20.4 Mechanics using the moment of inertia matrix

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from the inside cover. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. Linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center-of-mass of the system of interest. One would like something so simple for the calculation of angular momentum.

We are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies, the scalar moment of inertia from Chapter 7 is all we need. The luck is not so great for 3-D rigid bodies but still there is some simplification①.The simplification is to use the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$ from the previous section. One may have to do a sum or integral to find $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\mathrm{cm}}]$ if an adequate table is not handy. But once you know $[\boldsymbol{I}^{\mathrm{cm}}]$, say, you need not work with the integrals to evaluate angular momentum and its rate of change. Assuming that you are comfortable calculating and looking-up moments of inertia, we proceed to use it for the purposes of studying mechanics.

①For general motion of *non*-rigid bodies, a topic not covered in this book, the simplification for linear momentum still holds; linear momentum and its rate of change are given by the system mass times the velocity and acceleration of the center-of-mass. But there is no general simplification for the sums needed to evaluate angular momentum and energy or their rates of change.

Let's now consider again the conically swinging rod of fig. 20.18 on page 1081. Method 1 for evaluating $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ was evaluating the sums directly. If we accept the formulae presented for rigid bodies in Table I at the back of the book, we can find all of the motion quantities by setting $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{0}}$.

Method 2 of evaluating $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$: using the xyz moment of inertia matrix about point O

In section 20.1 we examined the conical swinging of a straight uniform rod. Now let's look at that example again using its moment of inertia matrix. For a rigid body in constant rate circular motion about an axis through O,

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}}$$

because the $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ vector rotates with the body. For a rigid body rotating about point O,

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}}.$$

We assumed at the outset that

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}.$$

So, the only trick is to find $[\boldsymbol{I}^{\text{O}}]$ for a rod in the configuration shown. We recall, look up, or believe for now that

$$[\boldsymbol{I}^{\text{O}}] = \begin{bmatrix} I^O_{xx} & I^O_{xy} & I^O_{xz} \\ I^O_{yx} & I^O_{yy} & I^O_{yz} \\ I^O_{zx} & I^O_{zy} & I^O_{zz} \end{bmatrix}.$$

$$\begin{aligned}
I^O_{xy} &= -\int \underbrace{x}_{0}\, y\, dm = 0 && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
I^O_{xz} &= -\int \underbrace{x}_{0}\, z\, dm = 0 && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
I^O_{yy} &= \int (\underbrace{x^2}_{0} + z^2) dm && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
&= \int_0^\ell \underbrace{(-s \cdot \cos\phi)^2}_{z} \rho\, ds \\
&= \cos^2\phi\, \rho \frac{\ell^3}{3} = \cos^2\phi\, m \frac{\ell^2}{3} \\
I^O_{zz} &= \int (\underbrace{x^2}_{0} + y^2) dm && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
&= \int \underbrace{(s \cdot \sin\phi)^2}_{y} \rho\, ds \\
&= \sin^2\phi\, \rho \frac{\ell^3}{3} = \sin^2\phi\, m \frac{\ell^2}{3} \\
I^O_{xx} &= \int (y^2 + z^2) dm && (\text{both integrals have been evaluated above}) \\
&= I^O_{yy} + I^O_{zz} && (\text{perpendicular axis theorem}) \\
&= (\cos^2\phi + \sin^2\phi) m \frac{\ell^2}{3} \\
&= m \frac{\ell^2}{3} \\
I^O_{yz} &= -\int yz\, dm \\
&= -\int (s \cdot \sin\phi)(-s \cdot \cos\phi)\, \rho\, ds \\
&= \sin\phi \cos\phi \cdot \rho \frac{\ell^3}{3} = \sin\phi \cos\phi \cdot m \frac{\ell^2}{3}.
\end{aligned}$$

Putting these terms all together in the matrix, we get

$$[\boldsymbol{I}^{\rm O}] = m\frac{\ell^2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi\sin\phi \\ 0 & \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}.$$

