20.2 Dynamics of fixed-axis rotation

We now address mechanics questions concerning objects which are known *a priori* to spin about a fixed axis. We would like to calculate forces and moments if the motion is known. And we would like to determine the details of the motion, the angular acceleration in particular, if the applied forces and moments are known. Once the angular acceleration is known (as a function of some combination of time, angle and angular rate) the angular rate and angular position can be found by integration or solution of an ordinary differential equation.

The full content of the subject follows from the basic mechanics equations

$$\text{linear momentum balance,} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}}$$

$$\text{angular momentum balance,} \qquad \sum \vec{\boldsymbol{M}}_{i/C} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}},$$

$$\text{and power balance:} \qquad P = \dot{E}_\mathrm{K} + \dot{E}_\mathrm{P} + \dot{E}_\mathrm{int}.$$

The quantities $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ are defined in terms of the position and acceleration of the system's mass (see the second page of the inside cover). To evaluate $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ for fixed-axis rotation we can use the kinematics relations from the previous chapter which determine velocity and acceleration of points on a body spinning about a fixed axis in terms of the position $\vec{\boldsymbol{r}}$ of the point of interest relative to any point on the axis.

$$\begin{aligned} \vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}, \\ \vec{\boldsymbol{a}} &= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}. \end{aligned}$$

For fixed-axis rotation $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\lambda}}$ and $\dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{\lambda}}$ with $\hat{\boldsymbol{\lambda}}$ a constant unit vector along the axis of rotation.

To solve problems we draw a free-body diagram, write the equations of linear and angular momentum balance, and evaluate the terms using the kinematics relations. In general this will lead to the evaluation of a sum or an integral. A short cut, the moment of inertia matrix, will be introduced in later sections.

Before proceeding to more difficult three-dimensional problems, let's review a simple 2D problem.

Example: Spinning disk

The round flat uniform disk in fig. 20.17 is in the xy plane spinning at the constant rate $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ about its center. It has mass m_tot and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} = m_\mathrm{tot}\vec{\boldsymbol{a}}_{cm} = \vec{\boldsymbol{0}},$$

which we could also have calculated by evaluating the integral $\dot{\vec{\boldsymbol{L}}} \equiv \int \vec{\boldsymbol{a}}\, dm$ instead of using the general result that $\dot{\vec{\boldsymbol{L}}} = m_\mathrm{tot}\vec{\boldsymbol{a}}_{cm}$. From angular momentum balance we have:

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{i/O} &= \dot{\vec{\boldsymbol{H}}}_{/O} \\
\Rightarrow \quad \vec{\boldsymbol{M}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm \\
&= \int_0^{R_0} \int_0^{2\pi} (R\hat{\boldsymbol{e}}_R) \times (-R\omega^2 \hat{\boldsymbol{e}}_R) \underbrace{\frac{m_{\text{tot}}}{\pi R_O^2} \overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\
&= \int\int \vec{\boldsymbol{0}}\, d\theta\, dR \\
&= \vec{\boldsymbol{0}}.
\end{aligned}
$$

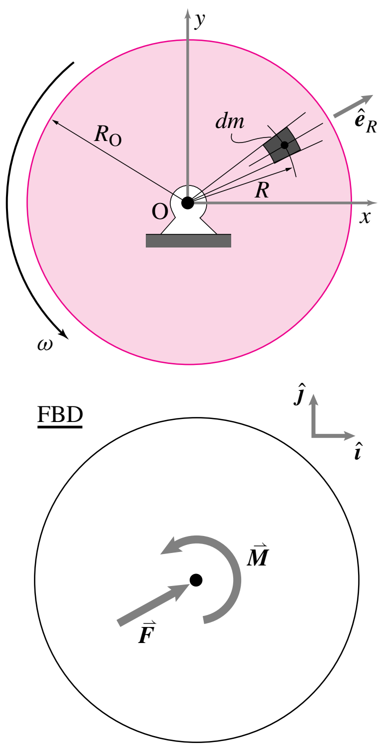


Figure 20.17: A uniform disk turned by a motor at a constant rate.

Power balance is of limited use for constant rate circular motion. If all parts of a system move at constant angular rate at a constant radius then they all have constant speed. Thus the kinetic energy of the system is constant. So the power balance equation just says that the net power into the system is the amount dissipated inside (assuming no energy storage).

Example: Spinning disk with power balance
Consider the spinning disk from fig. 20.17 and the previous example. The power balance equation III gives

$$
P = \dot{E}_K + \underbrace{\dot{E}_P}_{0} + \underbrace{\dot{E}_{\text{int}}}_{0} \quad \Rightarrow \quad P = \int \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}\, dm = \int 0\, dm = 0. \qquad (20.16)
$$

In this example there is no force or torque acting on the disk so the power P must turn out to be zero. In other constant rate problems the force and moment will *not* turn out to be zero, but the kinetic energy of the system will still be constant and so, assuming no energy storage or dissipation, we will still have $P = 0$.

A stick sweeps out a cone

Now we consider a genuinely three-dimensional problem involving fixed-axis, rigid body rotation. Consider a long narrow stick swinging in circles so that it sweeps out a cone (fig. 20.18). Each point on the stick is moving in circles around the z-axis at a constant rate ω. What is the relation between ω and the angle of the stick ϕ? The approach to this problem is, as usual, to draw a free-body diagram, write momentum balance equations, evaluate the left and right hand sides, and then solve for quantities of interest. The hard part of this problem is evaluating the right hand side of the angular momentum balance equations.

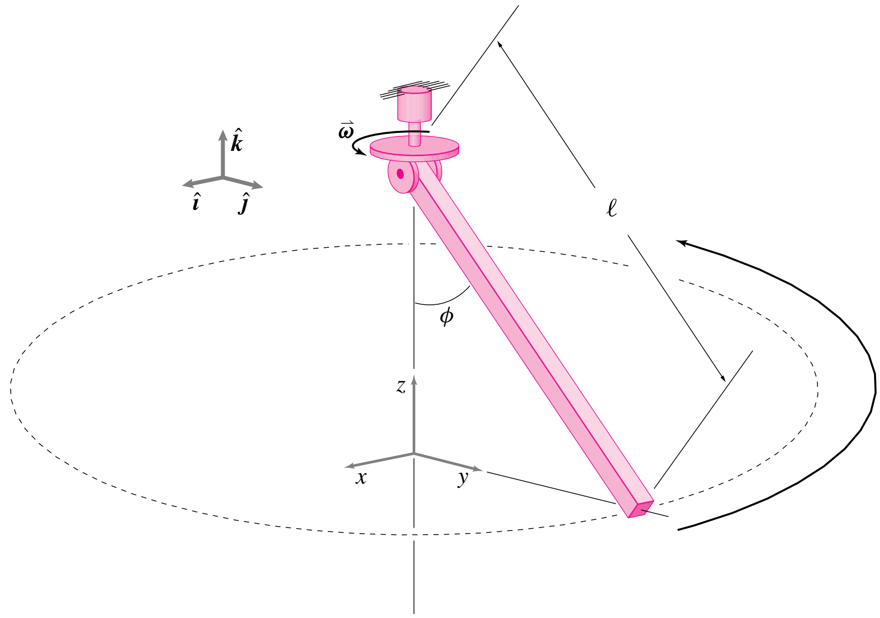


Figure 20.18: A spherical rigid body pendulum (uniform thin rod) going in circles at constant rate $\vec{\omega}$.

To simplify calculation, we look at the pendulum at the instant it passes through the yz-plane, assuming the xyz axes are fixed in space.

The free-body diagram shown in fig. 20.19 shows the gravity force at the center-of-mass, the reaction force at point O, and, consistent with the shown construction of the hinge, the moments at O perpendicular to the hinge.

Because we are interested in the relation between ϕ and ω and not the reaction force, at least for now, we look at angular momentum balance about point O.

$$\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{/\mathrm{O}}$$

First, we show and discuss the results of evaluating the equation of angular momentum balance. Then, we will show the details of calculating $\sum \vec{M}_{\mathrm{O}}$ and the details of several methods for calculating $\dot{\vec{H}}_{/\mathrm{O}}$.

