20.4 Mechanics using $[I^{cm}]$ and $[I^{O}]$

20.4.1 Find the angle between $\vec{\omega}$ and $\vec{H} = [I^{cm}]\vec{\omega}$, if $\vec{\omega} = 3\,\text{rad/s}\,\hat{k}$ and

$$[I^{cm}] = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{kg}\cdot\text{m}^2.$$

20.4.2 Find the projection of $\vec{H} = [I^{cm}]\vec{\omega}$ in the direction of $\vec{\omega}$, if $\vec{\omega} = (2\hat{\imath} - 3\hat{\jmath})\,\text{rad/s}$ and

$$[I^{cm}] = \begin{bmatrix} 5.4 & 0 & 0 \\ 0 & 5.4 & 0 \\ 0 & 0 & 10.8 \end{bmatrix} \text{lbm ft}^2.$$

20.4.3 Compute $\vec{\omega} \times \left([I^{cm}]\cdot\vec{\omega}\right)$ for the following two cases. In each case $\vec{\omega} = 2.5\,\text{rad/s}\,\hat{\jmath}$.

a) $$[I^{cm}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{kg m}^2, ^{*}$$

b) $$[I^{cm}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & -4 & 8 \end{bmatrix} \text{kg m}^2.$$

*

20.4.4 Find the z component of $\vec{H} = [I^{cm}]\vec{\omega}$ for $\vec{\omega} = 2\,\text{rad/s}\,\hat{k}$ and

$$[I^{cm}] = \begin{bmatrix} 10.20 & -2.56 & 0 \\ -2.56 & 10.20 & 0 \\ 0 & 0 & 14.00 \end{bmatrix} \text{lbm ft}^2.$$

20.4.5 For $\vec{\omega} = 5\,\text{rad/s}\hat{\lambda}$, find $\hat{\lambda}$ such that the angle between $\vec{\omega}$ and $\vec{H} = [I^{cm}]\cdot\vec{\omega}$ is zero if $[I^{cm}] = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 6 & 0 \\ 2 & 0 & 3 \end{bmatrix} \text{kg m}^2.$

20.4.6 Calculation of $\vec{H}_{/O}$ and $\dot{\vec{H}}_{/O}$. You will consider a disc spinning about various axes through the center of the disc O. For the most part, you are concerned with calculating $\dot{\vec{H}}_{/O}$. But, you should keep in mind that the purpose of the calculation might be to calculate reaction forces and moments. Every point on the disk moves in circles about the same axis at the same number of radians per second. These problems can be done at least three different ways: using the$[I]$'s from problem 20.3.6, direct integration of the definition of $\dot{\vec{H}}_{/O}$, and using the equivalent dumbbell (with the definition of $\dot{\vec{H}}_{/O}$). You should do them all three ways, first using the way you are most comfortable with.

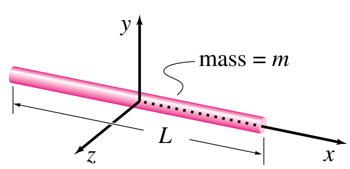
a) Assume that the configuration is as in problem 20.3.6(a) and that the disk is spinning about the z axis at the constant rate of $\omega\mathbf{k}$. What are $\vec{H}_{/O}$ and $\dot{\vec{H}}_{/O}$?

b) Assume that the configuration is as in problem 20.3.6(b) and that the disk is spinning about the z axis at the constant rate of $\omega\mathbf{k}$. What are $\vec{H}_{/O}$ and $\dot{\vec{H}}_{/O}$? (In this problem there are two ways to use [I]: one is in the xyz coordinate system, the other is in a coordinate system aligned nicely with the disk.)

20.4.7 What is the angular momentum $\vec{H}$ for:

a) A uniform sphere rotating with angular velocity $\vec{\omega}$ (radius R, mass m)?

b) A rod shown in the figure rotating about its center of mass with these angular velocities: $\vec{\omega} = \omega\hat{\imath}$, $\vec{\omega} = \omega\hat{\jmath}$, and $\vec{\omega} = \omega(\hat{\imath} + \hat{\jmath})/\sqrt{2}$?

