SAMPLE 20.5 **Going on a carnival ride at a constant rate.** A carnival ride with roof AB and carriage BC is rotating about the vertical axis with constant angular velocity $\vec{\omega} = \omega\hat{\jmath}$. If the carriage with its occupants has mass $m = 100\,\text{kg}$, find the tension in the inextensible and massless rod BC when $\theta = 30^o$. What is the required angular speed ω (in revolutions/minute) to maintain this angle?

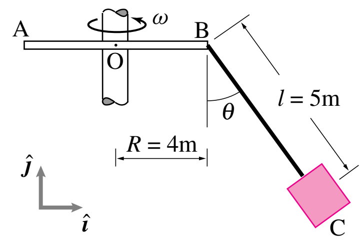
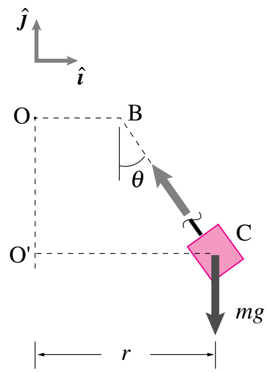
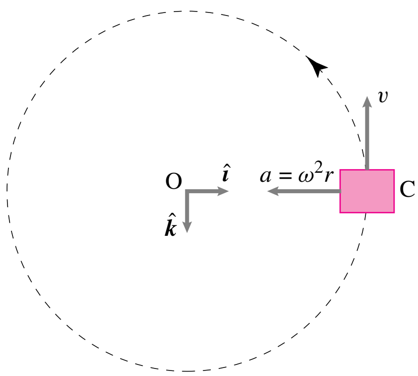


Figure 20.22: A carnival ride rotating at a constant speed

Solution The free-body diagram of the carriage is shown in Fig. 20.23(a). The geometry of motion of the carriage is shown in Fig. 20.23(b). The carriage goes around a circle of radius $r = O'C$ with constant speed $v = \omega r$. The only acceleration that the carriage has is the centripetal acceleration and at the instant of interest $\vec{a} = -\omega^2 r\hat{\imath}$.



(a) FBD of C



(b) Geometry of motion looking down $\hat{\jmath}$-axis

Figure 20.23:

The linear momentum balance $(\sum \vec{F} = \dot{\vec{L}})$ for the carriage gives:

$$\begin{aligned} T\hat{\lambda}_{CB} - mg\hat{\jmath} &= m\vec{a} \\ \text{or} \quad T(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) - mg\hat{\jmath} &= -m\omega^2 r\hat{\imath} \end{aligned} \tag{20.19}$$

Scalar equations from eqn. (20.19) are:

$$\begin{aligned}
[\text{eqn. (20.19)}]\cdot\hat{\boldsymbol{\jmath}} \quad \Rightarrow \quad T\cos\theta - mg &= 0\\
\Rightarrow \quad T &= \frac{mg}{\cos\theta}\\
&= \frac{100\,\text{kg}\cdot 9.8\,\text{m/s}^2}{\sqrt{3}/2}\\
&= 1133\,\text{N}.\\
[\text{eqn. (20.19)}]\cdot\hat{\boldsymbol{\imath}} \quad \Rightarrow \quad -T\sin\theta &= -m\omega^2 r\\
\Rightarrow \quad \omega^2 &= \frac{T\sin\theta}{mr}\\
&= \frac{T\sin\theta}{m(R+l\sin\theta)}\\
&= \frac{1133\,\text{N}\cdot\frac{1}{2}}{100\,\text{kg}(4\,\text{m}+5\,\text{m}\cdot\frac{1}{2})}\\
&= 0.87\frac{1}{\text{s}^2}\\
\Rightarrow \quad \omega &= 0.93\frac{\text{rad}}{\text{s}}\\
&= 0.93\frac{1}{\cancel{\text{s}}}\cdot\frac{1\text{rev}}{2\pi}\cdot\frac{60\cancel{\text{s}}}{1\,\text{min}}\\
&= 8.9\,\text{rpm}.
\end{aligned}$$

