_z) ?
3. What are the reactions at the support points of the shaft?

Solution

1. A simple line sketch and the free-body diagram of the system are shown in Fig. 20.60. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the system gives:



(a) (b)

Figure 20.60: (a) A simple line diagram of the system. (b) Free-Body Diagram of the system.

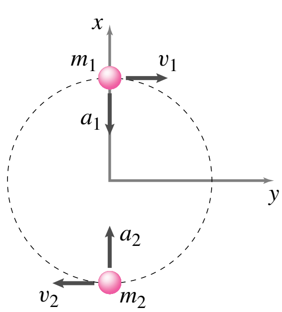


Figure 20.61: Accelerations of m_1 and m_2.

$$
\begin{aligned}
(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + A_z\hat{k} &= m_1\vec{a}_1 + m_2\vec{a}_2 \\
&= m_1\omega^2 h(-\hat{\imath}) + m_2\omega^2 h\hat{\imath} \\
\Rightarrow \quad A_x + B_x &= \omega^2 h(m_2 - m_1) && (20.51) \\
A_y + B_y &= 0 && (20.52) \\
A_z &= 0
\end{aligned}
$$

Clearly, $\sum \vec{F} \neq \vec{0}$ which means the system is not statically balanced.

2. M_z can be easily calculated by writing angular momentum balance about axis AB. In fact, we do not even need to write the equation in this case; since all the forces pass through this axis and the accelerations of m_1 and m_2 also pass through it,

$$\sum M_{/AB}(\text{due to reaction forces}) = 0 \quad \text{and} \quad \dot{H}_{/AB} = 0.$$

But

$$
\begin{aligned}
\sum M_{/AB} &= \dot{H}_{/AB} \\
\text{or} \quad M_z &+ \sum M_{/AB}(\text{due to reaction forces}) = \dot{H}_{/AB} \\
\Rightarrow \quad M_z &= 0.
\end{aligned}
$$

$$\boxed{M_z = 0}$$

3. For the four unknown reactions at A and B (we have already found A_z and M_z) we have two scalar equations so far (from Linear Momentum Balance). Angular Momentum Balance about point A gives:

$$\sum \vec{\boldsymbol{M}}_{/A} = \dot{\vec{\boldsymbol{H}}}_{/A}$$

Now,

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/A} &= \vec{\boldsymbol{r}}_{B/A} \times \sum \vec{\boldsymbol{F}}_B \\
&= (2\ell + b)\hat{\boldsymbol{k}} \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}}) \\
&= B_x(2\ell + b)\hat{\boldsymbol{\jmath}} - B_y(2\ell + b)\hat{\boldsymbol{\imath}} \\
\dot{\vec{\boldsymbol{H}}}_{/A} &= \vec{\boldsymbol{r}}_{C/A} \times m_1 \vec{\boldsymbol{a}}_1 + \vec{\boldsymbol{r}}_{D/A} \times m_2 \vec{\boldsymbol{a}}_2 \\
&= (\ell \hat{\boldsymbol{k}} + h\hat{\boldsymbol{\imath}}) \times m_1(-\omega^2 h \hat{\boldsymbol{\imath}}) + ((\ell + b)\hat{\boldsymbol{k}} - h\hat{\boldsymbol{\imath}}) \times m_2(\omega^2 h \hat{\boldsymbol{\imath}}) \\
&= \omega^2 h(m_2(\ell + b) - m_1 \ell)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Equating $\sum \vec{\boldsymbol{M}}_{/A}$ and $\dot{\vec{\boldsymbol{H}}}_{/A}$ we get

$$B_x(2\ell + b)\hat{\boldsymbol{\jmath}} - B_y(2\ell + b)\hat{\boldsymbol{\imath}} = \omega^2 h(m_2(\ell + b) - m_1 \ell)\hat{\boldsymbol{\jmath}}$$

Dotting both sides of the equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, we get

$$\begin{aligned}
B_y &= 0 \\
\text{and} \qquad B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1 \ell].
\end{aligned}$$