Now we can calculate $\vec{\boldsymbol{H}}_{/\rm O}$ as

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\rm O} &= [\boldsymbol{I}^{\rm O}]\cdot\vec{\boldsymbol{\omega}} \\
&= m\frac{\ell^2}{3}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi\sin\phi \\ 0 & \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}}_{[\boldsymbol{I}^{\rm O}]}\cdot\underbrace{\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}}_{\vec{\boldsymbol{\omega}}} \\
&= m\frac{\ell^2}{3}\begin{bmatrix} 0 \\ \omega\cos\phi\sin\phi \\ \omega\sin^2\phi \end{bmatrix} \\
&= m\omega\frac{\ell^2}{3}\sin\phi(\cos\phi\,\hat{\boldsymbol{\jmath}} + \sin\phi\,\hat{\boldsymbol{k}}).
\end{aligned}$$

You may notice, by the way, that for this problem, $\vec{\boldsymbol{H}}_{/\rm O}$ (in the direction of $\cos\phi\,\hat{\boldsymbol{\jmath}} + \sin\phi\,\hat{\boldsymbol{k}}$) is perpendicular to the rod (in the direction of $\sin\phi\,\hat{\boldsymbol{\jmath}} - \cos\phi\,\hat{\boldsymbol{k}}$). So $\vec{\boldsymbol{H}}_{/\rm O}$ is *not* in the direction of $\vec{\boldsymbol{\omega}}$. ②

②$\vec{\boldsymbol{H}}_{/\rm O}$ is only parallel to $\vec{\boldsymbol{\omega}}$ if rotation is about one of the principal axes (eigenvector directions) of the inertia matrix.

Now we calculate $\dot{\vec{\boldsymbol{H}}}_{/\rm O}$ as

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\rm O} &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{H}}_{/\rm O} \\
&= (\omega\hat{\boldsymbol{k}})\times\left[\omega m\frac{\ell^2}{3}(\cos\phi\sin\phi\,\hat{\boldsymbol{\jmath}} + \sin^2\phi\,\hat{\boldsymbol{k}})\right] \\
&= -m\frac{\ell^2}{3}\cos\phi\sin\phi\,\omega^2\hat{\boldsymbol{\imath}},
\end{aligned}$$

the same result we got before.

Method 3 of calculating $\dot{\vec{\boldsymbol{H}}}_{/\rm O}$: using $[I^{\rm O}]$ and a coordinate system lined up with the rod

Here, as suggested above, we redo the problem using a rotated set of axes better aligned with the rod. This method makes calculation of the moment of inertia matrix quite a bit easier — we can even look it up in a table — but makes the determination of $\vec{\boldsymbol{\omega}}$ a little harder. We are stuck finding the components of $\vec{\boldsymbol{\omega}}$ in a rotated coordinate system. Referring to the table of moment of inertias on the inside of the back cover and taking care because different coordinates are used, the moment of inertia matrix about point O for a thin uniform rod, in terms of the rotated coordinates

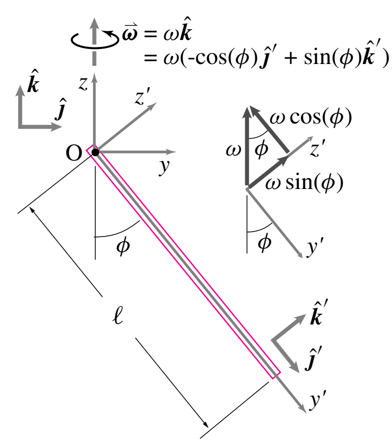


Figure 20.36: The spherical pendulum using $x'y'z'$ axes aligned with the rod.

is

$$[\boldsymbol{I}^{\mathrm{O}}]_{x'y'z'} = m\frac{\ell^2}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

and the angular velocity in rotated coordinates is $\vec{\boldsymbol{\omega}} = -\omega\cos\phi\,\hat{\boldsymbol{\jmath}}' + \omega\sin\phi\hat{\boldsymbol{k}}'$, which can be written in component form as

$$[\vec{\boldsymbol{\omega}}]_{x'y'z'} = \omega\begin{bmatrix}0\\ -\cos\phi\\ \sin\phi\end{bmatrix}.$$