$$\boxed{T = 1133\,\text{N},\ \omega = 8.9\,\text{rpm}}$$

Alternatively,

we could also find the angular speed using angular momentum balance. The angular momentum balance about point B gives

$$\sum \vec{\boldsymbol{M}}_{/B} = \dot{\vec{\boldsymbol{H}}}_{/B}$$

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/B} &= \vec{\boldsymbol{r}}_{C/B}\times(-mg\hat{\boldsymbol{\jmath}})\\
&= -mgl\sin\theta\hat{\boldsymbol{k}}\\
\dot{\vec{\boldsymbol{H}}}_{/B} &= \vec{\boldsymbol{r}}_{C/B}\times(-m\omega^2 r\hat{\boldsymbol{\imath}})\\
&= -m\omega^2 rl\cos\theta\hat{\boldsymbol{k}}
\end{aligned}$$

Equating the two quantities, we get

$$\begin{aligned}
\cancel{m}\omega^2 r\cancel{l}\cos\theta &= \cancel{m}g\cancel{l}\sin\theta\\
\Rightarrow \quad \omega^2 &= \frac{g}{r}\tan\theta\\
&= \frac{g\tan\theta}{R+l\sin\theta}\\
&= \frac{9.8\,\text{m/s}^2\cdot 0.577}{4\,\text{m}+5\,\text{m}\frac{1}{2}}\\
&= 0.87\frac{1}{\text{s}^2}\\
\omega &= 0.93\,\text{s}^{-1} = 8.9\,\text{rpm}
\end{aligned}$$

which is the same value as we found using the linear momentum balance .

SAMPLE 20.6 A crooked bar rotating with a shaft in space. A uniform rod CD of mass $m = 2\,\mathrm{kg}$ and length $\ell = 1\,\mathrm{m}$ is fastened to a shaft AB by means of two strings: AC of length $R_1 = 30\,\mathrm{cm}$, and BD of length $R_2 = 50\,\mathrm{cm}$. The shaft is rotating at a constant angular velocity $\vec{\omega} = 5\,\mathrm{rad/s}\hat{\boldsymbol{k}}$. There is no gravity. At the instant shown, find the tensions in the two strings.

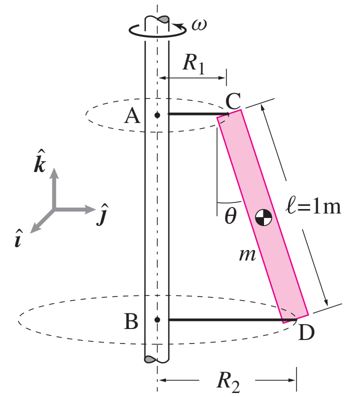


Figure 20.24: A bar held by two strings rotates in 3-D.

Solution The free-body diagram of the rod is shown in Fig. 20.25. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the rod gives:

$$T_1 + T_2 = m\omega^2 r_G. \tag{20.20}$$

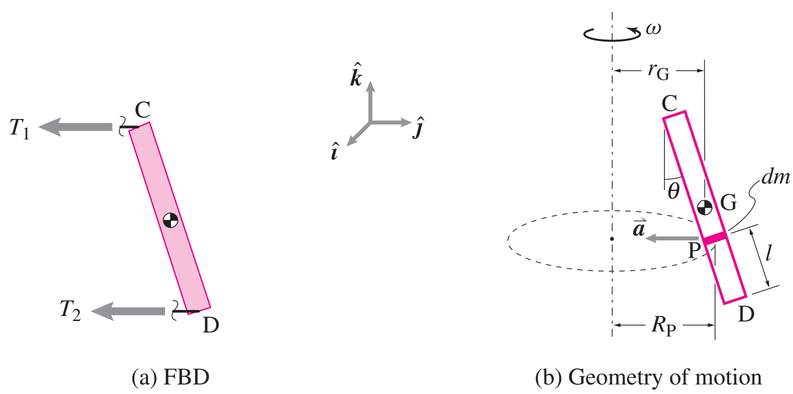


Figure 20.25:

Usually, linear momentum balance gives us two scalar equations in 2-D and three scalar equations in 3-D. Unfortunately, in this case, it gives only one equation for two unknowns T_1 and T_2. Therefore, we need one more equation.

