SAMPLE 20.19 Static and dynamic balance in 2-D motion. A system with two equal masses at A and B rotates about point O at a constant rate $\omega = 10\,\text{rpm}$. The system is shown in the figure. The rod connecting the two masses has negligible mass.

1. Is the system statically balanced? If not, suggest a way to balance it.
2. Is the system dynamically balanced? If not, suggest a way to balance it.

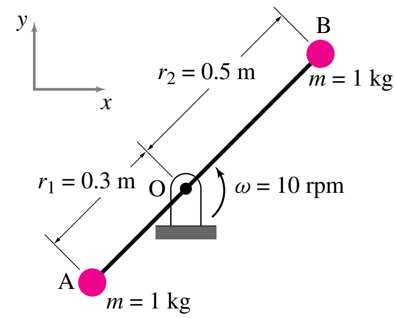

Figure 20.56

Solution

1. Since the two masses are equal, the center-of-mass of the system is at the geometric center of rod AB, i.e., at a point O′, halfway between A and B. Clearly, O′ is not on the axis of rotation which passes through O and is perpendicular to the plane of motion. Therefore, the system is *not* statically balanced.

   To balance the system, we must move the center-of-mass to O or pivot the system at O′. Say we cannot move the pivot point. So, to move the center-of-mass O, we add mass m' to m at A. From the definition of center-of-mass:

$$
\begin{aligned}
(m+m')r_1 &= mr_2 \\
\Rightarrow \qquad m' &= \frac{m'(r_2-r_1)}{r_1} \\
&= \frac{1\,\text{kg}(0.5\,\text{m}-0.3\,\text{m})}{0.3\,\text{m}} \\
&= 0.67\,\text{kg}.
\end{aligned}
$$

Add 0.67 kg to mass at A.

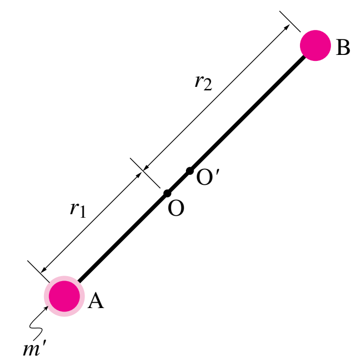

Figure 20.57

2. The system, as given, is *not* dynamically balanced since it is not statically balanced. Let us check if it is dynamically balanced after adding m' to A as suggested above.

$$\sum \vec{\boldsymbol{M}}_{\text{O}} = \dot{\vec{\boldsymbol{H}}}_{/\text{O}}.$$

   Let us calculate $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$ to see if it is zero:

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\text{O}} &= \vec{\boldsymbol{r}}_1 \times m_1\vec{\boldsymbol{a}}_1 \quad + \quad \vec{\boldsymbol{r}}_2 \times m_2\vec{\boldsymbol{a}}_2 \\
&= (-r_1\hat{\boldsymbol{e}}_R)\times m_1(\omega^2 r_1\hat{\boldsymbol{e}}_R) \quad + \quad (r_2\hat{\boldsymbol{e}}_R)\times(-\omega^2 r_2\hat{\boldsymbol{e}}_R)m_2 \\
&= -m_1\omega^2 r_1^2\underbrace{(\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_R)}_{\vec{\boldsymbol{0}}} \quad - \quad m_2\omega^2 r_2^2\underbrace{(\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_R)}_{\vec{\boldsymbol{0}}} \\
&= \vec{\boldsymbol{0}}.
\end{aligned}
$$

   The net torque on the system is zero, therefore it is dynamically balanced. Note that $\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{0}}$ irrespective of the values of m_1, m_2, r_1, and r_2. Thus $\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{0}}$ even for the system as given. But the system, as given, is not dynamically balanced because *static balance is a necessary condition for dynamic balance.*

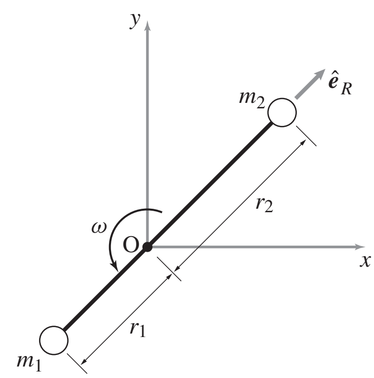

Figure 20.58

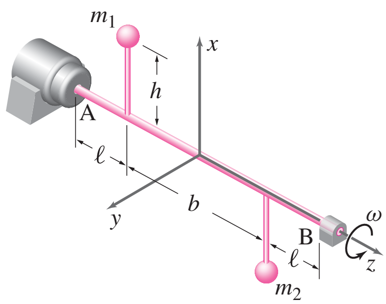

Figure 20.59: Imbalance of a rotor due to point masses spinning at a constant rate.

