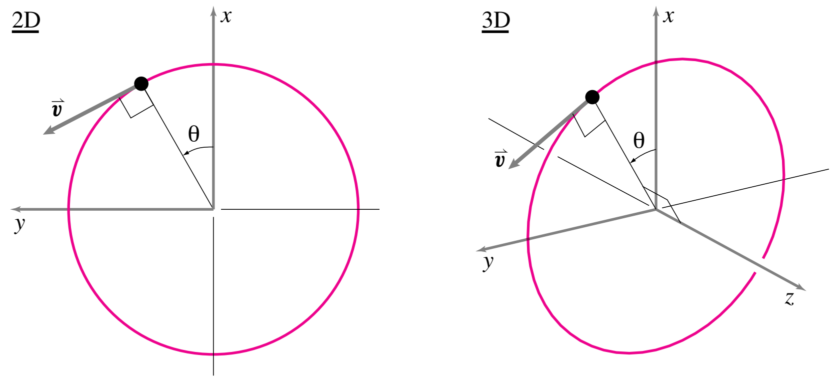
20.1 3-D description of circular motion

Let's first assume each particle is going in circles around the z axis, as in the previous chapter. The figure below shows this two-dimensional situation first two-dimensionally (left) and then as a two-dimensional motion in a three-dimensional world (right).



Either way, the velocity and acceleration are the same:

- the velocity is tangent to the circle it is going around and is proportional in magnitude to the radius of the circle and also to its angular speed. That is, the direction of the velocity is in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude ωR, where ω is the angular rate of rotation and R is the radius of the circle that the particle is going around.
- the acceleration can be constructed as the sum of two vectors. One is pointed to the center of the circle and proportional in magnitude to both the square of the angular speed and to the radius. The other vector is tangent to the circle and equal in magnitude to the rate of increase of speed.

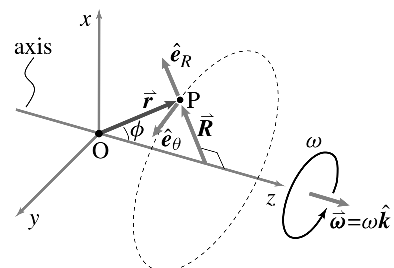


Figure 20.2: When a point P is going in circles about the z axis, we define the unit vector $\hat{\boldsymbol{e}}_R$ to be pointed from the axis to the point P. We define the unit vector $\hat{\boldsymbol{e}}_\theta$ to be tangent to the circle at P. Both of these vectors change in time as the point moves along its circular path.

These two ideas are summarized by the following formulas:

$$\vec{\boldsymbol{v}} = \omega R \hat{\boldsymbol{e}}_\theta \qquad \text{and} \qquad \vec{\boldsymbol{a}} = -\underbrace{\omega^2 R}_{v^2/R}\,\hat{\boldsymbol{e}}_R + \underbrace{R\ddot{\theta}}_{\dot{v}}\,\hat{\boldsymbol{e}}_\theta \tag{20.1}$$

with

$$v = \omega R. \tag{20.2}$$

The axis of rotation might not be the z-axis of a convenient xyz coordinate system. So the xy plane of circles might not be the xy plane of the coordinate system you might want to use for some other reasons. Fortunately, we can write the formulas 20.1 in a way that rids us of these problems.

Here are some formulas which are equivalent to the formulas 20.1 but which do not make use of the polar coordinate base vectors.

$$\vec{\boldsymbol{v}} = \vec{\omega} \times \vec{\boldsymbol{r}}, \tag{20.3}$$

$$\vec{\boldsymbol{a}} = \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) + \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}. \tag{20.4}$$

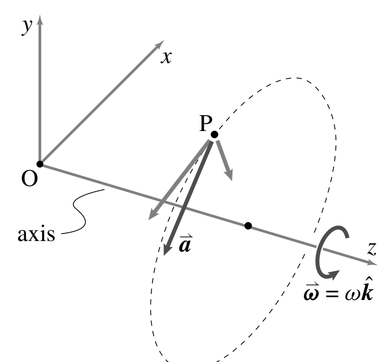


Figure 20.3: The acceleration is the sum of two components. One directed towards the center of the circle in the $-\hat{\boldsymbol{e}}_R$ direction, and one tangent to the circle in the $\hat{\boldsymbol{e}}_\theta$ direction.

To check that equations 20.3 and 20.4 are really equivalent to 20.1 we need to verify that the vector $\vec{\omega}\times\vec{r}$ is equal to $\omega R\hat{e}_\theta$, that the vector $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is equal to the vector $-\omega^2 R\hat{e}_R$, and that $\dot{\vec{\omega}}\times\vec{r}$ is equivalent to $R\ddot{\theta}\hat{e}_\theta$.

First, define $\vec{r}$ as the position of the point of interest relative to any point on the axis of rotation. If this point happens to be the center of the circle then $\vec{r}=\vec{R}$. But, in general, $\vec{r}\neq\vec{R}$.