Substituting B_x and B_y in (20.51) and (20.52) we find

$$\begin{aligned}
A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2 \ell - m_1(\ell + b)] \\
A_y &= 0
\end{aligned}$$

$$\boxed{\begin{aligned}
A_x &= \tfrac{\omega^2 h}{(2\ell + b)}[m_2 \ell - m_1(\ell + b)] \\
A_y &= A_z = 0 \\
B_x &= \tfrac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1 \ell] \\
B_y &= 0 \\
M_z &= 0
\end{aligned}}$$

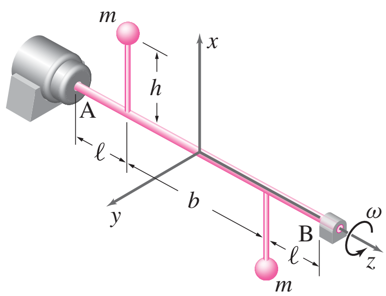
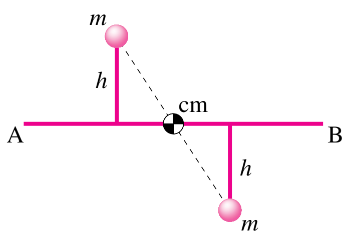


Figure 20.62: Imbalance of a rotor due to point masses spinning at a constant rate.



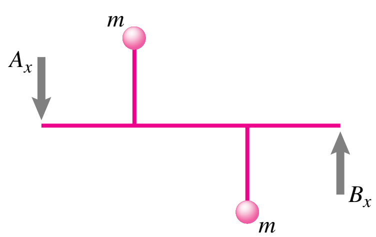


Figure 20.63: Reaction forces at A and B are required to keep the motion going.

SAMPLE 20.21 Balancing a rotor with spinning masses. Consider the same system as in the previous sample problem (Sample 20.20). Assume that $m_1 = m_2 = m$.

1. Set $m_1 = m_2$ in the reactions calculated. Is the system statically balanced now? Explain.
2. Is the system *dynamically balanced*? Explain.
3. Balance the system dynamically by (a) adjusting the geometry, (b) adding two masses to the system.

Solution

1. Recall from the solution of Sample 20.20 that

$$\begin{aligned} A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2\ell - m_1(\ell + b)] \\ B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1\ell] \\ A_y &= B_y = A_z = M_z = 0 \end{aligned}$$

Setting $m_1 = m_2 = m$ in the above expressions for the reactions, we get

$$A_x = \frac{\omega^2 h}{(2\ell + b)}(-mb) \quad \text{and} \quad B_x = \frac{\omega^2 h}{(2\ell + b)}(mb).$$

Thus, $A_x + B_x = 0$. Therefore, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ which means the system is statically balanced.

Static Balance: When the two masses are equal, the center-of-mass of the system is on the axis of rotation. Therefore, the static reactions are zero irrespective of the orientations of the two masses.

The requirement for static balance is $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ in any static orientation of the system. As long as the center-of-mass of the system lies on the axis of rotation, the system will be in static balance.

Dynamic Balance: When the masses are rotating at constant speed, reaction forces at A and B are required to keep the motion going. Clearly, $\sum \vec{\boldsymbol{M}} \neq \vec{\boldsymbol{0}}$ now. Therefore, the system is not in dynamic balance.

At the instant shown the reaction forces are A_x and B_x as shown in Fig. 20.63, but they change directions as the position of the masses changes. For example, if the masses are in the y–z plane the reaction forces will be A_y and $-B_y$. The magnitudes of these forces will remain the same. Thus, we have rotating reaction forces at A and B. These rotary forces cause wear in the bearings and induce vibrations in the frame of the machine and the supporting structure. Therefore, these forces are undesirable.

If we can somehow make these *dynamic reactions* zero, the bearings, the machine frame, the supporting structure, the machine operator and the company will all be very happy. So, how do we do it? Read on.

2. There are many ways in which the rotor can be dynamically balanced:
   - **Trivial solution:** Remove both the masses. Of course, there is nothing left to produce any non-zero $\dot{\vec{\boldsymbol{H}}}$. Thus,

$$\sum \vec{\boldsymbol{M}}_{/\text{any point}} = \vec{\boldsymbol{0}}. \quad \text{Also, } \sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$$

.