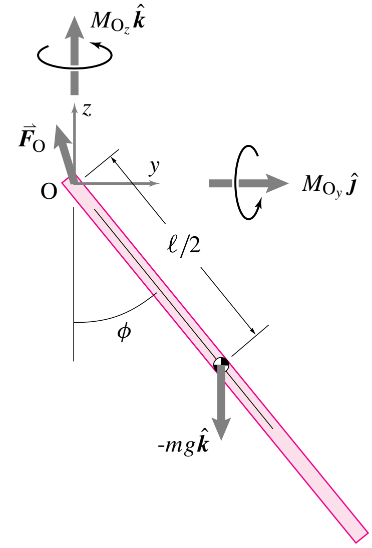


Figure 20.19: Free-body diagram of the rod.

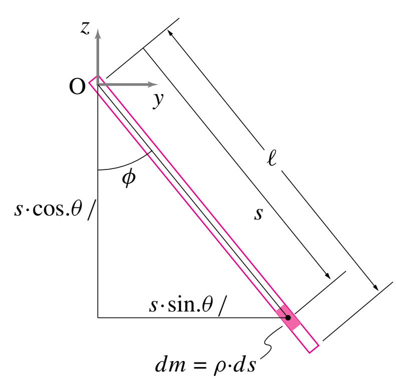


Figure 20.20: The rod is shown on the yz plane. We use this figure to locate the bit of mass dm corresponding to the bit of length of rod ds.

Evaluation of $\sum \vec{M}_{\mathrm{O}}$

To find $\sum \vec{M}_{\mathrm{O}}$, we had to find the moment of the gravity force. The most direct method is to use the definition

$$\begin{aligned}
\sum \vec{M}_{\mathrm{O}} &= \vec{r}_{/O} \times \vec{F} \\
&= \frac{\ell}{2}[\cos\phi(-\hat{k}) + \sin\phi\,\hat{\jmath}] \times (-mg\hat{k}) \\
&= -\frac{mg\ell\sin\phi}{2}\hat{\imath}
\end{aligned}$$

One could also 'slide' the gravity force to the level of O (a force displaced in its direction of action is mechanically equivalent). Then you can see from the figure that the force is perpendicular to its position relative to O. Moment is then force (mg) times distance ($\frac{\ell}{2}\sin\phi$) in the direction given by the right hand rule ($-\hat{\boldsymbol{\imath}}$).

Evaluation of $\dot{\vec{H}}_{/O}$

We now evaluate $\dot{\vec{\boldsymbol{H}}}_{/O}$ by adding up the contribution to the sum from each bit of mass.

$\vec{\boldsymbol{r}}_{/O} = -s\cos\phi\hat{\boldsymbol{k}} + s\sin\phi\hat{\boldsymbol{\jmath}}$

$dm = \rho\, ds$, where $\rho =$ mass per unit length

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \int \overbrace{\vec{\boldsymbol{r}}_{/O}} \times \underbrace{\vec{\boldsymbol{a}}}\ \overbrace{dm}$$

For constant rate circular motion,

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O}) \\
&= (\omega\hat{\boldsymbol{k}}) \times \big[(\omega\hat{\boldsymbol{k}}) \times \\
&\quad s(\sin\phi\,\hat{\boldsymbol{\jmath}} - \cos\phi\,\hat{\boldsymbol{k}})\big] \\
&= -\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\ell \underbrace{(-s\cos\phi\hat{\boldsymbol{k}} + s\sin\phi\,\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{/O}} \times \underbrace{(-\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{a}}} (\rho\, ds) \\
&= \int_0^\ell -s^2\cos\phi\sin\phi\ \omega^2\hat{\boldsymbol{\imath}}\rho ds && \text{(evaluating the cross product)} \\
&= -\cos\phi\sin\phi\ \omega^2\rho\hat{\boldsymbol{\imath}} \int_0^\ell s^2\, ds && (\phi, \rho, \text{ and } \omega \text{ do not vary with } s) \\
&= -\cos\phi\sin\phi\ \omega^2(\rho\frac{\ell^3}{3})\hat{\boldsymbol{\imath}} && \text{(evaluating the integral)} \\
\dot{\vec{\boldsymbol{H}}}_{/O} &= -\cos\phi\sin\phi\ \omega^2(\frac{m\ell^2}{3})\hat{\boldsymbol{\imath}} && \text{(because } m = \rho\ell\text{).}
\end{aligned}$$

We could have taken a short-cut in the calculation of acceleration $\vec{\boldsymbol{a}}$. Instead of using $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$, we could have used $\vec{\boldsymbol{a}} = -\omega^2\vec{\boldsymbol{R}}$ where $\vec{\boldsymbol{R}}$ is the radius of the circle each particle is traveling on. It is evident from the picture that the appropriate radius is $\vec{\boldsymbol{R}} = s\sin\phi\,\hat{\boldsymbol{\jmath}}$, so $\vec{\boldsymbol{a}} = -\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}}$.

We will show two more methods for calculating $\dot{\vec{\boldsymbol{H}}}_{/O}$ in section 20.4 on page 1102 using the moment of inertia matrix.

The results for the conically swinging stick

We can now evaluate the terms in the angular momentum balance equation as

$$\overbrace{-mg\sin\phi\frac{\ell}{2}\hat{\boldsymbol{\imath}} + M_{O_y}\hat{\boldsymbol{\jmath}} + M_{O_z}\hat{\boldsymbol{k}}}^{\sum \vec{\boldsymbol{M}}_O} = \overbrace{-\sin\phi\cos\phi\frac{m\ell^2}{3}\omega^2\hat{\boldsymbol{\imath}}}^{\dot{\vec{\boldsymbol{H}}}_{/O}}. \qquad (20.17)$$

We can get three scalar equations from eqn. 20.17 by dotting it with $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{\imath}}$ to get

$$\boxed{M_{O_y} = 0} \quad \text{and} \quad \boxed{M_{O_z} = 0}$$

and

$$\boxed{\omega^2 = \frac{3g}{2\ell\cos\phi}}.$$

Note that $M_{O_y} = M_{O_z} = 0$. That is, for this special motion, the hinge joint at O could be replaced with a ball-and-socket joint.

Note that the solution for a point mass spherical pendulum is $\omega^2 = \frac{g}{\ell\cos\phi}$. That is, this stick would rotate at the same rate and angle as a point mass at the end of a rod of length $\frac{2\ell}{3}$. One could not easily anticipate this result. We point it out here to emphasize that the analysis of this rigid-body problem cannot be reduced *a priori* to any simple particle mechanics problem.

In fig. 20.21, non-dimensional rotational speed $\frac{\omega^2\ell}{g}$ is plotted versus hang angle ϕ. As one might expect intuitively unless ω is high enough, ($\omega^2 > \frac{3g}{2\ell}$), the only solution is hanging straight down ($\phi = 0$). At the critical speed ($\omega^2 = \frac{3g}{2\ell}$), the curve is nearly flat, implying that a range of hang angles ϕ is possible all with nearly the same angular velocity. As is also intuitively plausible, as the bar gets close to the horizontal (close to $\frac{\pi}{2}$), the spin rate goes to infinity.

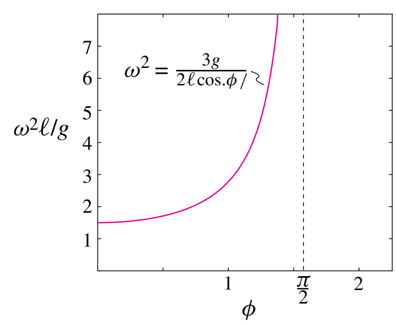


Figure 20.21: Plot of non-dimensional rotational speed $\frac{\omega^2\ell}{g}$ versus hang angle ϕ. For $\omega^2\ell/g < 3/2$ the only solution is $\phi = 0$ (hanging straight down). At or very close to $\omega^2\ell/g = 3/2$ a range of ϕ's is possible. As $\phi \to \pi/2$ and the rod becomes close to horizontal, the spin rate ω goes to infinity.