Now look at $\vec{v}=\vec{\omega}\times\vec{r}$ with respect to fig. 20.2. Using the right hand rule, it is clear that the direction of the cross product $\vec{\omega}\times\vec{r}$ is in fact the $\hat{e}_\theta$ direction. What about the magnitude? The magnitude $|\vec{\omega}\times\vec{r}| = |\vec{\omega}||\vec{r}|\sin\phi$. But $|\vec{r}|\sin\phi = R$. So the magnitude of $\vec{\omega}\times\vec{r}$ is ωR. That is,

$$\begin{aligned}
\vec{\omega}\times\vec{r} &= (|\vec{\omega}\times\vec{r}|)\cdot(\text{unit vector in the direction of } \vec{\omega}\times\vec{r}) && (20.5)\\
&= (\omega R)\hat{e}_\theta && (20.6)\\
&= \vec{v} && (20.7)
\end{aligned}$$

So $\vec{v}=\vec{\omega}\times\vec{r}$ is correct. The check of the second term in the acceleration formula follows the same reasoning. But the check of the first term involves the triple cross product.

Triple cross product

The formula for acceleration of a point on a rigid body includes the centripetal term $\vec{\omega}\times(\vec{\omega}\times\vec{r})$. This expression is a special case of the general vector expression

$$\vec{A}\times(\vec{B}\times\vec{C})$$

which is sometimes called the 'vector triple product' because its value is a vector (as opposed to the scalar value of the 'scalar triple product'). The primary useful identity with vector triple products (see (17.29)) is:

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}. \qquad (20.8)$$

This formula may be remembered by the semi-mnemonic device 'cab minus bac' since $\vec{A}\cdot\vec{C}=\vec{C}\cdot\vec{A}$ and $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$. This formula is discussed in box 17.4 on page 865.

So now we can write

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{\omega}\times(\omega R)\cdot\hat{e}_\theta = -\omega^2\underbrace{R\hat{e}_R}_{\vec{R}} = -\omega^2\vec{R} = \vec{a}. \qquad (20.9)$$

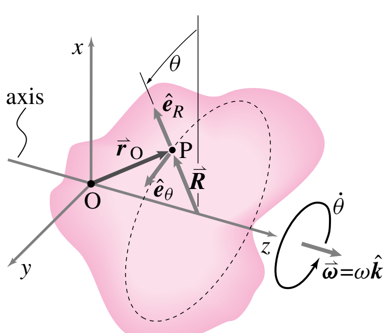


Figure 20.4: A rigid body spinning about the z axis. Every point on the body, like point P at $\vec{r}$, is going in circles. All of these circles have centers on the axis of rotation. All the points are going around at the same angular rate, $\dot{\theta}=\omega$.

Note that equations 20.3 and 20.4 are *vector* equations. They do not make use of any coordinate system. So, for example, we can use them even if $\vec{\omega}$ is not in the z direction.

Angular velocity of a rigid body in 3D

If a rigid body is constrained to rotate about an axis then all points on the body have the same angular rate about that axis. Hence one says that the *body* has an angular velocity. So the measure of rotation rate of a three-dimensional rigid body is the body's angular velocity vector $\vec{\boldsymbol{\omega}}$. At any instant in time a given body has one and only one angular velocity $\vec{\boldsymbol{\omega}}$. Although we only discuss fixed axis rotation in this chapter, a given body has a unique angular velocity for general motion.

For rotations about the z-axis, $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. ω is the $\dot{\theta}$ shown in fig. 20.4. Since all points of a rigid body have the same $\dot{\theta}$, even if they have different θ's, the definition is not ambiguous. We would like to make this idea precise enough to be useful for calculations. Why, one may ask, do we talk about rotation rate ω or $\vec{\boldsymbol{\omega}}$ instead of just using the derivative of an angle θ, namely $\dot{\theta}$? The answer is that for a rigid body one would have trouble deciding what angle θ to measure.

First recall the situation for a two-dimensional rigid body. Consider all possible θ_1, θ_2, θ_3, ... , the angles that all possible lines marked on a body could make with the positive x-axis, the positive y axis, or any other fixed line that does not rotate. As the body rotates all of these angles increment by the same amount. Therefore, each of these angles increases at the same rate. Because all these angular rates are the same, one need not define $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$, $\dot{\theta}_3 = \omega_3$, etc. for each of the lines. Every line attached to the body rotates at the same rate and we call this rate ω. So $\dot{\theta}_1 = \omega$, $\dot{\theta} = \omega$, $\dot{\theta}_3 = \omega$, etc. Rather than say the lengthy phrase 'the rate of rotation of every line attached to the rigid body is ω', we instead say 'the rigid body has angular velocity ω'. For use in vector equations, we define the angular velocity vector of a two-dimensional rigid body as $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and for a 3-D body rotating about an axis in the $\hat{\boldsymbol{\lambda}}$ direction as $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\lambda}}$.