- **Adjust geometry:** Set $b = 0$. The center-of-mass is still on the axis of rotation.

$$\Rightarrow \quad \sum \vec{F} = \vec{0}$$

In addition,

$$\begin{aligned} \dot{\vec{H}}_{/A} &= \vec{0} \\ \Rightarrow \quad \sum \vec{M}_{/A} &= \vec{0}. \end{aligned}$$

- **Add two masses:** If b is required to be non-zero, we can balance the rotor by adding two masses in any two selected transverse (to the shaft-axis) planes to make $\dot{\vec{H}}_{/A} = \vec{0}$. ① For example:
  - We can take two equal masses m and m, and place them in the opposite directions of the masses already on the shaft as shown in Fig. 20.64(a). It should be clear that the net angular momentum about any point on the shaft will be zero now.
  - We can add two masses m' and m' different from the masses attached to the shaft and place them a distance c apart on the shaft. Let the length of the connecting rods of the new masses be h'. For the net angular momentum to be zero we need

$$\begin{aligned} \overbrace{m'\omega^2 h'c}^{\dot{\vec{H}}\text{due to } m'\text{s}} &= \overbrace{m\omega^2 hb}^{\dot{\vec{H}}\text{due to } m\text{s}} \\ \Rightarrow \quad m'\,h'\,c &= m\,h\,b \end{aligned}$$

  Thus, we have the freedom ② of selecting any combination of m', h' and c to give the required product. Here are two examples:

  a) Let $m' = m/2$, $c = b/2$, then $h' = \frac{mhb}{m'c} = 4h$.

  b) Let $m' = m/2$, $c = 2b$, then $h' = \frac{mhb}{m'c} = h$.

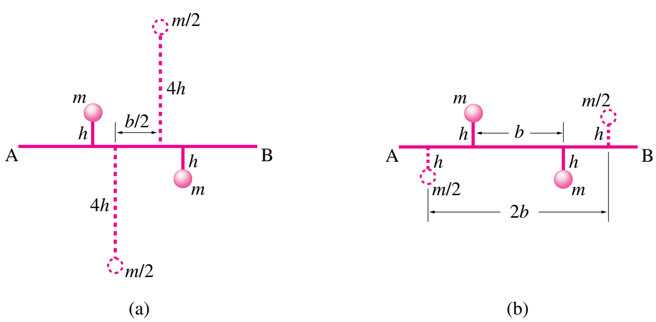


Figure 20.65: Dynamic balancing of the rotor by adding two masses. (a) $m' = m/2$, $c = b/2$ and $h' = 4h$. (b) $m' = m/2$, $c = 2b$ and $h' = h$.

① Balancing a rotor by adding two masses in two selected transverse (to the shaft axis) planes is quite a general method. Even if the rotor has more than two spinning masses the net angular momentum due to all masses can be reduced to the angular momentum produced by an equivalent system with just two masses such as the one under consideration.

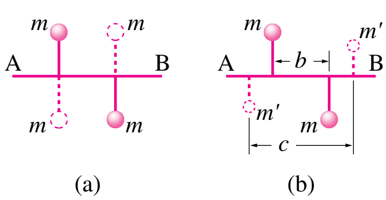


Figure 20.64: Dynamic balancing of the rotor by adding two masses. (a) The two added masses (shown by dotted circles) are the same as the original masses on the shaft. (b) The two added masses are different from the original masses ($m' \neq m$).

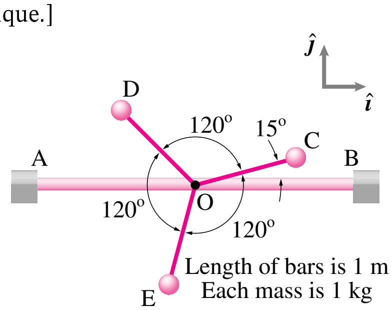
② In practice, this freedom is usually restricted by geometric and space constraints.

20.5 Dynamic balance

20.5.1 Is $\vec{\omega} = (2\hat{\imath} + 5\hat{k})$ rad/s an eigenvector of $[I^{\mathrm{cm}}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ kg m^2?

20.5.2
Find the eigenvalues and eigenvectors of $[I^{\mathrm{cm}}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 10 \end{bmatrix}$ kg m^2.

