20.2 Dynamics of fixed-axis rotation

We now address mechanics questions concerning objects which are known *a priori* to spin about a fixed axis. We would like to calculate forces and moments if the motion is known. And we would like to determine the details of the motion, the angular acceleration in particular, if the applied forces and moments are known. Once the angular acceleration is known (as a function of some combination of time, angle and angular rate) the angular rate and angular position can be found by integration or solution of an ordinary differential equation.

The full content of the subject follows from the basic mechanics equations

$$\text{linear momentum balance,} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}}$$

$$\text{angular momentum balance,} \qquad \sum \vec{\boldsymbol{M}}_{i/C} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}},$$

$$\text{and power balance:} \qquad P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}.$$

The quantities $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ are defined in terms of the position and acceleration of the system's mass (see the second page of the inside cover). To evaluate $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ for fixed-axis rotation we can use the kinematics relations from the previous chapter which determine velocity and acceleration of points on a body spinning about a fixed axis in terms of the position $\vec{\boldsymbol{r}}$ of the point of interest relative to any point on the axis.

$$\begin{aligned} \vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}, \\ \vec{\boldsymbol{a}} &= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}. \end{aligned}$$

For fixed-axis rotation $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\lambda}}$ and $\dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{\lambda}}$ with $\hat{\boldsymbol{\lambda}}$ a constant unit vector along the axis of rotation.

To solve problems we draw a free-body diagram, write the equations of linear and angular momentum balance, and evaluate the terms using the kinematics relations. In general this will lead to the evaluation of a sum or an integral. A short cut, the moment of inertia matrix, will be introduced in later sections.

Before proceeding to more difficult three-dimensional problems, let's review a simple 2D problem.

Example: Spinning disk

The round flat uniform disk in fig. 20.17 is in the xy plane spinning at the constant rate $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ about its center. It has mass m_{tot} and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{cm} = \vec{\boldsymbol{0}},$$

which we could also have calculated by evaluating the integral $\dot{\vec{\boldsymbol{L}}} \equiv \int \vec{\boldsymbol{a}}\, dm$ instead of using the general result that $\dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{cm}$. From angular momentum balance we have:

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{i/O} &= \dot{\vec{\boldsymbol{H}}}_{/O} \\
\Rightarrow \quad \vec{\boldsymbol{M}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm \\
&= \int_0^{R_0} \int_0^{2\pi} (R\hat{\boldsymbol{e}}_R) \times (-R\omega^2 \hat{\boldsymbol{e}}_R) \underbrace{\frac{m_{\text{tot}}}{\pi R_O^2} \overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\
&= \int \int \vec{\boldsymbol{0}}\, d\theta\, dR \\
&= \vec{\boldsymbol{0}}.
\end{aligned}$$

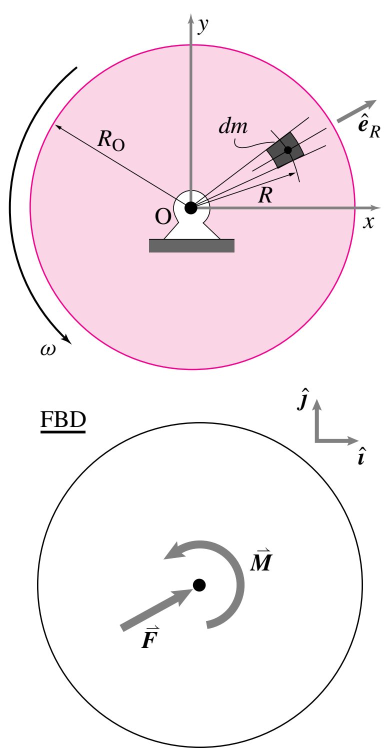

Figure 20.17: A uniform disk turned by a motor at a constant rate.

Power balance is of limited use for constant rate circular motion. If all parts of a system move at constant angular rate at a constant radius then they all have constant speed. Thus the kinetic energy of the system is constant. So the power balance equation just says that the net power into the system is the amount dissipated inside (assuming no energy storage).

Example: Spinning disk with power balance

Consider the spinning disk from fig. 20.17 and the previous example. The power balance equation III gives

$$P = \dot{E}_K + \underbrace{\dot{E}_P}_{0} + \underbrace{\dot{E}_{\text{int}}}_{0} \quad \Rightarrow \quad P = \int \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}\, dm = \int 0\, dm = 0. \qquad (20.16)$$

In this example there is no force or torque acting on the disk so the power P must turn out to be zero. In other constant rate problems the force and moment will *not* turn out to be zero, but the kinetic energy of the system will still be constant and so, assuming no energy storage or dissipation, we will still have $P = 0$.