The angular momentum balance about point D gives:

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/D} &= \dot{\vec{\boldsymbol{H}}}_{/D}, \\
\text{where} \quad \sum \vec{\boldsymbol{M}}_{/D} &= \vec{\boldsymbol{r}}_{C/D} \times (-T_1 \hat{\boldsymbol{\jmath}}) \\
&= \ell(-\sin\theta \hat{\boldsymbol{\jmath}} + \cos\theta \hat{\boldsymbol{k}}) \times (-T_1 \hat{\boldsymbol{\jmath}}) \\
&= \ell\, T_1 \cos\theta \hat{\boldsymbol{\imath}},
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/D} &= \int_m \vec{\boldsymbol{r}}_{P/D} \times (-\omega^2 R_P \hat{\boldsymbol{\jmath}}) dm \\
&= \int_0^\ell \overbrace{l(-\sin\theta \hat{\boldsymbol{\jmath}} + \cos\theta \hat{\boldsymbol{k}})}^{\vec{\boldsymbol{r}}_{P/D}} \times (-\omega^2 \overbrace{(R_2 - l\sin\theta)}^{R_P} \hat{\boldsymbol{\jmath}}) \overbrace{\frac{m}{\ell} dl}^{dm} \\
&= \frac{m\omega^2}{\ell} \left(R_2 \cos\theta \int_0^\ell l\, dl - \cos\theta \sin\theta \int_0^\ell l^2\, dl \right) \hat{\boldsymbol{\imath}} \\
&= \frac{m\omega^2}{\ell} \left(R_2 \cos\theta \frac{\ell^2}{2} - \cos\theta \sin\theta \frac{\ell^3}{3} \right) \hat{\boldsymbol{\imath}} \\
&= m\omega^2 \ell \cos\theta (\frac{1}{2} R_2 - \frac{1}{3} \ell \sin\theta) \hat{\boldsymbol{\imath}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\cancel{\ell}\, T_1 \cancel{\cos}\theta &= m\omega^2 \cancel{\ell} \cancel{\cos}\theta (\frac{1}{2} R_2 - \frac{1}{3} \ell \sin\theta) \\
\Rightarrow \quad T_1 &= m\omega^2 (\frac{1}{2} R_2 - \frac{1}{3} \ell \sin\theta).
\end{aligned}
$$

Substituting in (20.20) we get

$$
T_2 = m\omega^2 (r_G - \frac{1}{2} R_2 + \frac{1}{3} \ell \sin\theta).
$$

Plugging in the given numerical values and noting that $r_G = (R_1 + R_2)/2 = 40\,\text{cm}$ and $\ell \sin\theta = R_2 - R_1 = 20\,\text{cm}$, we get

$$
\begin{aligned}
T_1 &= 2\,\text{kg} \cdot (5\frac{1}{s})^2 \cdot (0.4\,\text{m} - \frac{1}{2} 0.5\,\text{m} + \frac{1}{3} 0.2\,\text{m}) \\
&= 9.17 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = 9.17\,\text{N} \\
\text{and} \quad T_2 &= 10.83\,\text{N}.
\end{aligned}
$$

$T_1 = 9.1\,\text{N}, \quad T_2 = 10.9\,\text{N}$

SAMPLE 20.7 **A crooked plate rotating with a shaft in space.** A rectangular plate of mass m, length ℓ, and width b is welded to a shaft AB in the center. The long edge of the plate is parallel to the shaft axis but is tipped by an angle ϕ with respect to the shaft axis. The shaft rotates with a constant angular speed ω. The end B of the shaft is free to move in the z-direction. Assume there is no gravity. Find the reactions at the supports.