We calculate the angular momentum about point O as

$$\overbrace{\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}}^{\text{vector equation, independent of coordinates}}$$

$$\underbrace{\begin{bmatrix}H_{/\mathrm{O}_{x'}}\\ H_{/\mathrm{O}_{y'}}\\ H_{/\mathrm{O}_{z'}}\end{bmatrix} = m\frac{\ell^2}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}_{x'y'z'}\cdot\,\omega\begin{bmatrix}0\\ -\cos\phi\\ \sin\phi\end{bmatrix}_{x'y'z'}}_{\text{all components in } x'y'z' \text{ coordinate system}}$$

$$\begin{bmatrix}H_{/\mathrm{O}_{x'}}\\ H_{/\mathrm{O}_{y'}}\\ H_{/\mathrm{O}_{z'}}\end{bmatrix} = m\frac{\ell^2}{3}\omega\begin{bmatrix}0\\ 0\\ \sin\phi\end{bmatrix}_{x'y'z'}.$$

Finally, we calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{H}}_{/\mathrm{O}}\\ &= [\omega(-\cos\phi\,\hat{\boldsymbol{\jmath}}' + \sin\phi\hat{\boldsymbol{k}}'] \times (\omega m\frac{\ell^2}{3}\sin\phi\hat{\boldsymbol{k}}')\\ &= -\sin\phi\cos\phi\; m\omega^2\frac{\ell^2}{3}\hat{\boldsymbol{\imath}}'.\end{aligned}$$

This answer is again the same as what we got before because the x and x' axis are coincident so $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$.

The multitude of ways to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$

We just showed three ways to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ for a conically swinging stick, but there are many more. You can get a sense of the possibilities by studying table I summarizing momenta and energy in the back of the book.

Here are three basic choices.

$$
\begin{aligned}
(1)\, \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm, \\
(2)\, \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}}, \qquad \text{or}
\end{aligned}
$$

$$
(3)\, \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{a}}_{cm} m_{tot} + \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}.
$$

Choice (3) is the safest choice to make if you are in doubt, since it is the only one of the three choices that does not depend on point O being a fixed point (which it was for this example). For option (2) above, we can calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ various ways as

$$
\begin{aligned}
(a)\, \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm \qquad \text{or,} \\
(b)\, \vec{\boldsymbol{H}}_{/\mathrm{O}} &= [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}} \qquad \text{or,} \\
(c)\, \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{v}}_{cm} m_{tot} + \vec{\boldsymbol{H}}_{\mathrm{cm}}.
\end{aligned}
$$

For option (3) above, we can calculate $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ as

$$
\begin{aligned}
(d)\, \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} &= \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}_{/cm} dm \qquad \text{or,} \\
(e)\, \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{cm}}.
\end{aligned}
$$

For (c) and (e) above, we can calculate $\vec{\boldsymbol{H}}_{\mathrm{cm}}$ as

$$
\begin{aligned}
(f)\, \vec{\boldsymbol{H}}_{\mathrm{cm}} &= \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}_{/cm}\, dm \qquad \text{or,} \\
(g)\, \vec{\boldsymbol{H}}_{\mathrm{cm}} &= [\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}.
\end{aligned}
$$

For either of options (b) or (g), we can calculate $[I]$ relative to the xyz axes (as we did for the second method on the previous pages) or relative to some rotated axes better aligned with the rod (as we did for the third method on the previous pages.)

As you can surmise from all the choices above, the list of options for the calculation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ is too long and boring to show here.

The pros and cons of these methods depend on the problem at hand. If you want to avoid integration you are pretty much stuck using either $[\boldsymbol{I}^{\mathrm{O}}]$ or $[\boldsymbol{I}^{\mathrm{cm}}]$ with (e) and (g) above. Avoiding the integrals depends on your having a table of moments of inertia (like table IV in the back of this book).