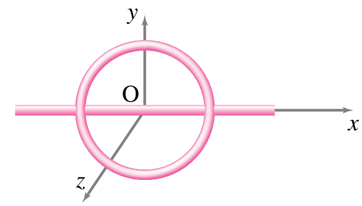


Problem 20.4.7

20.4.8 A hoop is welded to a long rigid massless shaft which lies on a diameter of the hoop. The shaft is parallel to the x-axis. At the instant of interest the hoop is in the xy-plane of a coordinate system which has its origin O at the center of the hoop. The hoop spins about the x-axis at 1 rad/s. There is no gravity.

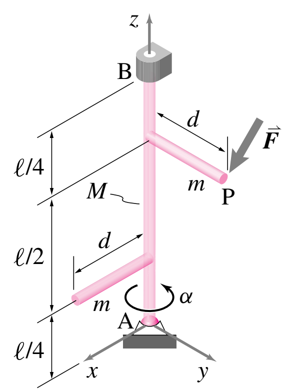
a) What is $[I^{cm}]$ of the hoop?

b) What is $\vec{H}_{/O}$ of the hoop?



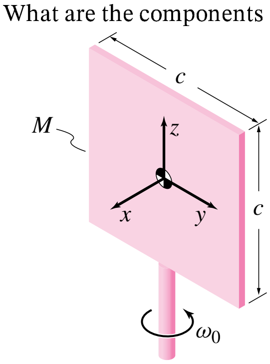
Problem 20.4.8

20.4.9 Two thin rods, each of mass m and length d, are welded perpendicularly to an axle of mass M and length ℓ, which is supported by a ball-and-socket joint at A and a journal bearing at B. A force $\vec{F} = -F\hat{\imath} + G\hat{k}$ acts at the point P as shown in the figure. Determine the initial angular acceleration α and initial reactions at point B.



Problem 20.4.9

20.4.10 The thin square plate of mass M and side c is mounted vertically on a vertical shaft which rotates with angular speed ω_0. In the fixed x, y, z coordinate system with origin at the center of mass (with which the plate is currently aligned) what are the vector components of $\vec{\omega}$? What are the components of $\vec{H}_{cm}$?



Problem 20.4.10

20.4.11 If you take any rigid body (with weight), skewer it with an axle, hold the axle with hinges that cause no moment about the axle, and hold it so that it cannot slide along the axle, it will swing like a pendulum. That is, it will obey the equations:

$$\ddot{\theta} = -C\sin\theta, \quad \text{or} \quad (a)$$
$$\ddot{\phi} = +C\sin\phi, \quad (b)$$

where

θ is the amount of rotation about the axis measuring zero when the center of mass is directly below the z'-axis; i. e., when the center of mass is in a plane containing the z'-axis and the gravity vector,

ϕ is the amount of rotation about the axis measuring zero when the center of mass is directly above the z'-axis,

$C = M_{total}\, gd\sin\gamma / I_{z'z'}$,

M_{total} is the total mass of the body,

g is the gravitational constant,

d is the perpendicular distance from the axis to the center of mass of the object,

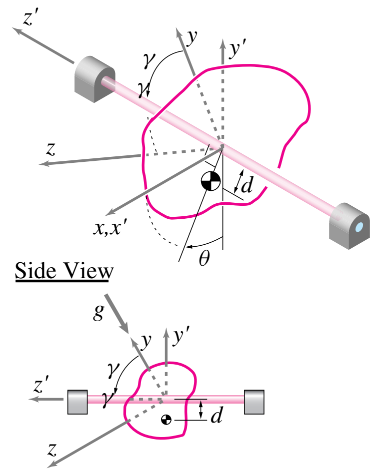
γ is the angle of tip of the axis from the vertical,

$I_{z'z'}$ is the moment of inertia of the object about the skewer axis.

a) How many examples of such pendula can you think of and what would be the values of the constants? (e.g. a simple pendulum?, a broom upside down on your hand?, a door?, a box tipping on one edge?, a bicycler stopped at a red light and stuck in her new toe clips?, a person who got *rigor mortis* and whose ankles went totally limp at the same time?,...)

b) A door with frictionless hinges is misaligned by half an inch (the top hinge is half an inch away from the vertical line above the other hinge). Estimate the period with which it will swing back and forth. Use reasonable magnitudes of any quantities you need.

c) Can you verify the formula (a simple pendulum and a swinging stick are special cases)?