The scalar equations governing rotation about an axis

For two-dimensional motion of flat hinged objects we had the simple relation "$M = I\alpha$". This formula captures our simple intuitions about angular momentum balance. When you apply torque to a body its rate of rotation increases. It turns out that, for three-dimensional motion of a rigid body about a fixed axis, the same result applies if we interpret the terms correctly.①

①**Caution:** For more general three-dimensional motion than rotation about a fixed axis the equation $M = I\alpha$ does not apply. Trying to vectorize by underlining various terms gives the wrong answer.

If the axis of rotation goes through C and is in the direction $\hat{\boldsymbol{\lambda}}$ we can define $M = \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_C$ as the moment about the axis of rotation. We can similarly look at the $\hat{\boldsymbol{\lambda}}$ component of $\dot{\vec{\boldsymbol{H}}}_{/C}$ (assume, for definiteness, that the system is continuous).

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} &= \hat{\boldsymbol{\lambda}} \cdot \int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\, dm \\
&= \hat{\boldsymbol{\lambda}} \cdot \int \vec{\boldsymbol{r}} \times \left[(\omega\hat{\boldsymbol{\lambda}}) \times \left((\omega\hat{\boldsymbol{\lambda}}) \times \vec{\boldsymbol{r}}\right) + (\dot{\omega}\hat{\boldsymbol{\lambda}}) \times \vec{\boldsymbol{r}}\right]\, dm. \\
&= \dot{\omega} \int R^2\, dm, \qquad (20.18)
\end{aligned}$$

where R is the distance of the mass points from the axis. The last line follows from the previous most simply by paying attention to directions and magnitudes when using the right-hand rule and the geometric definition of the cross product. We thus have derived the result that

$$M = I\alpha,$$

if by M we mean moment about the fixed axis and by I we mean $\int R^2\, dm$. Actually, the scalar we call I in the above equation is a manifestation of a more general matrix $[\boldsymbol{I}]$ that we will explore in the next section.

SAMPLE 20.5 Going on a carnival ride at a constant rate. A carnival ride with roof AB and carriage BC is rotating about the vertical axis with constant angular velocity $\vec{\omega} = \omega \hat{\jmath}$. If the carriage with its occupants has mass $m = 100\,\text{kg}$, find the tension in the inextensible and massless rod BC when $\theta = 30^o$. What is the required angular speed ω (in revolutions/minute) to maintain this angle?

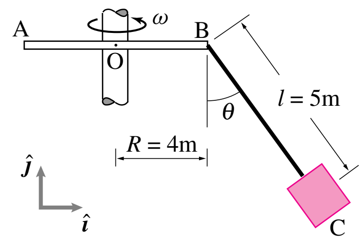
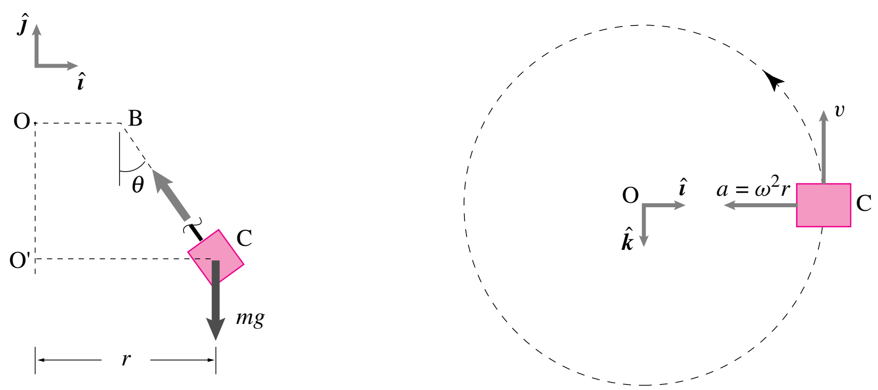


Figure 20.22: A carnival ride rotating at a constant speed

Solution The free-body diagram of the carriage is shown in Fig. 20.23(a). The geometry of motion of the carriage is shown in Fig. 20.23(b). The carriage goes around a circle of radius $r = O'C$ with constant speed $v = \omega r$. The only acceleration that the carriage has is the centripetal acceleration and at the instant of interest $\vec{a} = -\omega^2 r \hat{\imath}$.



(a) FBD of C

(b) Geometry of motion looking down $\hat{\jmath}$-axis

Figure 20.23:

The linear momentum balance $(\sum \vec{F} = \dot{\vec{L}})$ for the carriage gives:

$$\begin{aligned} T\hat{\lambda}_{CB} - mg\hat{\jmath} &= m\vec{a} \\ \text{or} \quad T(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) - mg\hat{\jmath} &= -m\omega^2 r\hat{\imath} \end{aligned} \tag{20.19}$$

Scalar equations from eqn. (20.19) are:

$$
\begin{aligned}
[\text{eqn. }(20.19)]\cdot\hat{\boldsymbol{\jmath}} \quad \Rightarrow \quad T\cos\theta - mg &= 0\\
\Rightarrow \quad T &= \frac{mg}{\cos\theta}\\
&= \frac{100\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}}{\sqrt{3}/2}\\
&= 1133\,\mathrm{N}.\\
[\text{eqn. }(20.19)]\cdot\hat{\boldsymbol{\imath}} \quad \Rightarrow \quad -T\sin\theta &= -m\omega^2 r\\
\Rightarrow \quad \omega^2 &= \frac{T\sin\theta}{mr}\\
&= \frac{T\sin\theta}{m(R+l\sin\theta)}\\
&= \frac{1133\,\mathrm{N}\cdot\frac{1}{2}}{100\,\mathrm{kg}(4\,\mathrm{m}+5\,\mathrm{m}\cdot\frac{1}{2})}\\
&= 0.87\frac{1}{\mathrm{s}^2}\\
\Rightarrow \quad \omega &= 0.93\frac{\mathrm{rad}}{\mathrm{s}}\\
&= 0.93\frac{1}{\cancel{\mathrm{s}}}\cdot\frac{1\mathrm{rev}}{2\pi}\cdot\frac{60\cancel{\mathrm{s}}}{1\,\mathrm{min}}\\
&= 8.9\,\mathrm{rpm}.
\end{aligned}
$$

$$\boxed{T = 1133\,\mathrm{N},\ \omega = 8.9\,\mathrm{rpm}}$$

Alternatively,

we could also find the angular speed using angular momentum balance. The angular momentum balance about point B gives

$$\sum \vec{\boldsymbol{M}}_{/B} = \dot{\vec{\boldsymbol{H}}}_{/B}$$

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/B} &= \vec{\boldsymbol{r}}_{C/B}\times(-mg\hat{\boldsymbol{\jmath}})\\
&= -mgl\sin\theta\hat{\boldsymbol{k}}\\
\dot{\vec{\boldsymbol{H}}}_{/B} &= \vec{\boldsymbol{r}}_{C/B}\times(-m\omega^2 r\hat{\boldsymbol{\imath}})\\
&= -m\omega^2 rl\cos\theta\hat{\boldsymbol{k}}
\end{aligned}
$$

Equating the two quantities, we get

$$
\begin{aligned}
\cancel{m}\omega^2 r\cancel{l}\cos\theta &= \cancel{m}g\cancel{l}\sin\theta\\
\Rightarrow \quad \omega^2 &= \frac{g}{r}\tan\theta\\
&= \frac{g\tan\theta}{R+l\sin\theta}\\
&= \frac{9.8\,\mathrm{m/s^2}\cdot 0.577}{4\,\mathrm{m}+5\,\mathrm{m}\frac{1}{2}}\\
&= 0.87\frac{1}{\mathrm{s}^2}\\
\omega &= 0.93\,\mathrm{s}^{-1} = 8.9\,\mathrm{rpm}
\end{aligned}
$$

which is the same value as we found using the linear momentum balance .