SAMPLE 20.20 **Imbalance of a rotor due to rotating masses.** Two masses m_1 and m_2 are attached to a massless shaft AB by massless rigid rods of length h each. The two masses are separated by distance b along the shaft and are in the same plane but on the opposite sides of the shaft. The shaft is rotating with a constant angular speed ω. The shaft is free to move along the z-axis at point B. Ignore gravity.

1. Is the system *statically balanced*?
2. What is the torque required to keep the motion going about the z-axis (i.e., M_z) ?
3. What are the reactions at the support points of the shaft?

Solution

1. A simple line sketch and the free-body diagram of the system are shown in Fig. 20.60. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the system gives:

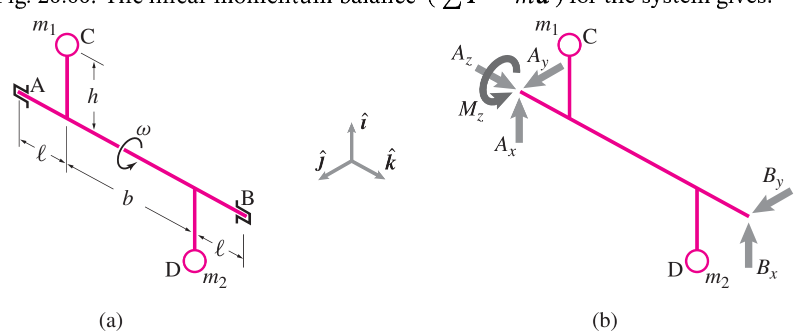

Figure 20.60: (a) A simple line diagram of the system. (b) Free-Body Diagram of the system.

$$
\begin{aligned}
(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + A_z\hat{k} &= m_1\vec{a}_1 + m_2\vec{a}_2 \\
&= m_1\omega^2 h(-\hat{\imath}) + m_2\omega^2 h\hat{\imath} \\
\Rightarrow \quad A_x + B_x &= \omega^2 h(m_2 - m_1) \qquad (20.51) \\
A_y + B_y &= 0 \qquad (20.52) \\
A_z &= 0
\end{aligned}
$$

Clearly, $\sum \vec{F} \neq \vec{0}$ which means the system is not statically balanced.

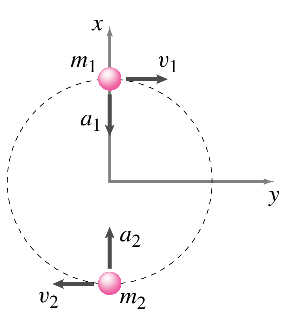

Figure 20.61: Accelerations of m_1 and m_2.

2. M_z can be easily calculated by writing angular momentum balance about axis AB. In fact, we do not even need to write the equation in this case; since all the forces pass through this axis and the accelerations of m_1 and m_2 also pass through it,

$$\sum M_{/AB}(\text{due to reaction forces}) = 0 \quad \text{and} \quad \dot{H}_{/AB} = 0.$$

But

$$
\begin{aligned}
\sum M_{/AB} &= \dot{H}_{/AB} \\
\text{or} \quad M_z &+ \sum M_{/AB}(\text{due to reaction forces}) = \dot{H}_{/AB} \\
\Rightarrow \quad M_z &= 0.
\end{aligned}
$$

$$\boxed{M_z = 0}$$

3. For the four unknown reactions at A and B (we have already found A_z and M_z) we have two scalar equations so far (from Linear Momentum Balance). Angular Momentum Balance about point A gives:

$$\sum \vec{\boldsymbol{M}}_{/A} = \dot{\vec{\boldsymbol{H}}}_{/A}$$

Now,

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/A} &= \vec{\boldsymbol{r}}_{B/A} \times \sum \vec{\boldsymbol{F}}_B \\
&= (2\ell + b)\hat{\boldsymbol{k}} \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}}) \\
&= B_x(2\ell + b)\hat{\boldsymbol{\jmath}} - B_y(2\ell + b)\hat{\boldsymbol{\imath}} \\
\dot{\vec{\boldsymbol{H}}}_{/A} &= \vec{\boldsymbol{r}}_{C/A} \times m_1 \vec{\boldsymbol{a}}_1 + \vec{\boldsymbol{r}}_{D/A} \times m_2 \vec{\boldsymbol{a}}_2 \\
&= (\ell \hat{\boldsymbol{k}} + h\hat{\boldsymbol{\imath}}) \times m_1(-\omega^2 h\hat{\boldsymbol{\imath}}) + ((\ell + b)\hat{\boldsymbol{k}} - h\hat{\boldsymbol{\imath}}) \times m_2(\omega^2 h\hat{\boldsymbol{\imath}}) \\
&= \omega^2 h(m_2(\ell + b) - m_1 \ell)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Equating $\sum \vec{\boldsymbol{M}}_{/A}$ and $\dot{\vec{\boldsymbol{H}}}_{/A}$ we get

$$B_x(2\ell + b)\hat{\boldsymbol{\jmath}} - B_y(2\ell + b)\hat{\boldsymbol{\imath}} = \omega^2 h(m_2(\ell + b) - m_1 \ell)\hat{\boldsymbol{\jmath}}$$

Dotting both sides of the equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, we get

$$\begin{aligned}
B_y &= 0 \\
\text{and} \qquad B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1 \ell].
\end{aligned}$$

Substituting B_x and B_y in (20.51) and (20.52) we find

$$\begin{aligned}
A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2 \ell - m_1(\ell + b)] \\
A_y &= 0
\end{aligned}$$

$$\boxed{\begin{aligned}
A_x &= \tfrac{\omega^2 h}{(2\ell + b)}[m_2 \ell - m_1(\ell + b)] \\
A_y &= A_z = 0 \\
B_x &= \tfrac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1 \ell] \\
B_y &= 0 \\
M_z &= 0
\end{aligned}}$$

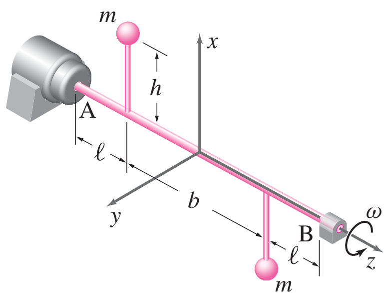

Figure 20.62: Imbalance of a rotor due to point masses spinning at a constant rate.

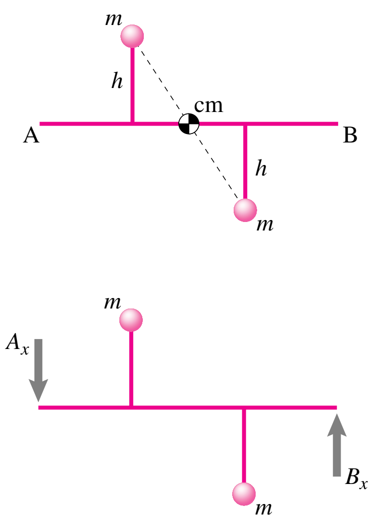

Figure 20.63: Reaction forces at A and B are required to keep the motion going.

SAMPLE 20.21 **Balancing a rotor with spinning masses.** Consider the same system as in the previous sample problem (Sample 20.20). Assume that $m_1 = m_2 = m$.

1. Set $m_1 = m_2$ in the reactions calculated. Is the system statically balanced now? Explain.
2. Is the system *dynamically balanced*? Explain.
3. Balance the system dynamically by (a) adjusting the geometry, (b) adding two masses to the system.

Solution

1. Recall from the solution of Sample 20.20 that

$$\begin{aligned} A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2\ell - m_1(\ell + b)] \\ B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1\ell] \\ A_y &= B_y = A_z = M_z = 0 \end{aligned}$$

Setting $m_1 = m_2 = m$ in the above expressions for the reactions, we get

$$A_x = \frac{\omega^2 h}{(2\ell + b)}(-mb) \quad \text{and} \quad B_x = \frac{\omega^2 h}{(2\ell + b)}(mb).$$

Thus, $A_x + B_x = 0$. Therefore, $\sum \vec{F} = \vec{0}$ which means the system is statically balanced.

Static Balance: When the two masses are equal, the center-of-mass of the system is on the axis of rotation. Therefore, the static reactions are zero irrespective of the orientations of the two masses.

The requirement for static balance is $\sum \vec{F} = \vec{0}$ in any static orientation of the system. As long as the center-of-mass of the system lies on the axis of rotation, the system will be in static balance.