Problem 20.4.11: **-3-D pendulum on a hinge.**

20.4.12 A hanging rod going in circles. A uniform rod with length ℓ and mass m hangs from a well greased ball and socket joint at O. It swings in circles at constant rate $\vec{\omega} = \omega\boldsymbol{k}$ while the rod makes an angle ϕ with the vertical (so it sweeps a cone).

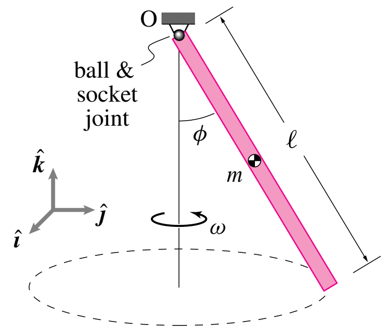
a) Calculate $\vec{\boldsymbol{H}}_{/O}$ and $\dot{\vec{\boldsymbol{H}}}_{/O}$.

b) Using $\dot{\vec{\boldsymbol{H}}}_{/O}$ and angular momentum balance, find the rotation rate ω in terms of ϕ, g, ℓ, and m. How long does it take to make one revolution? *

c) If another stick rotates twice as fast while making the same angle ϕ, what must be its length compared to ℓ?

d) Is there a simple way to describe what two uniform sticks of different length and different cone angle ϕ both have in common if their rate of rotation ω is the same? (If you accurately draw two such sticks, you should see the relation.)

e) Find the reaction force at the ball and socket joint. Is this force parallel to the stick?



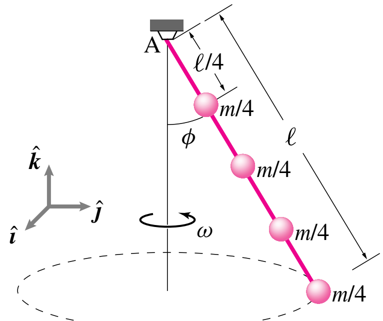
Problem 20.4.12

20.4.13 From discrete to continuous. n equal beads of mass m/n each are glued to a rigid massless rod at equal distances $l = \ell/n$ from each other. For $n = 4$, the system is shown in the figure, but you are to consider the general problem with n masses. The rod swings around the vertical axis maintaining angle ϕ.

a) Find the rate of rotation of the system as a function of g, ℓ, n, and ϕ for constant rate circular motion. [Hint: You may need these series sums: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ and $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.] *

b) Set $n = 1$ in the expression derived in (a) and check the value with the rate of rotation of a *conical pendulum* of mass m and length ℓ.

c) Now, put $n \to \infty$ in the expression obtained in part (a). Does this rate of rotation make sense to you? [Hint: How does it compare with the rate of rotation for the uniform stick of Problem 20.4.12?]



Problem 20.4.13

20.4.14 Spinning dumbbell. Two point masses are connected by a rigid rod of negligible mass and length $2\sqrt{2}\ell$. The rod is welded to a shaft that is along the y axis. The shaft is spinning at a constant rate ω driven by a motor at A (Not shown). At the time of interest the dumbbell is in the yz-plane. Neglect gravity.

a) What is the moment of inertia matrix about point O for the dumbbell at the instant shown? *

b) What is the angular momentum of the dumbbell at the instant shown about point O. *

c) What is the rate of change of angular momentum of the dumbbell at the instant shown? Calculate the rate of change of angular momentum two different ways:

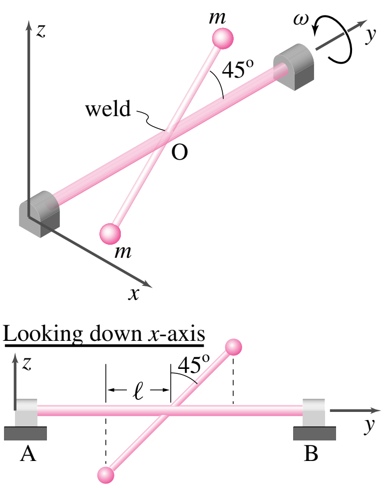
   (i) by adding up the contribution of both of the masses to the rate of change of angular momentum (i.e. $\dot{\vec{H}} = \sum \vec{r}_i \times (m_i \vec{a}_i)$) and

   (ii) by calculating I_{xy} and I_{yz}, calculating $\vec{H}$, and then calculating $\dot{\vec{H}}$ from $\vec{\omega} \times \vec{H}$. *

d) What is the total force and moment (at the CM of the dumbbell) required to keep this motion going? Do the calculation in the following ways:

   (i) Using angular momentum balance (about any point of your choice) for the shaft dumbbell system.