SAMPLE 20.6 A crooked bar rotating with a shaft in space. A uniform rod CD of mass $m = 2\,\mathrm{kg}$ and length $\ell = 1\,\mathrm{m}$ is fastened to a shaft AB by means of two strings: AC of length $R_1 = 30\,\mathrm{cm}$, and BD of length $R_2 = 50\,\mathrm{cm}$. The shaft is rotating at a constant angular velocity $\vec{\omega} = 5\,\mathrm{rad/s}\hat{\boldsymbol{k}}$. There is no gravity. At the instant shown, find the tensions in the two strings.

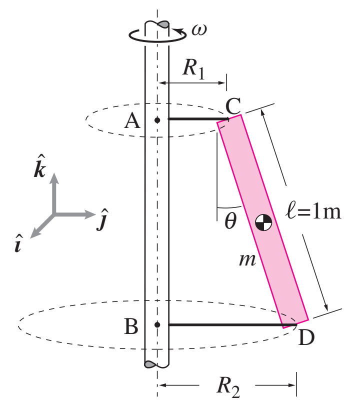


Figure 20.24: A bar held by two strings rotates in 3-D.

Solution The free-body diagram of the rod is shown in Fig. 20.25. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the rod gives:

$$T_1 + T_2 = m\omega^2 r_G. \qquad (20.20)$$

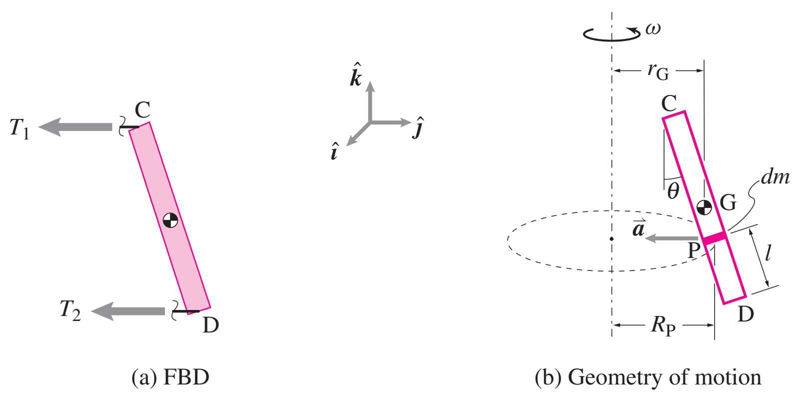


Figure 20.25:

Usually, linear momentum balance gives us two scalar equations in 2-D and three scalar equations in 3-D. Unfortunately, in this case, it gives only one equation for two unknowns T_1 and T_2. Therefore, we need one more equation.

The angular momentum balance about point D gives:

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/D} &= \dot{\vec{\boldsymbol{H}}}_{/D}, \\
\text{where} \quad \sum \vec{\boldsymbol{M}}_{/D} &= \vec{\boldsymbol{r}}_{C/D} \times (-T_1 \hat{\boldsymbol{\jmath}}) \\
&= \ell(-\sin\theta \hat{\boldsymbol{\jmath}} + \cos\theta \hat{\boldsymbol{k}}) \times (-T_1 \hat{\boldsymbol{\jmath}}) \\
&= \ell\, T_1 \cos\theta \hat{\boldsymbol{\imath}},
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/D} &= \int_m \vec{\boldsymbol{r}}_{P/D} \times (-\omega^2 R_P \hat{\boldsymbol{\jmath}}) dm \\
&= \int_0^\ell \overbrace{l(-\sin\theta \hat{\boldsymbol{\jmath}} + \cos\theta \hat{\boldsymbol{k}})}^{\vec{\boldsymbol{r}}_{P/D}} \times (-\omega^2 \overbrace{(R_2 - l\sin\theta)}^{R_P} \hat{\boldsymbol{\jmath}}) \overbrace{\frac{m}{\ell} dl}^{dm} \\
&= \frac{m\omega^2}{\ell} \left(R_2 \cos\theta \int_0^\ell l\, dl - \cos\theta \sin\theta \int_0^\ell l^2\, dl \right) \hat{\boldsymbol{\imath}} \\
&= \frac{m\omega^2}{\ell} \left(R_2 \cos\theta \frac{\ell^2}{2} - \cos\theta \sin\theta \frac{\ell^3}{3} \right) \hat{\boldsymbol{\imath}} \\
&= m\omega^2 \ell \cos\theta (\frac{1}{2} R_2 - \frac{1}{3} \ell \sin\theta) \hat{\boldsymbol{\imath}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\cancel{\ell}\, T_1 \cancel{\cos\theta} &= m\omega^2 \cancel{\ell} \cancel{\cos\theta} (\frac{1}{2} R_2 - \frac{1}{3} \ell \sin\theta) \\
\Rightarrow \quad T_1 &= m\omega^2 (\frac{1}{2} R_2 - \frac{1}{3} \ell \sin\theta).
\end{aligned}
$$

Substituting in (20.20) we get

$$
T_2 = m\omega^2 (r_G - \frac{1}{2} R_2 + \frac{1}{3} \ell \sin\theta).
$$

Plugging in the given numerical values and noting that $r_G = (R_1 + R_2)/2 = 40\,\text{cm}$ and $\ell \sin\theta = R_2 - R_1 = 20\,\text{cm}$, we get

$$
\begin{aligned}
T_1 &= 2\,\text{kg} \cdot (5 \frac{1}{s})^2 \cdot (0.4\,\text{m} - \frac{1}{2} 0.5\,\text{m} + \frac{1}{3} 0.2\,\text{m}) \\
&= 9.17 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = 9.17\,\text{N} \\
\text{and} \quad T_2 &= 10.83\,\text{N}.
\end{aligned}
$$

$T_1 = 9.1\,\text{N}, \quad T_2 = 10.9\,\text{N}$

SAMPLE 20.7 A crooked plate rotating with a shaft in space. A rectangular plate of mass m, length ℓ, and width b is welded to a shaft AB in the center. The long edge of the plate is parallel to the shaft axis but is tipped by an angle ϕ with respect to the shaft axis. The shaft rotates with a constant angular speed ω. The end B of the shaft is free to move in the z-direction. Assume there is no gravity. Find the reactions at the supports.

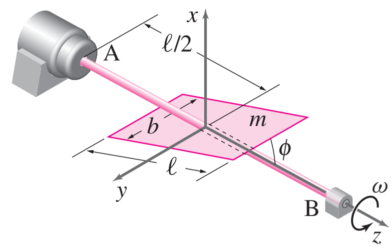


Figure 20.26: A rectangular plate, mounted rigidly at an angle ϕ on a shaft, wobbles as the shaft rotates at a constant speed.

Solution A simple line sketch and the Free-Body Diagram of the system are shown in Fig. 20.27ab.

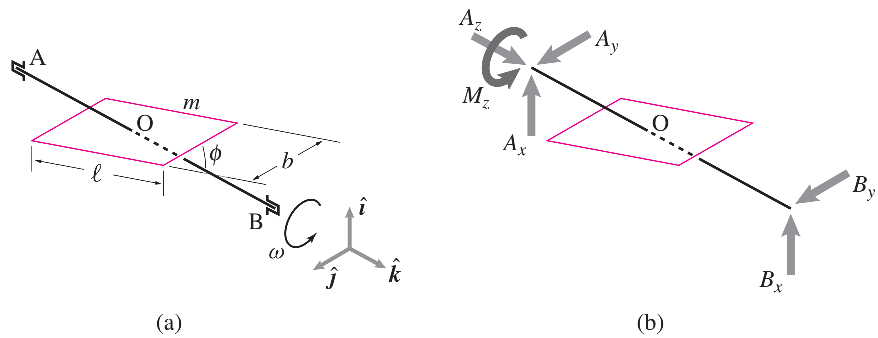


Figure 20.27:

The linear momentum balance equation for the shaft and the plate system is:

$$\sum \vec{F} = m_{total}\,\vec{a}_{cm}.$$

Since the center-of-mass is on the axis of rotation, $\vec{a}_{cm} = \vec{0}$. Therefore,

$$(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + A_z\hat{k} = \vec{0}$$

$$\Rightarrow \qquad A_x + B_x = 0, \quad A_y + B_y = 0, \quad A_z = 0. \tag{20.21}$$