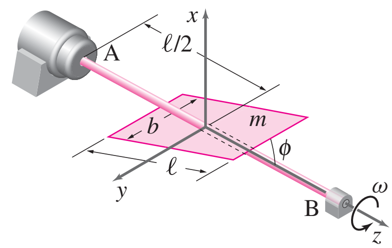
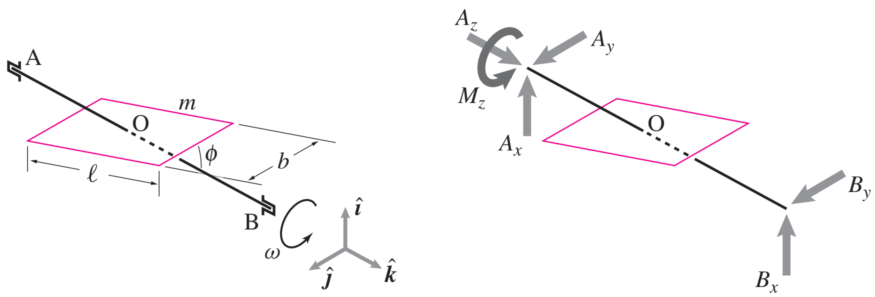


Figure 20.26: A rectangular plate, mounted rigidly at an angle ϕ on a shaft, wobbles as the shaft rotates at a constant speed.

Solution A simple line sketch and the Free-Body Diagram of the system are shown in Fig. 20.27ab.



(a) (b)

Figure 20.27:

The linear momentum balance equation for the shaft and the plate system is:

$$\sum \vec{F} = m_{total}\,\vec{a}_{cm}.$$

Since the center-of-mass is on the axis of rotation, $\vec{a}_{cm} = \vec{0}$. Therefore,

$$(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + A_z\hat{k} = \vec{0}$$

$$\Rightarrow \qquad A_x + B_x = 0, \quad A_y + B_y = 0, \quad A_z = 0. \qquad (20.21)$$

The angular momentum balance about the center-of-mass O is:

$$\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{/\mathrm{O}}$$

- **Calculation of $\sum \vec{M}_{\mathrm{O}}$:**

$$\begin{aligned}
\sum \vec{M}_{\mathrm{O}} &= \vec{r}_{A/O} \times \vec{F}_A + \vec{r}_{B/O} \times \vec{F}_B + M_z\hat{k} \\
&= -\frac{\ell}{2}\hat{k} \times (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + \frac{\ell}{2}\hat{k} \times (B_x\hat{\imath} + B_y\hat{\jmath}) + M_z\hat{k} \\
&= \frac{\ell}{2}(A_y - B_y)\hat{\imath} + \frac{\ell}{2}(B_x - A_x)\hat{\jmath} + M_z\hat{k} \qquad (20.22)
\end{aligned}$$

- **Calculation of $\dot{\vec{H}}_{/\mathrm{O}}$:** $\dot{\vec{H}}_{/\mathrm{O}}$ can be computed in various ways. ① Here, to compute $\dot{\vec{H}}_{/\mathrm{O}}$, we use

$$\dot{\vec{H}}_{/\mathrm{O}} = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm}\; dm,$$

the formula which we have used so far. To carry out this integration for the plate, we take, as usual, an infinitesimal mass dm of the body, calculate its angular momentum about O, and then integrate over the entire mass of the body:

$$\dot{\vec{H}}_{/\mathrm{O}} = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm}\; dm$$

① $\dot{\vec{H}}$ could also be computed using the moment of inertia matrix of the body. See the next two text sections.

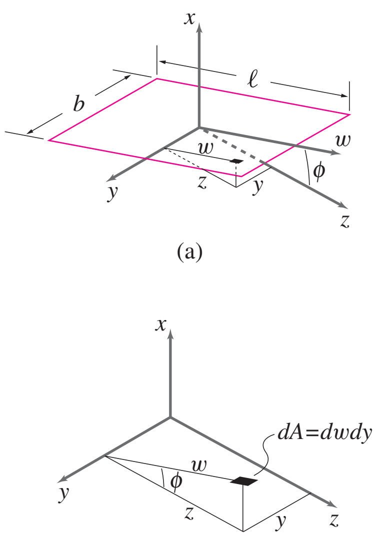


Figure 20.28: Calculation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$: (a) mass element dm is shown on the plate, (b) the mass element as an area element and its geometry.