20.5.3 Three equal length (1 m) massless bars are welded to the rigid shaft AB which spins at constant rate $\vec{\omega} = \omega\hat{\imath}$. Three equal masses (1 kg) are attached to the ends of the bars. At the instant of interest the masses are all in the xy plane as shown. Where should what masses be added to make the system shown dynamically balanced? [The solution is not unique.]



Problem 20.5.3

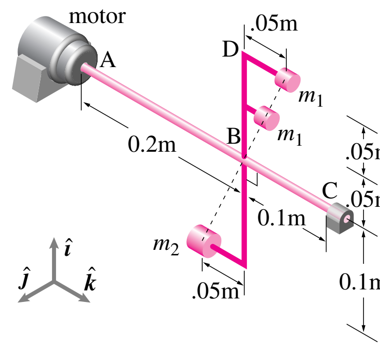
20.5.4 Dynamic balance of a system of particles. Three masses are mounted on a massless shaft AC with the help of massless rigid rods. Mass $m_1 = 1$ kg and mass $m_2 = 1.5$ kg. The shaft rotates with a constant angular velocity $\vec{\omega} = 5$ rad/s$\hat{k}$. At the instant shown, the three masses are in the xz-plane. Assume frictionless bearings and ignore gravity.

a) Find the net reaction force (the total of all the forces) on the shaft. Is the system statically balanced?

b) Find the rate of change of angular momentum $\dot{\vec{H}}_{/A}$, using $\dot{\vec{H}}_{/A} = \sum_i \vec{r}_{i/A} \times m_i \vec{a}_i$.

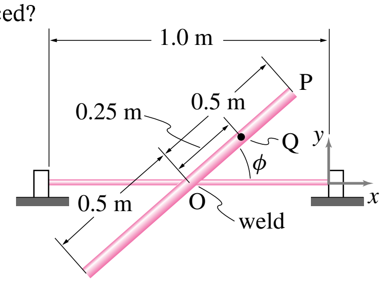
c) Find the angular momentum about point A, $\vec{H}_{/A}$. Use this intermediate result to find the rate of change of angular momentum $\dot{\vec{H}}_{/A}$, using $\dot{\vec{H}}_{/A} = \vec{\omega} \times \vec{H}_{/A}$ and verify the result obtained in (b).

d) What is the value of the resultant unbalanced torque on the shaft? Propose a design to balance the system by adding an appropriate mass in an appropriate location.



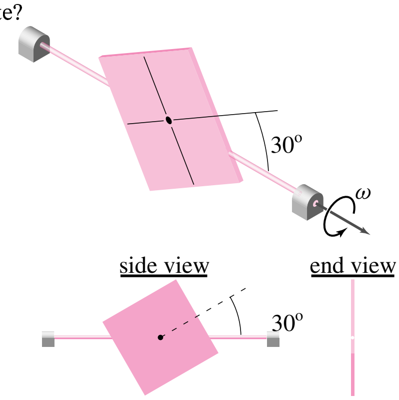
Problem 20.5.4

20.5.5 For what orientation of the rod ϕ is the system shown dynamically balanced?



Problem 20.5.5

20.5.6 A uniform square plate with 1 m sides and 1 kg mass is mounted (as shown) with a shaft that is in the plane of the square but not parallel to the sides. What is the net force and moment required to maintain rotation at a constant rate?



Problem 20.5.6

20.5.7 An equilateral uniform triangular plate with sides of length ℓ plate spins at constant rate ω about an axis through its center of mass. The axis is coplanar with the triangle. The angle ϕ is arbitrary. When $\phi = 0°$, the base of the triangle is parallel to the axis of rotation. Find the net force and moment required to maintain rotation of the plate at a constant rate for any ϕ. For what angle ϕ is the plate dynamically balanced?