   (ii) By drawing free-body diagrams of the masses alone and calculating the forces on the masses. Then use action and reaction and draw a free-body diagram of the remaining (massless) part of the shaft-dumbbell system. Then use moment balance for this system. *



Problem 20.4.14: Crooked spinning dumbbell.

20.4.15 A uniform rectangular plate of mass m, height h, and width b spins about its diagonal at a constant angular speed ω, as shown in the figure. The plate is supported by frictionless bearings at O and C. The plate lies in the YZ-plane at the instant shown.

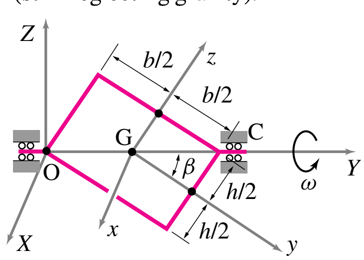
[Note: $\cos(\beta) = b/\sqrt{(h^2 + b^2)}$, and $\sin(\beta) = h/\sqrt{(h^2 + b^2)}$.]

a) Compute the angular momentum of the plate about G (with respect to the body axis $Gxyz$ shown).

b) Ignoring gravity determine the net torque (moment) that the supports impose on the body at the instant shown. (Express your solution using base vectors and components associated with the inertial XYZ frame.)

c) Calculate the reactions at O or C for the configuration shown. *

d) In the case when $h = b = 1\,\text{m}$ what is the vertical reaction at C (still neglecting gravity).



$OXYZ$ - Fixed Frame
(base vectors, $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$)

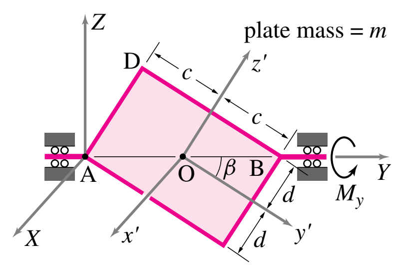
$Gxyz$ - Moving "Body" Frame
(base vectors, $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$)

Problem 20.4.15

20.4.16 A uniform rectangular plate spins about a fixed axis. The constant torque M_y is applied about the Y axis. The plate is held by bearings at the corners of the plate at A and B. At the instant shown the plate and the $y'z'$ axis attached to the plate are in the YZ plane. Neglect gravity. At $t = 0$ the plate is at rest.

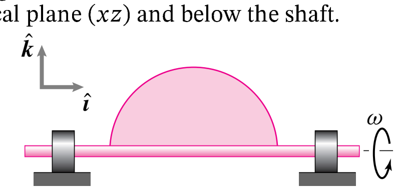
a) At $t = 0^+$ (i.e., *just* after the start), what is the acceleration of point D? Give your answer in terms of any or all of $c, d, m, M_y, \hat{\imath}, \hat{\jmath}$, and $\hat{k}$ (the base vectors aligned with the fixed XYZ axes), and I_{YY}^{cm} [You may use I_{YY}^{cm} in your answer. Or you could use this result from the change of coordinates formulae in linear algebra: $I_{YY}^{cm} = (\cos^2\beta) I_{y'y'}^{cm} + (\sin^2\beta) I_{z'z'}^{cm}$ to get an answer in terms of c, d, m, and M_y.] *

b) At $t = 0^+$, what is the reaction at B? *



Problem 20.4.16

20.4.17 A uniform semi-circular disk of radius $r = 0.2\,\text{m}$ and mass $m = 1.5\,\text{kg}$ is attached to a shaft along its straight edge. The shaft is tipped very slightly from a vertical upright position. Find the dynamic reactions on the bearings supporting the shaft when the disk is in the vertical plane (xz) and below the shaft.