We need to write carefully each term in the integrand. Let us define an axis w (don't confuse this dummy variable w with ω) along the length of the plate (see Fig. 20.28(a)). We take an area element $dA = dw\,dy$ on the plate as our infinitesimal mass. Fig. 20.28(b) shows this element and its coordinates.

$$
\begin{aligned}
dm &= \rho\, dA = \frac{m}{\ell b} dw\, dy \quad (\rho = \text{mass per unit area})\\
\vec{\boldsymbol{a}}_{dm} &= \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_{dm/O})\\
\vec{\boldsymbol{r}}_{dm/O} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}
\end{aligned}
$$

where

$$
x = w\sin\phi, \quad y = y, \quad z = w\cos\phi. \tag{20.23}
$$

Therefore,

$$
\begin{aligned}
\vec{\boldsymbol{a}}_{cm} &= \omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times (w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + w\cos\phi\hat{\boldsymbol{k}}))\\
&= \omega\hat{\boldsymbol{k}} \times (\omega\, w\sin\phi\,\hat{\boldsymbol{\jmath}} - \omega\, y\hat{\boldsymbol{\imath}}) = -\omega^2(w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}),\\
\vec{\boldsymbol{r}}_{dm/O} \times \vec{\boldsymbol{a}}_{dm} &= (w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + w\cos\phi\hat{\boldsymbol{k}}) \times [-\omega^2(w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})]\\
&= \omega^2(-w^2\sin\phi\cos\phi\,\hat{\boldsymbol{\jmath}} + wy\cos\phi\,\hat{\boldsymbol{\imath}}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \int_{-b/2}^{b/2}\int_{-\ell/2}^{\ell/2} \overbrace{\omega^2(-w^2\sin\phi\cos\phi\,\hat{\boldsymbol{\jmath}} + wy\cos\phi\,\hat{\boldsymbol{\imath}})}^{\vec{\boldsymbol{r}}_{dm/O}\times\vec{\boldsymbol{a}}_{dm}}\,\overbrace{\frac{m}{\ell b}dw\,dy}^{dm}\\
&= \frac{m}{\ell b}\omega^2 \int_{-b/2}^{b/2}\left(\int_{-\ell/2}^{\ell/2}(-w^2\sin\phi\cos\phi\,\hat{\boldsymbol{\jmath}} + wy\cos\phi\,\hat{\boldsymbol{\imath}})dw\right)dy\\
&= \frac{m}{\ell b}\omega^2 \int_{-b/2}^{b/2}\left(-\sin\phi\cos\phi\left.\frac{w^3}{3}\right|_{-\ell/2}^{\ell/2} + y\cos\phi\,\underbrace{\left.\frac{w^2}{2}\right|_{-\ell/2}^{\ell/2}}_{0}\right)dy\\
&= -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\,\hat{\boldsymbol{\jmath}}.
\end{aligned}\tag{20.24}
$$

- **Now, back to angular momentum balance:** Now equating (20.22) and (20.24) and dotting both sides with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ we get

$$
A_y - B_y = 0, \quad B_x - A_x = -\frac{m\omega^2\ell}{6}\sin\phi\cos\phi, \quad M_z = 0, \tag{20.25}
$$

respectively. Solving (20.21) and (20.25) simultaneously we get

$$
A_y = B_y = 0, \quad A_x = \frac{m\omega^2\ell}{12}, \quad \sin\phi\cos\phi B_x = -\frac{m\omega^2\ell}{12}\sin\phi\cos\phi.
$$

$$
\boxed{A_x = \frac{m\omega^2\ell}{12}\sin\phi\cos\phi,\ B_x = -\frac{m\omega^2\ell}{12}\sin\phi\cos\phi,\ A_y = B_y = A_z = M_z = 0}
$$

SAMPLE 20.8 **A short rod as a 3-D pendulum.** A uniform rod AB of mass m and length 2ℓ is welded to a massless, inextensible thin rod OA at point A. Rod OA is attached to a ball and socket joint at point O. The rods are going around in a circle with constant speed maintaining a constant angle θ with the vertical axis. Assume that θ is small.