The angular momentum balance about the center-of-mass O is:

$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}$$

- **Calculation of $\sum \vec{M}_O$:**

$$\begin{aligned}\sum \vec{M}_O &= \vec{r}_{A/O} \times \vec{F}_A + \vec{r}_{B/O} \times \vec{F}_B + M_z\hat{k} \\ &= -\frac{\ell}{2}\hat{k} \times (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + \frac{\ell}{2}\hat{k} \times (B_x\hat{\imath} + B_y\hat{\jmath}) + M_z\hat{k} \\ &= \frac{\ell}{2}(A_y - B_y)\hat{\imath} + \frac{\ell}{2}(B_x - A_x)\hat{\jmath} + M_z\hat{k} \end{aligned} \tag{20.22}$$

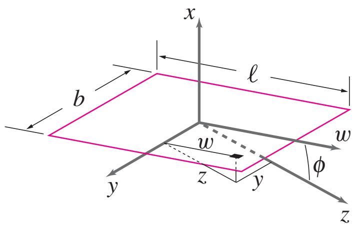
- **Calculation of $\dot{\vec{H}}_{/O}$:** $\dot{\vec{H}}_{/O}$ can be computed in various ways. ① Here, to compute $\dot{\vec{H}}_{/O}$, we use

$$\dot{\vec{H}}_{/O} = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm,$$

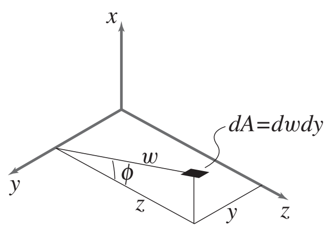
the formula which we have used so far. To carry out this integration for the plate, we take, as usual, an infinitesimal mass dm of the body, calculate its angular momentum about O, and then integrate over the entire mass of the body:

$$\dot{\vec{H}}_{/O} = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm$$

① $\dot{\vec{H}}$ could also be computed using the moment of inertia matrix of the body. See the next two text sections.



(a)



(b)

Figure 20.28: Calculation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$: (a) mass element dm is shown on the plate, (b) the mass element as an area element and its geometry.

We need to write carefully each term in the integrand. Let us define an axis w (don't confuse this dummy variable w with ω) along the length of the plate (see Fig. 20.28(a)). We take an area element $dA = dw\,dy$ on the plate as our infinitesimal mass. Fig. 20.28(b) shows this element and its coordinates.

$$\begin{aligned}
dm &= \rho\, dA = \frac{m}{\ell b} dw\, dy \quad (\rho = \text{mass per unit area}) \\
\vec{\boldsymbol{a}}_{dm} &= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{dm/O}) \\
\vec{\boldsymbol{r}}_{dm/O} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}
\end{aligned}$$

where

$$x = w\sin\phi, \quad y = y, \quad z = w\cos\phi. \tag{20.23}$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{cm} &= \omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times (w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + w\cos\phi\hat{\boldsymbol{k}})) \\
&= \omega\hat{\boldsymbol{k}} \times (\omega\, w\sin\phi\hat{\boldsymbol{\jmath}} - \omega\, y\hat{\boldsymbol{\imath}}) = -\omega^2(w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}), \\
\vec{\boldsymbol{r}}_{dm/O} \times \vec{\boldsymbol{a}}_{dm} &= (w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + w\cos\phi\hat{\boldsymbol{k}}) \times [-\omega^2(w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})] \\
&= \omega^2(-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \int_{-b/2}^{b/2}\int_{-\ell/2}^{\ell/2} \overbrace{\omega^2(-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}})}^{\vec{\boldsymbol{r}}_{dm/O}\times\vec{\boldsymbol{a}}_{dm}} \overbrace{\frac{m}{\ell b} dw\, dy}^{dm} \\
&= \frac{m}{\ell b}\omega^2 \int_{-b/2}^{b/2} \left(\int_{-\ell/2}^{\ell/2} (-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}}) dw \right) dy \\
&= \frac{m}{\ell b}\omega^2 \int_{-b/2}^{b/2} \left(-\sin\phi\cos\phi \left.\frac{w^3}{3}\right|_{-\ell/2}^{\ell/2} + y\cos\phi \underbrace{\left.\frac{w^2}{2}\right|_{-\ell/2}^{\ell/2}}_{0} \right) dy \\
&= -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}.
\end{aligned} \tag{20.24}$$

- **Now, back to angular momentum balance:** Now equating (20.22) and (20.24) and dotting both sides with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ we get

$$A_y - B_y = 0, \quad B_x - A_x = -\frac{m\omega^2\ell}{6}\sin\phi\cos\phi, \quad M_z = 0, \tag{20.25}$$

respectively. Solving (20.21) and (20.25) simultaneously we get

$$A_y = B_y = 0, \quad A_x = \frac{m\omega^2\ell}{12}, \quad \sin\phi\cos\phi B_x = -\frac{m\omega^2\ell}{12}\sin\phi\cos\phi.$$

$$A_x = \frac{m\omega^2\ell}{12}\sin\phi\cos\phi,\ B_x = -\frac{m\omega^2\ell}{12}\sin\phi\cos\phi,\ A_y = B_y = A_z = M_z = 0$$

SAMPLE 20.8 A short rod as a 3-D pendulum. A uniform rod AB of mass m and length 2ℓ is welded to a massless, inextensible thin rod OA at point A. Rod OA is attached to a ball and socket joint at point O. The rods are going around in a circle with constant speed maintaining a constant angle θ with the vertical axis. Assume that θ is small.

1. How many revolutions does the system make in one second?

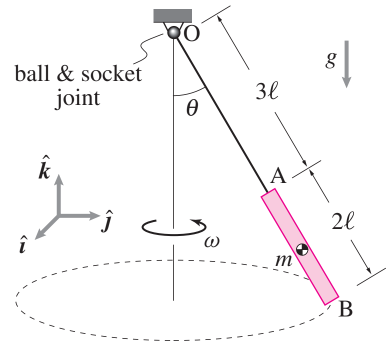


Figure 20.29: A short rod swings in 3-D.

Solution

1. The free-body diagram of the system (rod OA + rod AB) is shown in Fig. 20.30. Let ω be the angular speed of the system. Then, the number of revolutions in one second is $n = \omega/(2\pi)$. Therefore, to find the answer we need to calculate ω.

   The angular momentum balance about point O gives $\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$. Now,

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \vec{\boldsymbol{r}}_{G/O} \times (-mg\hat{\boldsymbol{k}}) \\
&= 4\ell(\sin\theta\,\hat{\boldsymbol{\jmath}} - \cos\theta\,\hat{\boldsymbol{k}}) \times (-mg\hat{\boldsymbol{k}}) \\
&= -4\ell\, mg\sin\theta\,\hat{\boldsymbol{\imath}}.
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \qquad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \int_m \vec{\boldsymbol{r}}_{dm/O} \times \vec{\boldsymbol{a}}_{dm}\, dm \\
&= \int_{3\ell}^{5\ell} \overbrace{\ell(\sin\theta\,\hat{\boldsymbol{\jmath}} - \cos\theta\,\hat{\boldsymbol{k}})}^{\vec{\boldsymbol{r}}_{dm/O}} \times \overbrace{(-\omega^2 l \sin\theta\,\hat{\boldsymbol{\jmath}})}^{\vec{\boldsymbol{a}}_{dm}} \overbrace{\frac{m}{2\ell}\, d\ell}^{dm} \\
&= -\frac{m}{2\ell}\,\omega^2 \sin\theta\cos\theta\,\hat{\boldsymbol{\imath}} \int_{3\ell}^{5\ell} \ell^2\, d\ell \\
&= -\frac{49}{3}\ell^2 m\omega^2 \sin\theta\cos\theta\,\hat{\boldsymbol{\imath}}.
\end{aligned}
$$

   By equating the two quantities ($\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$), we get