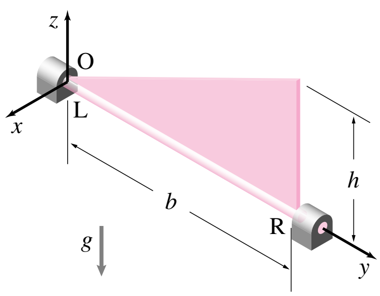
Problem 20.4.17

20.4.18 A thin triangular plate of base b, height h, and mass m is attached to a massless shaft which passes through the bearings L and R. The plate is initially vertical and then, provoked by a teensy disturbance, falls, pivoting around the shaft. The bearings are frictionless. Given,

$$\begin{aligned}
\vec{r}_{cm} &= (0, \tfrac{2}{3}b, \tfrac{1}{3}h),\\
I^O_{xx} &= \frac{m}{2}(b^2 + h^2/3),\\
I^O_{yy} &= mh^2/6,\\
I^O_{zz} &= mb^2/2,\\
I^{cm}_{yy} &= mh^2/18,\\
I^O_{zy} &= -mbh/4,
\end{aligned}$$

a) what is the plate's angular velocity as it passes through the downward vertical position?

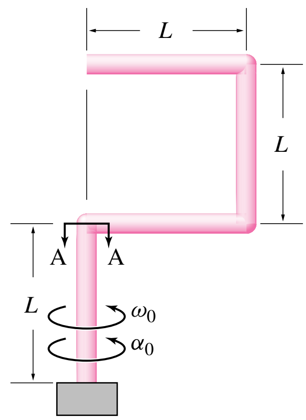
b) Compute the bearing reactions when the plate swings through the downward vertical position.



Problem 20.4.18

20.4.19 A welded structural tubing framework is proposed to support a heavy emergency searchlight. To select properly the tubing dimensions it is necessary to know the forces and moments at the critical section AA, when the framework is moving. The frame moves as shown.

a) Determine the magnitude and direction of the dynamic moments (bending and twisting) and forces (axial and shear) at section AA of the frame if, at the instant shown, the frame has angular velocity ω_o and angular acceleration α_o. Assume that the mass per unit length of the pipe is ρ.



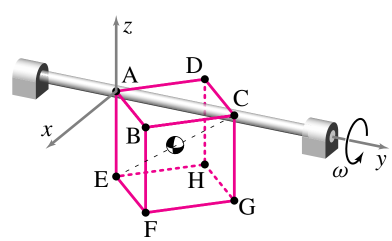
Problem 20.4.19

20.4.20 A uniform cube with side 0.25 m is glued to a pipe along the diagonal of one of its faces $ABCD$. The pipe rotates about the y-axis at a constant rate ω = 30 rpm. At the instant of interest, the face $ABCD$ is in the xy-plane.

a) Find the velocity and acceleration of point F on the cube.

b) Find the position of the center of mass of the cube and compute its acceleration $\vec{a}_{cm}$.

c) Suppose that the entire mass of the cube is concentrated at its center of mass. What is the angular momentum of the cube about point A?

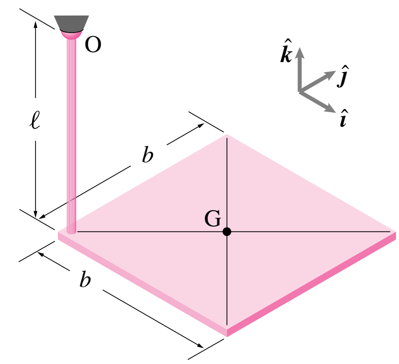


Problem 20.4.20

20.4.21 A uniform square plate of mass m and side length b is welded to a narrow shaft at one corner. The shaft is perpendicular to the plate and has length ℓ. The upper end of the shaft is connected to a ball and socket joint at O. The assembly is rotating about the z axis through O at the fixed rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ which happens to be just right to make the plate horizontal and the shaft vertical, as shown. The gravitational constant is g.

a) What is ω in terms of m, ℓ, b, and g?

b) Is the reaction at O on the rod-plate assembly in the direction of the line from G to O? (Why?)