Problem 20.5.7

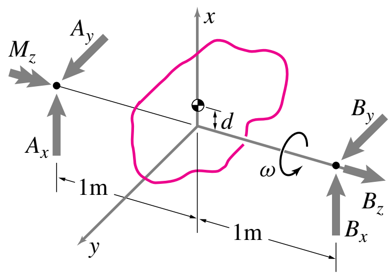
20.5.8 An abstract problem. This problem concerns a system rigidly attached to a massless rod spinning about a fixed axis at a fixed rate. That is, whether or not the system is a rigid body, it is moving as if it were a rigid body. Every point in the system is moving in circles about the same axis at the same number of radians per second. The position of a particle of the system relative to the axis is $\vec{R}$. The position relative to some point C is $\vec{r}_C$. In all cases, the body rotates about the axis shown. The total mass of all the objects is M_{tot}. Point O is at the center of mass of the system and is at a distance d from the rotation axis and is on the x-axis at the moment of interest. Assume there is no gravity. At the point A, the rod is free to slide in the z-direction. But, at point B, the rod is constrained from moving in the z-direction. The FBD of the system is shown in the figure. Referring to the free-body diagram, there are six scalar unknown reactions in this 3 dimensional problem.

a) **The general case:** Imagine that you know $\dot{\vec{H}}_{/O} = \dot{H}_{0x}\hat{\imath} + \dot{H}_{0y}\hat{\jmath} + \dot{H}_{0z}\hat{k}$ and also $\dot{\vec{L}} = \dot{L}_x\hat{\imath} + \dot{L}_y\hat{\jmath} + \dot{L}_z\hat{k}$.

How would you find the six unknown reactions A_x, A_y, M_{Az}, B_x, B_y, B_z? [Reduce the problem to one of solving six equations in six unknowns.]

b) **The General Result:** Is M_{Az} necessarily equal to zero? Prove your result or find a counter-example. *

c) **Static Balance:**. If $d = 0$ is it true that all reaction forces are equal to zero? Prove your result or find a counter example. *



Problem 20.5.8

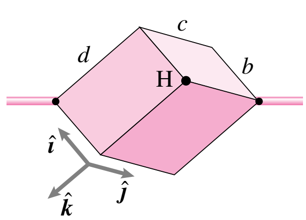
20.5.9 A uniform block spins at constant angular velocity. The magnitude of the angular velocity is ω. The center of mass G is on the axis of rotation. The $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ directions are parallel to the sides with lengths b, c and d, respectively.
Answer in terms of some or all of ω, m, b, c, d, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

a) Find $\vec{v}_H$.

b) Find $\vec{a}_H$.

c) What is the *sum* of all the moments about point G of all the reaction forces at A and B?

d) Are there any values of b, c, and d for which this block is dynamically balanced for rotation about an axis through a pair of corners? Why?



Problem 20.5.9

20.5.10 Dynamic Balance. Consider each of the objects shown in turn (so to speak), each spinning about the z axis which is horizontal. In each case the center of mass is half way between the bearings (not shown). Gravity is in the $-\hat{\jmath}$ direction. State whether it is possible to spin the object at constant rate with constant bearing reactions of $mg\hat{\jmath}/2$ *and* no other applied torques ('yes') or might not be ('no'). Give a brief explanation of your answer (enough to distinguish an informed answer from a guess).

a) Axis along centerline of a uniform rectangular plate.

b) Axis orthogonal to uniform rectangular plate and through its centroid.

c) Axis orthogonal to uniform rectangular plate and *not* through its centroid.

d) Axis orthogonal to uniform but totally irregular shaped plate and through its centroid.

e) Axis along the diagonal of a uniform rhombus.

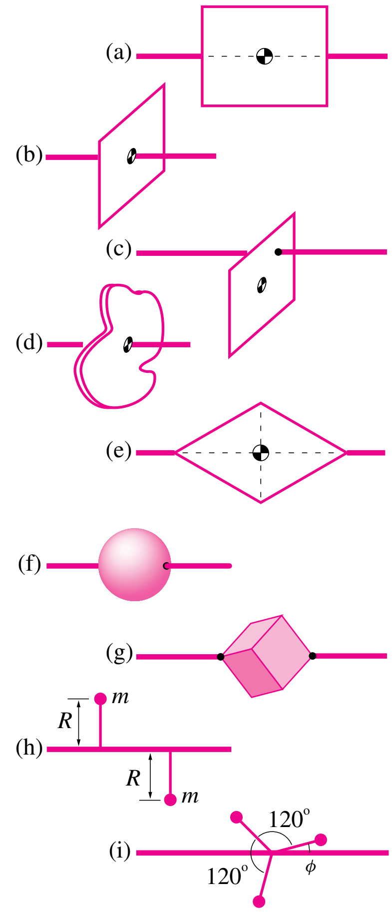
f) Axis through the center of a uniform sphere.