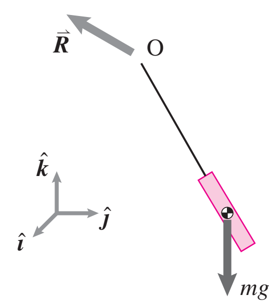
$$
\begin{aligned}
-4\ell\, mg\sin\theta\,\hat{\boldsymbol{\imath}} &= -\frac{49}{3}\ell^2 m\omega^2 \sin\theta\cos\theta\,\hat{\boldsymbol{\imath}} \\
\Rightarrow \qquad \omega^2 &= \frac{12g}{49\ell\cos\theta}.
\end{aligned}
$$

   But for small θ, $\cos\theta \approx 1$. Therefore,

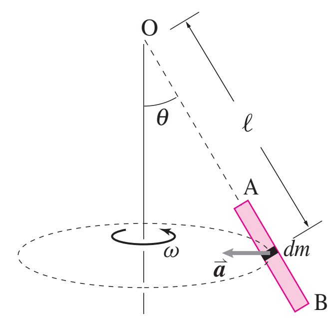
$$
\omega = \sqrt{\frac{12g}{49\ell}} = \frac{2\sqrt{3}}{7}\sqrt{\frac{g}{\ell}}
$$

   and the number of revolutions per unit time is $n = \frac{2\sqrt{3}}{14\pi}\sqrt{\frac{g}{\ell}}$.

$$
\boxed{n = \frac{2\sqrt{3}}{14\pi}\sqrt{\frac{g}{\ell}}}
$$



(a) FBD



(b) Calculation of $\dot{\vec{\boldsymbol{H}}}$

Figure 20.30:

2. Note, the natural frequency of this rod swinging back and forth as a simple pendulum turns out to be the same as the angular speed ω of the rotating system above for small θ.

20.2 Dynamics of fixed axis rotation

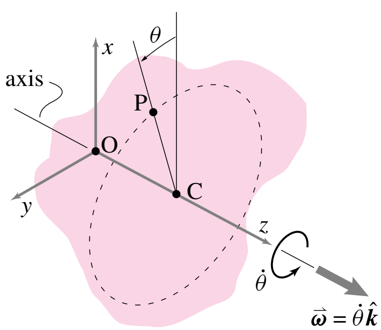
20.2.1 Let $P(x, y, z)$ be a point mass on an abstract massless body that rotates at a constant rate ω about the z-axis. Let $C(0, 0, z)$ be the center of the circular path that P describes during its motion.

a) Show that the rate of change of angular momentum of mass P about the origin O is given by

$$\dot{\vec{H}}_{/O} = -m\omega^2 Rz\hat{e}_\theta$$

where $R = \sqrt{x^2 + y^2}$ is the radius of the circular path of point P and m is its mass.

b) Is there any point on the axis of rotation (z-axis) about which $\dot{\vec{H}}$ is zero. Is this point unique?



Problem 20.2.1

20.2.2 System of Particles going in a circle. Assume for the problems below that you have two mass points going in circles about a common axis at a fixed common angular rate.

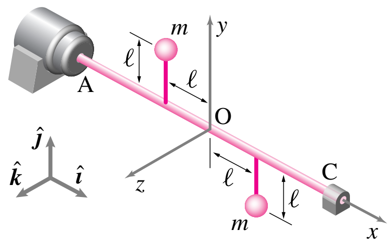
a) 2-D (particles in the plane, axis of rotation normal to the plane). Show, by calculating the sum in $\dot{\mathbf{H}}_C$ that the result is the same when using the two particles or when using a single particle at the center of mass (with mass equal to the sum of the two particles). You may do the general problem or you may pick particular appropriate locations for the masses, particular masses for the masses, a specific point C, a specific axis and a particular rotation rate.

b) 3-D (particles *not* in a common plane normal to the axis of rotation). Using particular mass locations that you choose, show that 2 particles give a *different* value for $\dot{\mathbf{H}}_C$ than a single particle at the center of mass. [Moral: In general, $\dot{\vec{H}}_{/C} \neq \vec{r}_{cm/C} \times m_{tot}\vec{a}_{cm}$.]

20.2.3 The rotating two-mass system shown in the figure has a constant angular velocity $\vec{\omega} = 12\,\text{rad/s}\,\hat{\imath}$. Assume that at the instant shown, the masses are in the xy-plane. Take $m = 1\,\text{kg}$ and $\ell = 1\,\text{m}$.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ for this system.

b) Find the angular momentum $\vec{H}_{/O}$ about point O and its rate of change $\dot{\vec{H}}_{/O}$ for this system.



Problem 20.2.3

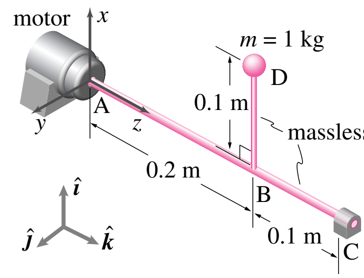
20.2.4 Particle attached to a shaft, no gravity. A shaft of negligible mass spins at constant rate. The motor at A imposes this rate if needed. The bearing at A prevents translation of its end of the shaft in any direction but causes no torques other than that of the motor (which causes a torque in the $\hat{k}$ direction). The frictionless bearing at C holds that end of the shaft from moving in the $\hat{\imath}$ and $\hat{\jmath}$ directions but allows slip in the $\hat{k}$ direction. The shaft is welded orthogonally to a bar, also of negligible mass, which is attached to a small ball of (non-negligible) mass of 1 kg. There is no gravity. The shaft is spinning counterclockwise (when viewed from the positive $\hat{k}$ direction) at 3 rad/s. In the configuration of interest the rod BD is parallel with the $\hat{\imath}$ direction.

a) What is the tension in the rod BD? *

b) What are all the reaction forces at A and C? *

c) What is the torque that the motor at A causes on the shaft? *

d) There are two ways to do problems (b) and (c): one is by treating the shaft, bar and mass as one system and writing the momentum balance equations. The other is to use the result of (a) with action-reaction to do statics on the shaft BC. Repeat problems (b) and (c) doing it the way you did not do it the first time.



Problem 20.2.4

20.2.5 Particle attached to a shaft, with gravity. Consider the same situation as in problem 20.2.4 with the following changes: There is gravity (pointing in the $-\mathbf{i}$ direction). At the time of interest the rod BD is parallel to the $+\mathbf{j}$ direction.

a) What is the force on D from the rod BD? (Note that the rod BD is *not* a 'two-force' member. Why not?)

b) What are all the reaction forces at A and C?

c) What is the torque that the motor at A causes on the shaft?

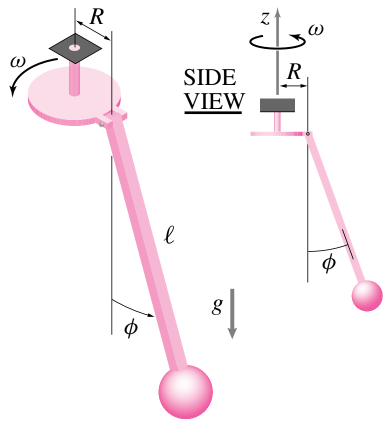
d) Do problems (b) and (c) again a different way.

20.2.6 A turntable rotates at the constant rate of $\omega = 2\,\text{rad/s}$ about the z axis. At the edge of the 2 m radius turntable swings a pendulum which makes an angle ϕ with the negative z direction. The pendulum length is $\ell = 10\,\text{m}$. The pendulum consists of a massless rod with a point mass $m = 2\,\text{kg}$ at the end. The gravitational constant is $g = 10\,\text{m/s}^2$ pointing in the $-z$ direction. A perspective view and a side view are shown.

a) What possible angle(s) could the pendulum make in a steady motion. In other words, find ϕ for steady circular motion of the system. It turns out that this quest leads to a transcendental equation. You are not asked to solve this equation but you should have the equation clearly defined. *

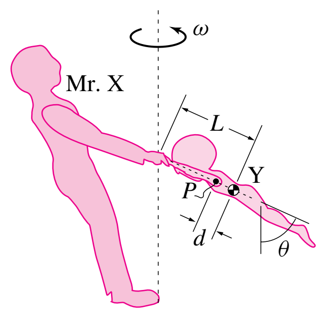
b) How many solutions are there? [hint, sketch the functions involved in your final equation.] *

c) Sketch very approximately any equilibrium solution(s) not already sketched on the side view in the picture provided.