Problem 20.4.21

20.4.22 The uniform cube shown is rotating at constant angular velocity about the fixed axis OA with angular speed 10 rad/s counterclockwise looking towards O from A. Find:

a) $\vec{\omega}$

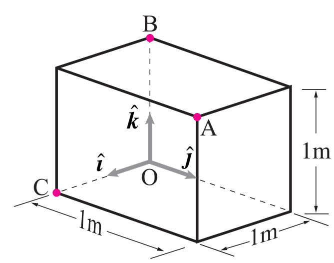
b) $\vec{v}_B$

c) $\vec{a}_C$

d) $\vec{H}_{/O}$

e) $\dot{\vec{H}}_{/O}$

f) The total torque applied to the block in the configuration shown during this motion.



Problem 20.4.22

20.4.23 A spinning box. A solid box $\mathcal{B}$ with dimensions a, a, and $2a$ (where a = .1 m) and mass = 1 kg. It is supported with ball and socket joints at opposite diagonal corners at A and B. It is spinning at constant rate of $(50/\pi)$ rev/sec and at the instant of interest is aligned with the fixed axis $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\boldsymbol{k}}$. Neglect gravity.

a) What is $\vec{\omega}_{\mathcal{B}}$?

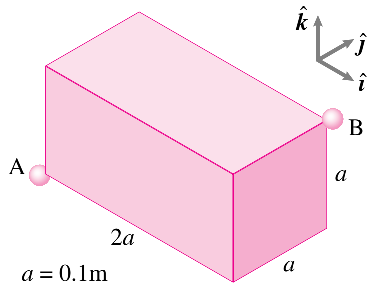
b) What is $\dot{\vec{\omega}}_{\mathcal{B}}$?

c) What is $\vec{H}_{\mathcal{B}/cm}$?

d) What is $\dot{\vec{H}}_{\mathcal{B}/cm}$?

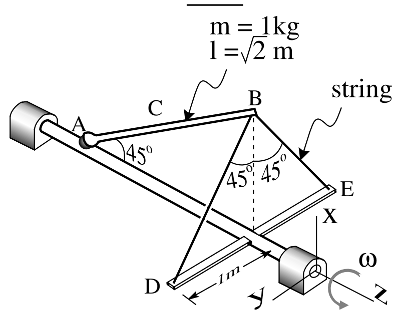
e) What is $\vec{r}_{B/A} \times \vec{F}_B$ (where $\vec{F}_B$ is the reaction force on $\mathcal{B}$ at B)? A dimensional vector is desired.

f) If the hinges were suddenly cut off when $\mathcal{B}$ was in the configuration shown, what would be $\dot{\vec{\omega}}_{\mathcal{B}}$ immediately afterwards?



Problem 20.4.23

20.4.24 Uniform rod attached to a shaft. A uniform 1 kg, $\sqrt{2}$ m long rod is attached to a shaft which spins at a constant rate of 1 rad/s. One end of the rod is connected to the shaft by a ball-and-socket joint. The other end is held by two strings (or pin jointed rods) that are connected to a rigid cross bar. The cross bar is welded to the main shaft. Ignore gravity. Find the reaction **force** at A.



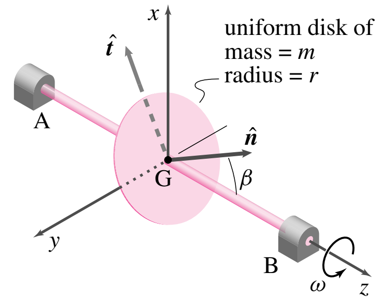
Problem 20.4.24

20.4.25 A round disk spins crookedly on a shaft. A thin homogeneous disk of mass m and radius r is welded to the horizontal axle AB. The normal to the plane of the disk forms an angle β with the axle. The shaft rotates at the constant rate ω.

a) What is the x component of the angular momentum of the disk about point G the center of mass of the disk?

b) What is the x component of the rate of change of angular momentum of the disk about G?

c) At the given instant, what is the total force and moment (about G) required to keep the disk spinning at constant rate?



Problem 20.4.25: A round disk spins crookedly on a shaft.