Energy of things going in circles at variable rate

The energy only depends on the speeds of the parts of a system, not their accelerations. So, as with constant rate motion,

$$\begin{aligned} E_{\mathrm{K}} &= \frac{1}{2}\int \underbrace{v^2}_{\vec{v}\cdot\vec{v}}\, dm \\ &= \frac{1}{2}\int (\vec{\omega}\times\vec{\boldsymbol{r}})\cdot(\vec{\omega}\times\vec{\boldsymbol{r}})\, dm \\ &= \frac{1}{2}\omega^2 \int R^2\, dm \end{aligned}$$

R is the distance from the axis to the mass

$$\begin{aligned} &= \frac{1}{2}\vec{\omega}\cdot[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega} \\ &= \underbrace{\frac{1}{2}\omega^2 I^o_{zz}} \end{aligned}$$

For rotation about the z axis the 9 terms in the matrix formula reduce to this one simple term.

The safest bet

The following are the most reliable (least prone to error) formulas for evaluating the motion quantities for a rigid body rotating about a fixed axes (they also apply to arbitrary motion of a rigid body).

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{cm/o}}\times m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} + [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega} \tag{20.38}$$

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{cm/o}}\times m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}} + \vec{\omega}\times\underbrace{[\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}}_{\vec{\boldsymbol{H}}_{\mathrm{cm}}} + [\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\omega}} \tag{20.39}$$

$$E_{\mathrm{K}} = \frac{1}{2}m_{\mathrm{tot}}v^2_{cm} + \vec{\omega}\cdot([\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}) \tag{20.40}$$

Please survey table I at the back of the book which summarizes the ways of evaluating momenta and energy.

The geometry of $\dot{\vec{H}}_{/\mathrm{O}}$

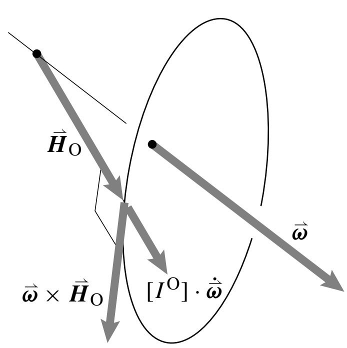


Figure 20.37: The two terms in the rate of change of angular momentum are shown.

It is possible to understand the formula for the change of angular momentum geometrically. Here is one way of looking at it. For this example it is most direct to look at $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ calculated using $[\boldsymbol{I}^{\mathrm{O}}]$, but the same discussion would work with $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ calculated using $[\boldsymbol{I}^{\mathrm{cm}}]$.

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \underbrace{\vec{\omega}\times\vec{\boldsymbol{H}}_{/\mathrm{O}}}_{\text{Contribution from rotation of } \vec{\boldsymbol{H}}_{/\mathrm{O}}} + \underbrace{[\boldsymbol{I}^{\mathrm{O}}]\cdot\dot{\vec{\omega}}}_{\text{Due to the changing length of } \vec{\boldsymbol{H}}_{/\mathrm{O}}}$$

The first term in the equation describes the rotation of the angular momentum vector. The second term describes its rate of change of length.

Since the body is spinning about a fixed axis, the orientation of the axis of rotation is not only fixed relative to the Newtonian frame, the room environment, but also relative to the body. At one instant of time we use a coordinate system to calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$. After the body has rotated a little, if we now used a coordinate system that rotated the same amount as the body, a coordinate system 'glued' to the body, we could calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ again.

This new calculation of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ will be almost identical to the calculation before the small rotation, however, because the moment of inertia matrix does not change in time relative to a coordinate system that moves with the body. Also, the coordinates of $\vec{\omega}$ will be unchanged except for possibly a multiplication by a constant because the direction of $\vec{\omega}$ doesn't change. Thus the only change of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ as represented in this rotated coordinate system, is a possible change in its length due to a change in the spinning rate.