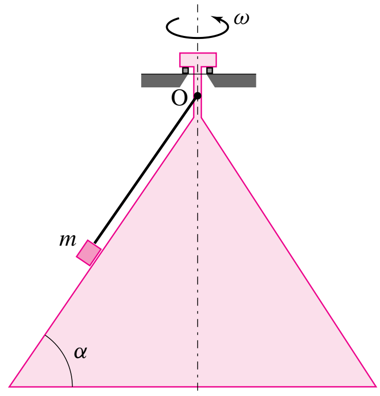
Problem 20.2.6

20.2.7 Mr. X swings his daughter Y about the vertical axis at a constant rate $\omega = 1\,\mathrm{rad/s}$. Y is merely one year old and weighs 12 kg. Assuming Y to be a point mass located at the center of mass, estimate the force on each of her shoulder joints.



Problem 20.2.7

20.2.8 A block of mass $m = 2\,\mathrm{kg}$ supported by a light inextensible string at point O sits on the surface of a rotating cone as shown in the figure. The cone rotates about its vertical axis of symmetry at constant rate ω. Find the value of ω in rpm at which the block loses contact with the cone.



Problem 20.2.8

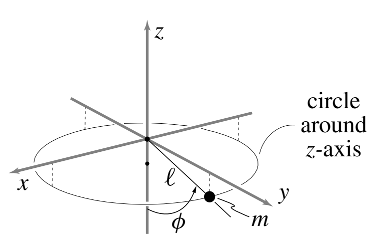
20.2.9 A small rock swings in circles at constant angular rate ω. It is hung by a string with length ℓ the other end of which is at the origin of a fixed coordinate system. The rock has mass m and the gravitational constant is g. The string makes an angle ϕ with the negative z-axis. The mass rotates so that if you follow it with the fingers of your right hand, your right thumb points up the z-axis. At the instant of interest the mass is passing through the yz-plane.

a) How long does the rock take to make one revolution? How does this compare with the period of oscillation of a simple pendulum. *

b) Find the following quantities in any order that pleases you. All solutions should be in terms of m, g, ℓ, ϕ, ω and the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

- The tension in the string (T).
- The velocity of the rock ($\vec{v}$).
- The acceleration of the rock ($\vec{a}$).
- The rate of change of the angular momentum about the $+x$ axis (dH_x/dt).

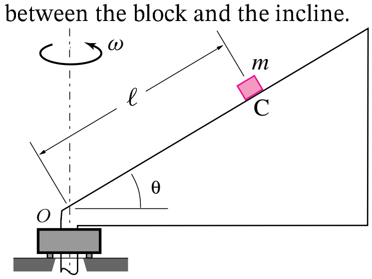
*



Problem 20.2.9

20.2.10 Consider the configuration in problem 20.2.9. What is the relation between the angle (measured from the vertical) at which the rock hangs and the speed at which it moves? Do an experiment with a string, weight, and stopwatch to check your calculation. *

20.2.11 A small block of mass $m = 5\,\mathrm{kg}$ sits on an inclined plane which is supported by a bearing at point O as shown in the figure. At the instant of interest, the block is at a distance $\ell = 0.5\,\mathrm{m}$ from where the surface of the incline intersects the axis of rotation. By experiments, the limiting rate of rotation of the turntable before the block starts sliding up the inclined plane is found to be $\omega = 125\,\mathrm{rpm}$. Find the coefficient of friction between the block and the incline.

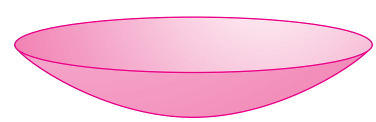


Problem 20.2.11

20.2.12 Particle sliding in circles in a parabolic bowl. As if in a James Bond adventure in a big slippery radar bowl, a particle-like human with mass m is sliding around in circles at speed v. The equation describing the bowl is $z = CR^2 = C(x^2 + y^2)$.

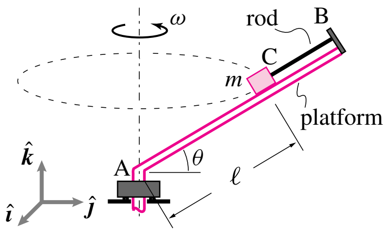
a) Find v in terms of any or all of R, g, and C.

b) Now say you are given ω, C and g. Find v and R if you can. Explain any oddities.



Problem 20.2.12

20.2.13 Mass on a rotating inclined platform. A block of mass $m = 5\,\mathrm{kg}$ is held on an inclined platform by a rod BC as shown in the figure. The platform rotates at a constant angular rate ω. The coefficient of friction between the block and the platform is μ. $\theta = 30^o$, $\ell = 1\,\mathrm{m}$, and $g = 9.81\,\mathrm{m/s^2}$. (a) Assuming $\omega = 5\,\mathrm{rad/s}$ and $\mu = 0$ find the tension in the rod. (b) If $\mu = .3$ what is the range of ω for which the block would stay on the ramp without slipping even if there were no rod? *

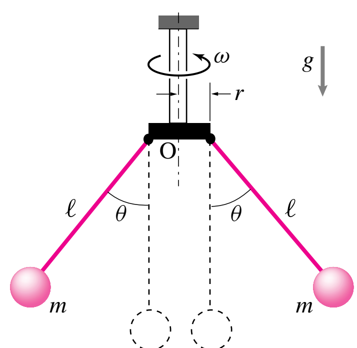


Problem 20.2.13

20.2.14 A circular plate of radius $r = 10\,\mathrm{cm}$ rotates in the horizontal plane about the vertical axis passing through its center O. Two identical point masses hang from two points on the plate diametrically opposite to each other as shown in the figure. At a constant angular speed ω, the two masses maintain a steady state angle θ from their vertical static equilibrium.

a) Find the linear speed and the magnitude of the centripetal acceleration of either of the two masses.

b) Find the steady state ω as a function of θ. Do the limiting values of θ, $\theta \to 0$ and $\theta \to \pi/2$ give sensible values of ω?

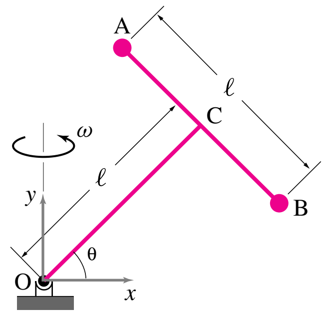


Problem 20.2.14

20.2.15 For the rod and mass system shown in the figure, assume that $\theta = 60°$, $\omega \equiv \dot{\theta} = 5\,\mathrm{rad/s}$, and $\ell = 1\,\mathrm{m}$. Find the acceleration of mass B at the instant shown by

a) calculating $\vec{a} = \vec{\omega}\times\vec{\omega}\times\vec{r}$ where $\vec{r}$ is the position vector of point B with respect to point O, i.e., $\vec{r} = \vec{r}_{B/O}$, and by

b) calculating $\vec{a} = -R\dot{\theta}^2\hat{e}_R$ where R is the radius of the circular path of mass B.



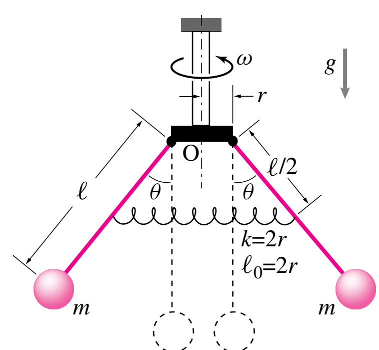
Problem 20.2.15

20.2.16 A spring of relaxed length ℓ_0 (same as the diameter of the plate) is attached to the mid point of the rods supporting the two hanging masses.

a) Find the steady angle θ as a function of the plate's rotation rate ω and the spring constant k. (Your answer may include other given constants.)