A stick sweeps out a cone

Now we consider a genuinely three-dimensional problem involving fixed-axis, rigid body rotation. Consider a long narrow stick swinging in circles so that it sweeps out a cone (fig. 20.18). Each point on the stick is moving in circles around the z-axis at a constant rate ω. What is the relation between ω and the angle of the stick ϕ? The approach to this problem is, as usual, to draw a free-body diagram, write momentum balance equations, evaluate the left and right hand sides, and then solve for quantities of interest. The hard part of this problem is evaluating the right hand side of the angular momentum balance equations.

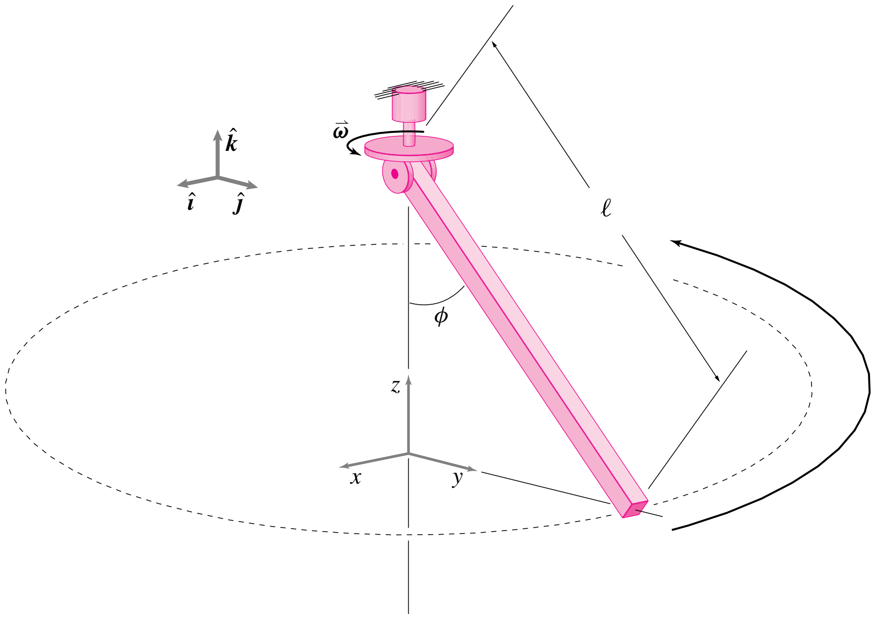

Figure 20.18: A spherical rigid body pendulum (uniform thin rod) going in circles at constant rate $\vec{\boldsymbol{\omega}}$.

To simplify calculation, we look at the pendulum at the instant it passes through the yz-plane, assuming the xyz axes are fixed in space.

The free-body diagram shown in fig. 20.19 shows the gravity force at the center-of-mass, the reaction force at point O, and, consistent with the shown construction of the hinge, the moments at O perpendicular to the hinge.

Because we are interested in the relation between ϕ and ω and not the reaction force, at least for now, we look at angular momentum balance about point O.

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$$

First, we show and discuss the results of evaluating the equation of angular momentum balance. Then, we will show the details of calculating $\sum \vec{\boldsymbol{M}}_{\mathrm{O}}$ and the details of several methods for calculating $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$.

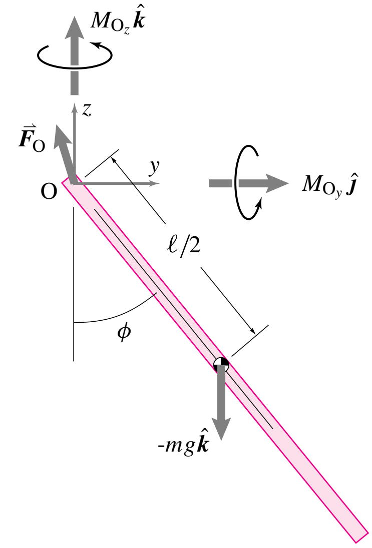

Figure 20.19: Free-body diagram of the rod.

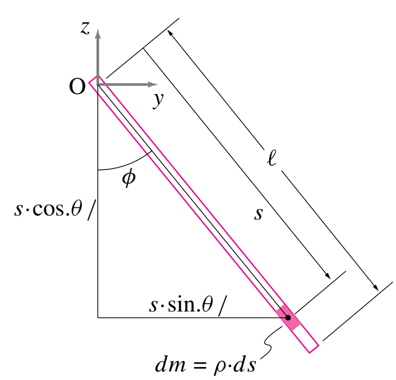

Figure 20.20: The rod is shown on the yz plane. We use this figure to locate the bit of mass dm corresponding to the bit of length of rod ds.