Dynamic Balance: When the masses are rotating at constant speed, reaction forces at A and B are required to keep the motion going. Clearly, $\sum \vec{M} \neq \vec{0}$ now. Therefore, the system is not in dynamic balance.

At the instant shown the reaction forces are A_x and B_x as shown in Fig. 20.63, but they change directions as the position of the masses changes. For example, if the masses are in the y–z plane the reaction forces will be A_y and $-B_y$. The magnitudes of these forces will remain the same. Thus, we have rotating reaction forces at A and B. These rotary forces cause wear in the bearings and induce vibrations in the frame of the machine and the supporting structure. Therefore, these forces are undesirable.

If we can somehow make these *dynamic reactions* zero, the bearings, the machine frame, the supporting structure, the machine operator and the company will all be very happy. So, how do we do it? Read on.

2. There are many ways in which the rotor can be dynamically balanced:
   - **Trivial solution:** Remove both the masses. Of course, there is nothing left to produce any non-zero $\dot{\vec{H}}$. Thus,

$$\sum \vec{M}_{/\text{any point}} = \vec{0}. \quad \text{Also, } \sum \vec{F} = \vec{0}$$

.

- **Adjust geometry:** Set $b = 0$. The center-of-mass is still on the axis of rotation.

$$\Rightarrow \quad \sum \vec{F} = \vec{0}$$

In addition,

$$\dot{\vec{H}}_{/A} = \vec{0}$$
$$\Rightarrow \quad \sum \vec{M}_{/A} = \vec{0}.$$

- **Add two masses:** If b is required to be non-zero, we can balance the rotor by adding two masses in any two selected transverse (to the shaft-axis) planes to make $\dot{\vec{H}}_{/A} = \vec{0}$. ① For example:
  - We can take two equal masses m and m, and place them in the opposite directions of the masses already on the shaft as shown in Fig. 20.64(a). It should be clear that the net angular momentum about any point on the shaft will be zero now.
  - We can add two masses m' and m' different from the masses attached to the shaft and place them a distance c apart on the shaft. Let the length of the connecting rods of the new masses be h'. For the net angular momentum to be zero we need

$$\overbrace{m'\omega^2 h' c}^{\dot{\vec{H}}\text{due to } m'\text{s}} = \overbrace{m\omega^2 hb}^{\dot{\vec{H}}\text{due to } m\text{s}}$$
$$\Rightarrow \quad m'\,h'\,c = m\,h\,b$$

Thus, we have the freedom ② of selecting any combination of m', h' and c to give the required product. Here are two examples:

a) Let $m' = m/2, \ c = b/2, \quad$ then $h' = \frac{mhb}{m'c} = 4h$.

b) Let $m' = m/2, \ c = 2b, \quad$ then $h' = \frac{mhb}{m'c} = h$.

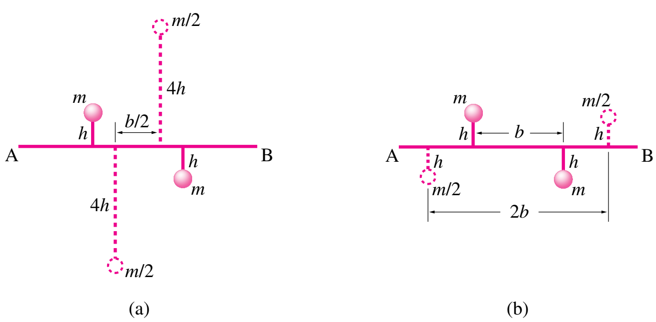

Figure 20.65: Dynamic balancing of the rotor by adding two masses. (a) $m' = m/2$, $c = b/2$ and $h' = 4h$. (b) $m' = m/2$, $c = 2b$ and $h' = h$.

① Balancing a rotor by adding two masses in two selected transverse (to the shaft axis) planes is quite a general method. Even if the rotor has more than two spinning masses the net angular momentum due to all masses can be reduced to the angular momentum produced by an equivalent system with just two masses such as the one under consideration.

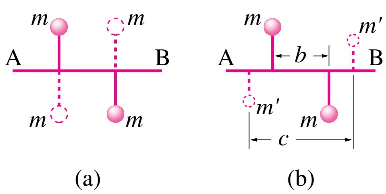

Figure 20.64: Dynamic balancing of the rotor by adding two masses. (a) The two added masses (shown by dotted circles) are the same as the original masses on the shaft. (b) The two added masses are different from the original masses ($m' \neq m$).

② In practice, this freedom is usually restricted by geometric and space constraints.