But this new coordinate system is a rotated coordinate system. To find the actual change in $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ we need to take this rotation into account. To picture this rotation consider the special case when the rotation rate is constant. Then the vector $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is constant in the rotating coordinate system. That is, the vector $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ rotates with the body.

So the net change in $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is a change due to rotation of the body added to a change due to the change in the rotation rate. For small angular changes, the direction of the first term is the same as the tangent to the circle that is traced by the tip of the angular momentum vector $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ as drawn on the body. To approximate the first term, consider the following reasoning. First, let's denote the first term by $(\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}})_{\mathrm{rot}} = \vec{\omega}\times\vec{\boldsymbol{H}}_{/\mathrm{O}}$, where the subscript 'rot' indicates that this term is the contribution to $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ from the rotation of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$. For small angular changes, $(\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}})_{rot} = (d\vec{\boldsymbol{H}}_{/\mathrm{O}}/dt)_{rot} = \vec{\omega}\times\vec{\boldsymbol{H}}_{/\mathrm{O}} \approx (\Delta\vec{\boldsymbol{H}}_{/\mathrm{O}})_{rot}/\Delta t$. Thus, the term due to rotation is approximately, for small Δt, $(\Delta\vec{\boldsymbol{H}}_{/\mathrm{O}})_{rot} = \Delta t(\vec{\omega}\times\vec{\boldsymbol{H}}_0)$.

The contribution to $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ due to change of length of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is $[\boldsymbol{I}^{\mathrm{O}}]\cdot\dot{\vec{\omega}}$. Similarly, this term is approximately, for small Δt, $\Delta t([\boldsymbol{I}^{\mathrm{O}}]\cdot\dot{\vec{\omega}})$.

Rotation of a rigid body about a fixed axis: the general case

Consider a general rigid body of mass m and moment of inertia matrix with respect to the center-of-mass $[\boldsymbol{I}^{\text{cm}}]$ rotating about a fixed axis. Without loss of generality, let the axis of rotation be the $\hat{\boldsymbol{k}}$ axis.

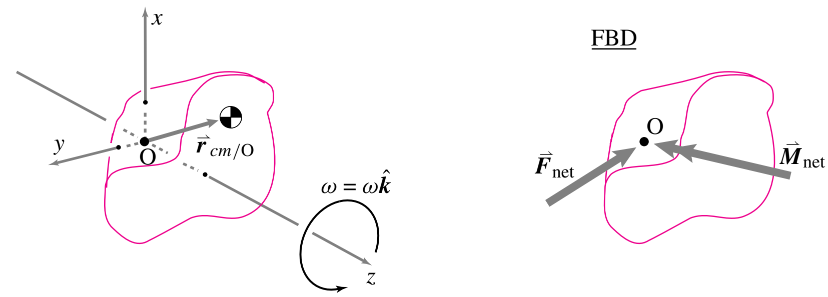
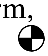


Figure 20.38: Free-body diagram of a general rigid body rotating about the z-axis.

Linear momentum balance

Referring to the free-body diagram of the rigid body, linear momentum balance gives

$$\begin{aligned}
\sum \vec{\boldsymbol{F}} &= \dot{\vec{\boldsymbol{L}}} \\
\vec{\boldsymbol{F}}_{net} &= m\vec{\boldsymbol{a}}_{cm} \\
&= m[\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}) + \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}].
\end{aligned} \quad (20.41)$$

The first term on the right hand side of equation 20.41, the centripetal term, is directed from the center-of-mass () through the axis of rotation; that is, it lies in the xy-plane (or has no $\hat{\boldsymbol{k}}$ component). The second term on the right hand side of equation 20.41, the tangential term, is normal to the plane determined by the axis and the center-of-mass. It is tangent to the circle that the center-of-mass travels on. It is zero if the center of mass is on the axis.