Evaluation of $\sum \vec{\boldsymbol{M}}_{\mathrm{O}}$

To find $\sum \vec{\boldsymbol{M}}_{\mathrm{O}}$, we had to find the moment of the gravity force. The most direct method is to use the definition

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{F}} \\
&= \frac{\ell}{2}[\cos\phi(-\hat{\boldsymbol{k}}) + \sin\phi\,\hat{\boldsymbol{\jmath}}] \times (-mg\hat{\boldsymbol{k}}) \\
&= -\frac{mg\ell\sin\phi}{2}\hat{\boldsymbol{\imath}}
\end{aligned}$$

One could also 'slide' the gravity force to the level of O (a force displaced in its direction of action is mechanically equivalent). Then you can see from the figure that the force is perpendicular to its position relative to O. Moment is then force (mg) times distance $(\frac{\ell}{2}\sin\phi)$ in the direction given by the right hand rule $(-\hat{\boldsymbol{\imath}})$.

Evaluation of $\dot{\vec{H}}_{/O}$

We now evaluate $\dot{\vec{\boldsymbol{H}}}_{/O}$ by adding up the contribution to the sum from each bit of mass.

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \int \overbrace{\vec{\boldsymbol{r}}_{/O}}^{\boxed{\vec{\boldsymbol{r}}_{/O} = -s\cos\phi\hat{\boldsymbol{k}} + s\sin\phi\hat{\boldsymbol{\jmath}}}} \times \underbrace{\vec{\boldsymbol{a}}}_{\boxed{\begin{array}{l}\text{For constant rate circular motion,}\\ \begin{aligned}\vec{\boldsymbol{a}} &= \vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/O})\\ &= (\omega\hat{\boldsymbol{k}})\times\big[(\omega\hat{\boldsymbol{k}})\times\\ &\quad s(\sin\phi\,\hat{\boldsymbol{\jmath}}-\cos\phi\,\hat{\boldsymbol{k}})\big]\\ &= -\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}}\end{aligned}\end{array}}} \overbrace{dm}^{\boxed{\begin{array}{l}dm = \rho\,ds\text{, where}\\ \rho = \text{mass per unit length}\end{array}}}$$

$$\begin{aligned}
&= \int_0^\ell \underbrace{(-s\cos\phi\hat{\boldsymbol{k}} + s\sin\phi\,\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{/O}} \times \underbrace{(-\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{a}}}(\rho\,ds)\\
&= \int_0^\ell -s^2\cos\phi\sin\phi\;\omega^2\hat{\boldsymbol{\imath}}\rho ds \qquad \text{(evaluating the cross product)}\\
&= -\cos\phi\sin\phi\;\omega^2\rho\hat{\boldsymbol{\imath}}\int_0^\ell s^2\,ds \qquad (\phi, \rho, \text{ and } \omega \text{ do not vary with } s)\\
&= -\cos\phi\sin\phi\;\omega^2(\rho\frac{\ell^3}{3})\hat{\boldsymbol{\imath}} \qquad \text{(evaluating the integral)}\\
\dot{\vec{\boldsymbol{H}}}_{/O} &= -\cos\phi\sin\phi\;\omega^2(\frac{m\ell^2}{3})\hat{\boldsymbol{\imath}} \qquad (\text{because } m = \rho\ell).
\end{aligned}$$

We could have taken a short-cut in the calculation of acceleration $\vec{\boldsymbol{a}}$. Instead of using $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}})$, we could have used $\vec{\boldsymbol{a}} = -\omega^2\vec{\boldsymbol{R}}$ where $\vec{\boldsymbol{R}}$ is the radius of the circle each particle is traveling on. It is evident from the picture that the appropriate radius is $\vec{\boldsymbol{R}} = s\sin\phi\,\hat{\boldsymbol{\jmath}}$, so $\vec{\boldsymbol{a}} = -\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}}$.

We will show two more methods for calculating $\dot{\vec{\boldsymbol{H}}}_{/O}$ in section 20.4 on page 1102 using the moment of inertia matrix.