1. How many revolutions does the system make in one second?

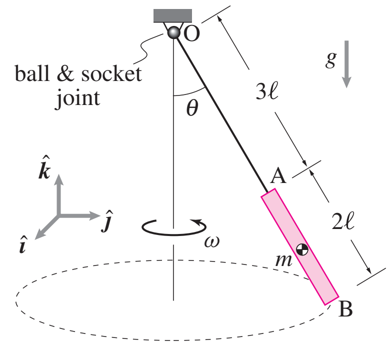


Figure 20.29: A short rod swings in 3-D.

Solution

1. The free-body diagram of the system (rod OA + rod AB) is shown in Fig. 20.30. Let ω be the angular speed of the system. Then, the number of revolutions in one second is $n = \omega/(2\pi)$. Therefore, to find the answer we need to calculate ω.

   The angular momentum balance about point O gives $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$. Now,

$$
\begin{aligned}
\sum \vec{M}_O &= \vec{r}_{G/O} \times (-mg\hat{k}) \\
&= 4\ell(\sin\theta\,\hat{\jmath} - \cos\theta\,\hat{k}) \times (-mg\hat{k}) \\
&= -4\ell\, mg\sin\theta\,\hat{\imath}.
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \qquad \dot{\vec{H}}_{/O} &= \int_m \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm \\
&= \int_{3\ell}^{5\ell} \overbrace{\ell(\sin\theta\,\hat{\jmath} - \cos\theta\,\hat{k})}^{\vec{r}_{dm/O}} \times \overbrace{(-\omega^2 l \sin\theta\,\hat{\jmath})}^{\vec{a}_{dm}} \overbrace{\frac{m}{2\ell}\, d\ell}^{dm} \\
&= -\frac{m}{2\ell}\,\omega^2 \sin\theta\cos\theta\,\hat{\imath} \int_{3\ell}^{5\ell} \ell^2\, d\ell \\
&= -\frac{49}{3}\ell^2 m\omega^2 \sin\theta\cos\theta\,\hat{\imath}.
\end{aligned}
$$

   By equating the two quantities ($\sum \vec{M}_O = \dot{\vec{H}}_{/O}$), we get

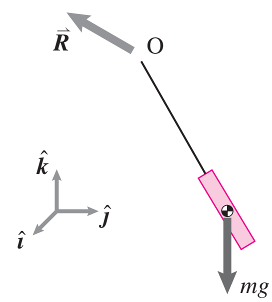
$$
\begin{aligned}
-4\ell\, mg\sin\theta\,\hat{\imath} &= -\frac{49}{3}\ell^2 m\omega^2 \sin\theta\cos\theta\,\hat{\imath} \\
\Rightarrow \qquad \omega^2 &= \frac{12g}{49\ell\cos\theta}.
\end{aligned}
$$

   But for small θ, $\cos\theta \approx 1$. Therefore,

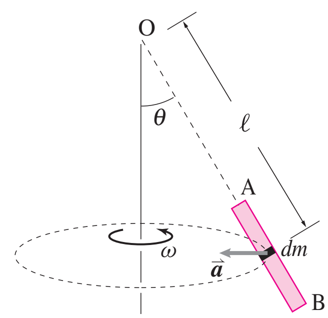
$$
\omega = \sqrt{\frac{12g}{49\ell}} = \frac{2\sqrt{3}}{7}\sqrt{\frac{g}{\ell}}
$$

   and the number of revolutions per unit time is $n = \frac{2\sqrt{3}}{14\pi}\sqrt{\frac{g}{\ell}}$.

$$
\boxed{n = \frac{2\sqrt{3}}{14\pi}\sqrt{\frac{g}{\ell}}}
$$



(a) FBD



(b) Calculation of $\dot{\vec{H}}$

Figure 20.30:

2. Note, the natural frequency of this rod swinging back and forth as a simple pendulum turns out to be the same as the angular speed ω of the rotating system above for small θ.