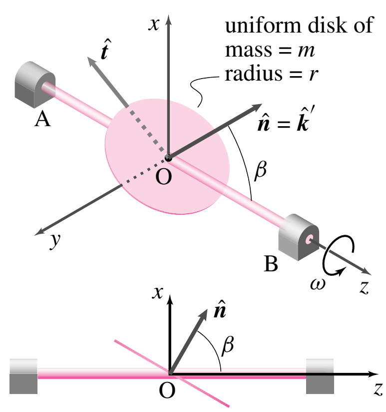
20.4.26 A uniform disk of mass $m = 2$ kg and radius $r = 0.2$ m is mounted rigidly on a massless axle. The normal to the disk makes an angle $\beta = 60°$ with the axle. The axle rotates at a constant rate $\omega = 60$ rpm. At the instant shown in the figure,

a) Find the angular momentum $\vec{H}_{/O}$ of the disk about point O. *

b) Draw the angular momentum vector indicating its magnitude and direction. *

c) As the disk rotates (with the axle), the angular momentum vector changes. Does it change in magnitude or direction, or both? *

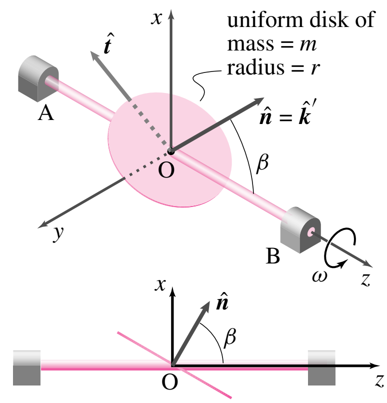
d) Find the direction of the net torque on the system at the instant shown. Note, only the *direction* is asked for. You can answer this question based on parts (b) and (c). *



Front View (xz-plane)

Problem 20.4.26

20.4.27 A uniform disk spins at constant angular velocity. The uniform disk of mass m and radius r spins about the z axis at the constant rate ω. Its normal $\hat{n}$ is inclined from the z axis by the angle β and is in the xz plane at the instant of interest. What is the total force and moment (relative to O) needed to keep this disk spinning at this rate? Answer in terms of m, r, ω, β and any base vectors you need.



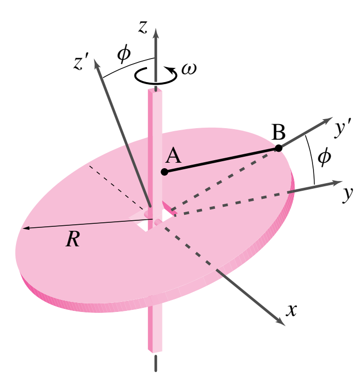
Front View (xz-plane)

Problem 20.4.27

20.4.28 Spinning crooked disk. A rigid uniform disk of mass m and radius R is mounted to a shaft with a hinge. An apparatus not shown keeps the shaft rotating around the z axis at a constant ω. At the instant shown the hinge axis is in the x direction. A massless wire, perpendicular to the shaft, from the shaft at A to the edge of the disk at B keeps the normal to the plate (the z' direction) at an angle ϕ with the shaft.

a) What is the tension in the wire? Answer in terms of some or all of R, m, ϕ and ω.

b) What is the reaction force at the hinge at O?



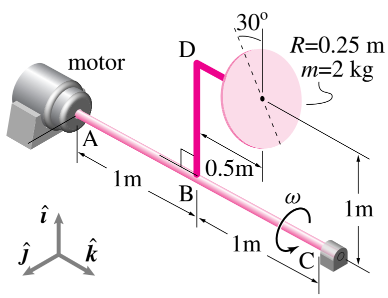
Problem 20.4.28

20.4.29 A disoriented disk rotating with a shaft. A uniform thin circular disk with mass $m = 2\,\text{kg}$ and radius $R = 0.25\,\text{m}$ is mounted on a massless shaft as shown in the figure. The shaft spins at a constant speed $\omega = 3\,\text{rad/s}$. At the instant shown, the disk is slanted at $30°$ with the xy-plane. Ignore gravity.

a) Find the rate of change of angular momentum of the disk about its center of mass. *

b) Find the rate of change of angular momentum about point A. *

c) Find the total force and moment required at the supports to keep the motion going. *



Problem 20.4.29