The results for the conically swinging stick

We can now evaluate the terms in the angular momentum balance equation as

$$\overbrace{-mg\sin\phi\frac{\ell}{2}\hat{\boldsymbol{\imath}} + M_{O_y}\hat{\boldsymbol{\jmath}} + M_{O_z}\hat{\boldsymbol{k}}}^{\sum \vec{\boldsymbol{M}}_O} = \overbrace{-\sin\phi\cos\phi\frac{m\ell^2}{3}\omega^2\hat{\boldsymbol{\imath}}}^{\dot{\vec{\boldsymbol{H}}}_{/O}}. \qquad (20.17)$$

We can get three scalar equations from eqn. 20.17 by dotting it with $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{\imath}}$ to get

$$\boxed{M_{O_y} = 0} \quad \text{and} \quad \boxed{M_{O_z} = 0}$$

and

$$\boxed{\omega^2 = \frac{3g}{2\ell\cos\phi}}.$$

Note that $M_{O_y} = M_{O_z} = 0$. That is, for this special motion, the hinge joint at O could be replaced with a ball-and-socket joint.

Note that the solution for a point mass spherical pendulum is $\omega^2 = \frac{g}{\ell\cos\phi}$. That is, this stick would rotate at the same rate and angle as a point mass at the end of a rod of length $\frac{2\ell}{3}$. One could not easily anticipate this result. We point it out here to emphasize that the analysis of this rigid-body problem cannot be reduced *a priori* to any simple particle mechanics problem.

In fig. 20.21, non-dimensional rotational speed $\frac{\omega^2\ell}{g}$ is plotted versus hang angle ϕ. As one might expect intuitively unless ω is high enough, ($\omega^2 > \frac{3g}{2\ell}$), the only solution is hanging straight down ($\phi = 0$). At the critical speed ($\omega^2 = \frac{3g}{2\ell}$), the curve is nearly flat, implying that a range of hang angles ϕ is possible all with nearly the same angular velocity. As is also intuitively plausible, as the bar gets close to the horizontal (close to $\frac{\pi}{2}$), the spin rate goes to infinity.

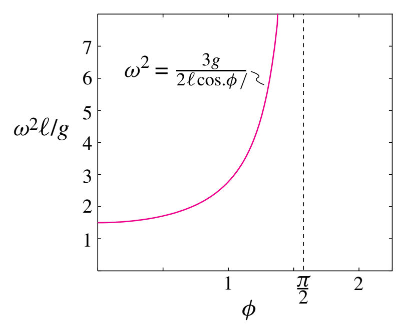

Figure 20.21: Plot of non-dimensional rotational speed $\frac{\omega^2\ell}{g}$ versus hang angle ϕ. For $\omega^2\ell/g < 3/2$ the only solution is $\phi = 0$ (hanging straight down). At or very close to $\omega^2\ell/g = 3/2$ a range of ϕ's is possible. As $\phi \to \pi/2$ and the rod becomes close to horizontal, the spin rate ω goes to infinity.

The scalar equations governing rotation about an axis

For two-dimensional motion of flat hinged objects we had the simple relation "$M = I\alpha$". This formula captures our simple intuitions about angular momentum balance. When you apply torque to a body its rate of rotation increases. It turns out that, for three-dimensional motion of a rigid body about a fixed axis, the same result applies if we interpret the terms correctly.①

① **Caution:** For more general three-dimensional motion than rotation about a fixed axis the equation $M = I\alpha$ does not apply. Trying to vectorize by underlining various terms gives the wrong answer.

If the axis of rotation goes through C and is in the direction $\hat{\boldsymbol{\lambda}}$ we can define $M = \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_C$ as the moment about the axis of rotation. We can similarly look at the $\hat{\boldsymbol{\lambda}}$ component of $\dot{\vec{\boldsymbol{H}}}_{/C}$ (assume, for definiteness, that the system is continuous).

$$
\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} &= \hat{\boldsymbol{\lambda}} \cdot \int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\, dm \\
&= \hat{\boldsymbol{\lambda}} \cdot \int \vec{\boldsymbol{r}} \times \left[(\omega \hat{\boldsymbol{\lambda}}) \times \left((\omega \hat{\boldsymbol{\lambda}}) \times \vec{\boldsymbol{r}} \right) + (\dot{\omega} \hat{\boldsymbol{\lambda}}) \times \vec{\boldsymbol{r}} \right] \, dm. \\
&= \dot{\omega} \int R^2 \, dm, \qquad (20.18)
\end{aligned}
$$

where R is the distance of the mass points from the axis. The last line follows from the previous most simply by paying attention to directions and magnitudes when using the right-hand rule and the geometric definition of the cross product. We thus have derived the result that

$$M = I\alpha,$$

if by M we mean moment about the fixed axis and by I we mean $\int R^2 \, dm$. Actually, the scalar we call I in the above equation is a manifestation of a more general matrix $[\boldsymbol{I}]$ that we will explore in the next section.