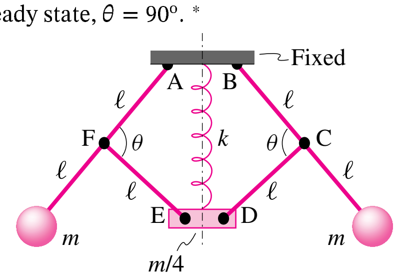
b) Verify that, for $k = 0$, your answer for θ reduces to that in problem 20.2.14.

c) Assume that the entire system sits inside a cylinder of radius 20 cm and the axis of rotation is aligned with the longitudinal axis of the cylinder. The maximum operating speed of the rotating shaft is 200 rpm (i.e., $\omega_{max} = 200\,\mathrm{rpm}$). If the two masses are not to touch the walls of the cylinder, find the required spring stiffness k.



Problem 20.2.16

20.2.17 Flyball governors are used to control the flow of working fluid in steam engines, diesel engines, steam turbines, etc. A spring-loaded flyball governor is shown schematically in the figure. As the governor rotates about the vertical axis, the two masses tend to move outwards. The rigid and massless links that connect the outer arms to the collar, in turn, try to lift the central collar (mass $m/4$) against the spring. The central collar can move along the vertical axis but the top of the governor remains fixed. The mass of each ball is $m = 5$ kg, and the length of each link is $\ell = 0.25$ m. There are frictionless hinges at points A, B, C, D, E, F where the links are connected. Find the amount the string is elongated, Δx, when the governor rotates at 100 rpm, given that the spring constant $k = 500$ N/m, and that in this steady state, $\theta = 90°$. *



Problem 20.2.17

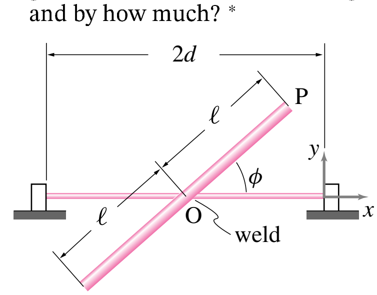
20.2.18 Crooked rod. A 2ℓ long uniform rod of mass m is welded at its center to a shaft at an angle ϕ as shown. The shaft is supported by bearings at its ends and spins at a constant rate ω.

a) What are $\dot{\vec{H}}_{/O}$ and $\dot{\vec{L}}$?

b) Calculate $\dot{\mathbf{H}}_{/O}$ using the integral definition of angular momentum. *

c) What are the six reaction components if there is no gravity? *

d) Repeat the calculations, including gravity g. Which reactions change and by how much? *



Problem 20.2.18

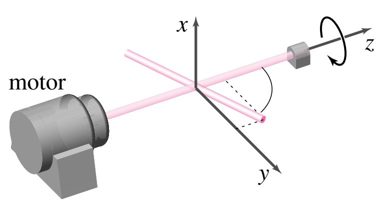
20.2.19 For the configuration in problem 20.2.18, find two positions along the rod where concentrating one-half the mass at each of these positions gives the same $\dot{\vec{\boldsymbol{H}}}_{/O}$.

20.2.20 Rod on a shaft. A uniform rod is welded to a shaft which is connected to frictionless bearings. The rod has mass m and length ℓ and is attached at an angle γ at its midpoint to the midpoint of the shaft. The length of the shaft is $2d$. A motor applies a torque M_{motor} along the axis of the shaft. At the instant shown in the figure, assume m, ℓ, d, ω and M_{motor} are known and the rod lies in the yz-plane.

a) Find the angular acceleration of the shaft.

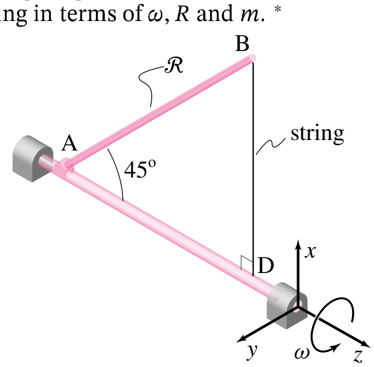
b) The moment that the shaft applies to the rod.

c) The reaction forces at the bearings.



Problem 20.2.20: Rod on a shaft.

20.2.21 Rod spins on a shaft. A uniform rod $\mathcal{R}$ with length R and mass m spins at constant rate ω about the z axis. It is held by a hinge at A and a string at B. Neglect gravity. Find the tension in the string in terms of ω, R and m. *

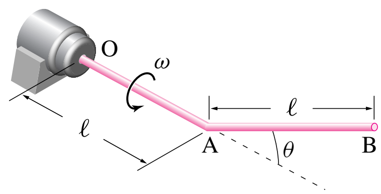


Problem 20.2.21

20.2.22 A uniform narrow shaft of length 2ℓ and mass $2M$ is bent at an angle θ half way down its length. It is spun at constant rate ω.

a) What is the magnitude of the force that must be applied at O to keep the shaft spinning?

b) (harder) What is the magnitude of the moment that must be applied at O to keep the shaft spinning?



Problem 20.2.22: **Spinning rod. No gravity.**

20.2.23 For the configuration in problem 20.2.22, find two positions along the bent portion of the shaft where concentrating one-half its mass at each of these positions gives the same $\dot{\vec{\boldsymbol{H}}}_{/O}$ and $\dot{\vec{\boldsymbol{L}}}$.

20.2.24 Rod attached to a shaft. A uniform 1 kg, $\sqrt{2}$ m rod is attached to a shaft which spins at a constant rate of 1 rad/s. One end of the rod is connected to the shaft by a ball-and-socket joint . The other end is held by two strings (or pin jointed rods) that are connected to a rigid cross bar. The cross bar is welded to the main shaft. Ignore gravity.

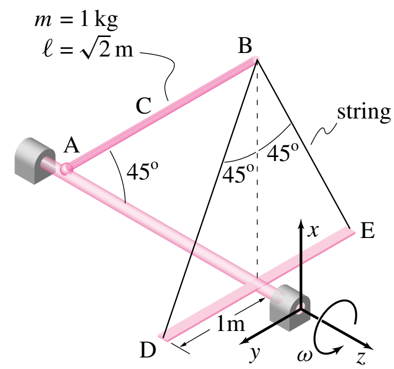
a) Draw a free-body diagram of the rod AB, including a bit of the strings.

b) Find the tension in the string DB. Two approaches:

   (i) Write the six equations of linear and angular momentum balance (about any point of your choosing) and solve them (perhaps using a computer) for the tension T_{DB}.

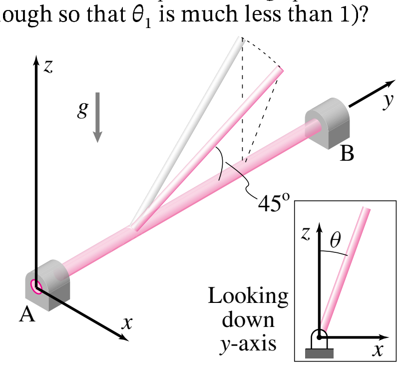
   (ii) Write the equation of angular momentum balance about the axis AE. (That is, take the equation of angular momentum balance about point A and dot both sides with a vector in the direction of $\vec{\boldsymbol{r}}_{E/A}$.) Use this calculation to find T_{DB} and compare to method ii.

c) Find the reaction <u>**force**</u> (a vector) at A.



Problem 20.2.24: Uniform rod attached to a shaft.

20.2.25 3D, crooked bar on shaft falls down. A massless, rigid shaft is connected to frictionless hinges at A and B. A bar with mass M_b and length ℓ is welded to the shaft at 45°. At time $t = 0$ the assembly is tipped a small angle θ_0 from the yz-plane but has no angular velocity. It then falls under the action of gravity. What is θ at $t = t_1$ (assuming t_1 is small enough so that θ_1 is much less than 1)?



Problem 20.2.25: A crooked rod welded to a shaft falls down.