Angular momentum balance

Angular momentum balance about point O gives

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \\
\vec{\boldsymbol{M}}_{net} &= \underbrace{\vec{\boldsymbol{r}}_{cm/O} \times m\vec{\boldsymbol{a}}_{cm}}_{\text{term (i)}} \\
&\quad + \underbrace{\omega\hat{\boldsymbol{k}} \times \{[\boldsymbol{I}^{\mathrm{cm}}] \cdot \omega\hat{\boldsymbol{k}}\}}_{\text{term (ii)}} \\
&\quad + \underbrace{[\boldsymbol{I}^{\mathrm{cm}}] \cdot (\dot{\omega}\hat{\boldsymbol{k}})}_{\text{term (iii)}}. \qquad (20.42)
\end{aligned}
$$

Let's look at each of the three terms on the right hand side in turn.

term (i)

The first term (i) on the right hand side of equation 20.42 is

$$
\begin{aligned}
\text{term (i)} &= \vec{\boldsymbol{r}}_{cm/O} \times m\, \overbrace{\vec{\boldsymbol{a}}_{cm}}^{\boxed{\begin{aligned}\vec{\boldsymbol{a}}_{cm} &= \omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}) \\ &+ \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}\end{aligned}}} \\
&= m[\vec{\boldsymbol{r}}_{cm/O} \times (\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O})) \\
&\quad + \quad \vec{\boldsymbol{r}}_{cm/O} \times (\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O})]
\end{aligned}
$$

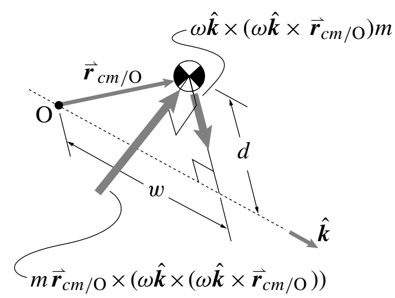


Figure 20.39: The first part of term (i)

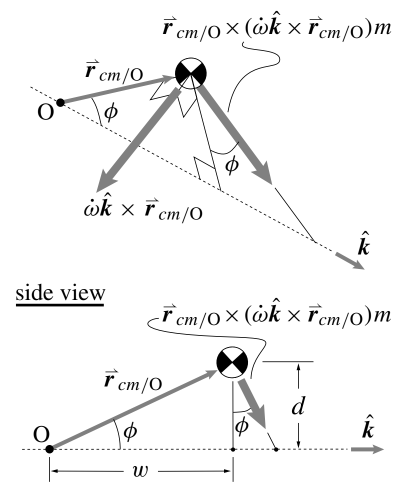


Figure 20.40: The second part of term (i)

Now, let's consider the two parts of term (i) in turn. The first part of term (i) is in the direction of $-\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}$. For example, if the center-of-mass is in the xz plane, this contribution to $\vec{\boldsymbol{M}}_{net}$ is in the $-\hat{\boldsymbol{\jmath}}$ direction and could be accommodated by reaction forces on the axis in the $\hat{\boldsymbol{\imath}}$ and $-\hat{\boldsymbol{\imath}}$ directions.
Now, the second part of term (i) can be decomposed into a part along the $\hat{\boldsymbol{k}}$ direction and a part perpendicular to $\hat{\boldsymbol{k}}$ (along d). The part along $\hat{\boldsymbol{k}}$ is

$$
\vec{\boldsymbol{r}}_{cm/O} \times (\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}) \cdot \hat{\boldsymbol{k}} = md^2\dot{\omega}\hat{\boldsymbol{k}}.
$$

The part of term (i) perpendicular to $\hat{\boldsymbol{k}}$ has magnitude $m\dot{\omega}dw$.

term (ii)

Now, let's look at the second term in equation 20.42, term (ii), and expand it.