g) Axis through the diagonal of a uniform cube.

h) Axis with two equal masses attached as shown.

i) Axis with three equal masses attached as shown, all in a common plane with the axis, with ϕ not equal to 30°.

j) An arbitrary rigid body which has center of mass on the axis and two of its moment-of-inertia eigenvectors perpendicular to the axis. (no picture)



Problem 20.5.10

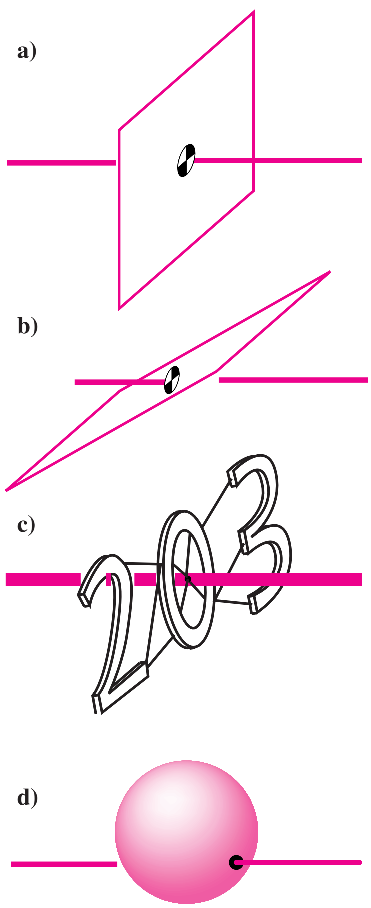
20.5.11 Static and Dynamic Balance A series of bodies, each of uniform density and each with total mass m, rotate at a constant angular speed ω about a fixed horizontal axis. Ignore gravity. For each body state whether the body is (i) *statically* balanced and whether it is (ii) *dynamically* balanced. Give clear arguments using words or equations to support your claims. (iii, iv) For each body you must *either* (a) add one point mass m or (b) add two point masses each of mass $m/2$ (your choice) that *maintain* static and dynamic balance if they are balanced, or that *make* the bodies statically and dynamically balanced. Justify your placement with words and/or equations. The masses need not be added to the bodies, but could be attached off the bodies by structures with negligible mass. [Hint: none of the placements are unique. You may draw a side view if that helps clarify your placement.]

a) A rectangular plate (height h, length ℓ) mounted with the axle perpendicular to the plate and through its center.

b) The same plate as in (a) above but mounted at an angle $\phi \neq \pi/2$ from the shaft.

c) The numerals '203' cut out of a plate and connected by massless rods. Each letter has mass $m/3$ and the three center-of-mass points of the individual letters are colinear and equally spaced. The shaft goes through the center of the '0' and is perpendicular to the plane of the letters.

d) A sphere with radius R where the shaft passes a distance $d < R$ from the center.

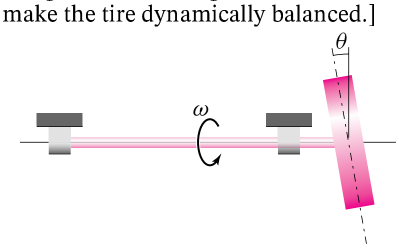


Problem 20.5.11

20.5.12 Dynamic tire balancing. Where should you place what weights in order to dynamically balance the tire?

Assume that a car tire can be modeled as a uniform disk of radius one foot and mass 20 lbm. Assume the tire has been mounted crooked by one degree (the plane of the disk makes an 89° angle with the car axle). This crooked wheel is *statically* balanced since its center of mass is on the axis of rotation. But it is dynamically unbalanced since it wobbles and will wobble the car. Car shops put little weights on unbalanced wheels to balance them. They figure out where to put the weights by using a little machine that measures the wheel wobble. But you don't need such a machine for this problem because you have been told where all the mass is located.

There are many approaches to solving this problem and there are also many correct solutions. You should find any correct solution. [one approach: guess a good location to put two masses and then figure out how big they have to be to make the tire dynamically balanced.]



Problem 20.5.12