$$
\begin{aligned}
\text{term (ii)} &= \omega\hat{\boldsymbol{k}} \times [\boldsymbol{I}^{\text{cm}}] \cdot \omega\hat{\boldsymbol{k}} \\
&= \omega\hat{\boldsymbol{k}} \times \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\
&= \omega\hat{\boldsymbol{k}} \times \left[I_{xz}^{cm}\omega\hat{\boldsymbol{\imath}} + I_{yz}^{cm}\omega\hat{\boldsymbol{\jmath}} + I_{zz}^{cm}\omega\hat{\boldsymbol{k}} \right] \\
&= I_{xz}^{cm}\omega^2\hat{\boldsymbol{\jmath}} - I_{yz}^{cm}\omega^2\hat{\boldsymbol{\imath}}
\end{aligned}
$$

So, the second term in equation 20.42, term (ii), has *no* $\hat{\boldsymbol{k}}$ component. The $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components are due to the off-diagonal terms in $[\boldsymbol{I}^{\text{cm}}]$. The off-diagonal terms are responsible for dynamic imbalance.

term (iii)

Now, the third term in equation 20.42, term (iii), is

$$
\begin{aligned}
[\boldsymbol{I}^{\text{cm}}] \cdot \dot{\omega}\hat{\boldsymbol{k}} &= \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix} \\
&= I_{xz}^{cm}\dot{\omega}\hat{\boldsymbol{\imath}} + I_{yz}^{cm}\dot{\omega}\hat{\boldsymbol{\jmath}} + I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}
$$

Putting terms (i), (ii), and (iii) back together

Now, let's put the three terms back together into the equation of angular momentum balance about point O, equation 20.42. To help with the interpretation, assume the center-of-mass is in the xz plane. So, we start with

$$\vec{\boldsymbol{M}}_{net} = \dot{\vec{\boldsymbol{H}}}_{/\text{O}}.$$

Breaking this vector equation into components, we get

$$
\begin{aligned}
M_{xx} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{\imath}} &= \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \cdot \hat{\boldsymbol{\imath}} \\
M_{yy} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{\jmath}} &= \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \cdot \hat{\boldsymbol{\jmath}} \\
M_{zz} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{k}} &= \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \cdot \hat{\boldsymbol{k}}.
\end{aligned}
$$

Now, using all the results so far, we find the torques about the x, y, and z axes. For the z-axis,

$$M_z = \underbrace{[m\,\overbrace{d^2}^{d\,=\,\text{distance of cm from axis of rotation}} + I_{zz}^{cm}]}_{\text{sometimes called `}[\boldsymbol{I}^{\text{O}}]\text{'}}\,\dot{\omega}. \qquad (20.43)$$

The only torque about the z-axis, the axis of rotation, is due to the acceleration about that axis.

Next, for the x-axis,

$$M_x = -I_{yz}^{cm}\omega^2 + I_{xz}^{cm}\dot{\omega} + m\dot{\omega}dw. \tag{20.44}$$

There is a torque about the x-axis due to off-diagonal terms in $[\boldsymbol{I}^{\mathrm{cm}}]$. One term is the dynamic imbalance term I_{yz}^{cm} and the other is associated with angular acceleration. (Recall, the center-of-mass is on the xz plane.) There is also a term due to the center of mass tangential acceleration since the center-of-mass is on a plane that does not contain point O.

Finally, the torque about the y-axis is

$$M_y = I_{xz}^{cm}\omega^2 + I_{yz}^{cm}\dot{\omega} + (-m\omega^2 dw). \tag{20.45}$$

Here, we have nearly the same types of terms as in equation 20.44. In this case, though, the centripetal acceleration causes the center-of-mass motion to contribute to the torque.

So, if

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} \text{ and } \dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{k}}$$

then

$$\begin{aligned}\vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \quad &= \quad \left(-I_{yz}^{cm}\omega^2 + (I_{xz}^{cm} + mdw)\dot{\omega}\right)\hat{\boldsymbol{\imath}} \\ &\quad +((I_{xz}^{cm} - mdw)\omega^2 + I_{yz}^{cm}\dot{\omega})\hat{\boldsymbol{\jmath}} \\ &\quad +(md^2 + I_{zz}^{cm})\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$