What do we mean by these angles θ_i for crooked lines in a three-dimensional body? We simply look at shadows of lines drawn in or on the body of interest onto a plane perpendicular to the axis of rotation; *i.e.,* perpendicular to $\hat{\boldsymbol{\lambda}}$. See fig. 20.5. The rate of change of their orientation ($\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3$) is ω, and $\vec{\boldsymbol{\omega}}$ is therefore $\omega\hat{\boldsymbol{\lambda}}$. This intuitive geometric definition of ω in terms of the rotation of shadows has run its course. It gives you a picture but is not very convenient for developing formulas.

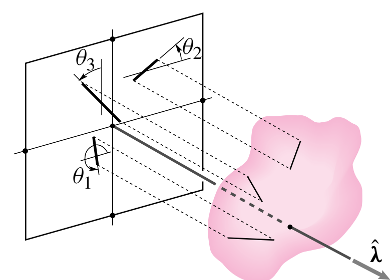


Figure 20.5: The shadows of lines marked in a 3-D rigid body are shown on a plane perpendicular to the axis of rotation. The shadows rotate on the plane at the rate $\dot{\theta}_1 = \dot{\theta}_2 = \omega$. The angular velocity vector is $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\lambda}}$.

Example: What are the velocity and acceleration of one corner of a cube that is spinning about a diagonal?

A one foot cube is spinning at 60 rpm about the diagonal OC. What are the velocity and acceleration of point B? First let's find the velocity using $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$:

$$\begin{aligned} \vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \\ &= (60\,\text{rpm}\,\boldsymbol{\lambda}_{OC}) \times \vec{\boldsymbol{r}}_{OB} \\ &= \left(2\pi\,\text{s}^{-1}\frac{(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})}{\sqrt{3}}\right) \times (1\,\text{ft}(\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})) \\ &= (2\pi/\sqrt{3})(-\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})\,\text{ft/s}. \end{aligned}$$

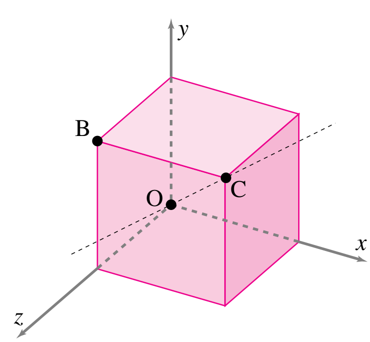


Figure 20.6: A spinning cube.

Now of course this equation could have been worked out with the first of equations 20.1 but it would have been quite tricky to find the vectors $\hat{\boldsymbol{e}}_\theta$, $\vec{\boldsymbol{R}}$, and $\hat{\boldsymbol{e}}_R$! More simply, to find the acceleration we just plug in the formula $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$

as follows:

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= \vec{\omega}\times[\vec{\omega}\times\vec{\boldsymbol{r}}]\\
&= (60\,\text{rpm}\,\boldsymbol{\lambda}_{OC})\times[(60\,\text{rpm}\,\boldsymbol{\lambda}_{OC})\times\vec{\boldsymbol{r}}_{OB}]\\
&= (2\pi\,\text{s}^{-1}\frac{(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}})}{\sqrt{3}})\\
&\qquad\qquad \times\left[(\frac{2\pi\,\text{s}^{-1}(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}})}{\sqrt{3}})\times(1\,\text{ft}(\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}})\right]\\
&= (2\pi/\sqrt{3})^2(2\hat{\boldsymbol{\imath}}-\hat{\boldsymbol{\jmath}}-\hat{\boldsymbol{k}})\,\text{ft/s}^2.
\end{aligned}
$$

The last line of calculation is eased by the calculation of velocity above where the term in square brackets, the velocity, was already calculated.

Relative motion of points on a rigid body

The relative velocity of two points A and B is defined to be

$$\vec{\boldsymbol{v}}_{\mathrm{B/A}} \equiv \vec{\boldsymbol{v}}_{\mathrm{B}} - \vec{\boldsymbol{v}}_{\mathrm{A}}$$

So, the relative velocity of two points glued to one rigid body, as observed from a Newtonian frame, is given by

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{B/A}} &\equiv \vec{\boldsymbol{v}}_{\mathrm{B}} - \vec{\boldsymbol{v}}_{\mathrm{A}} && (20.10)\\
&= \vec{\omega}\times\vec{\boldsymbol{r}}_{\mathrm{B/O}} - \vec{\omega}\times\vec{\boldsymbol{r}}_{\mathrm{A/O}} && (20.11)\\
&= \vec{\omega}\times(\vec{\boldsymbol{r}}_{\mathrm{B/O}} - \vec{\boldsymbol{r}}_{\mathrm{A/O}}) && (20.12)\\
&= \vec{\omega}\times\vec{\boldsymbol{r}}_{\mathrm{B/A}}, && (20.13)
\end{aligned}
$$

where point O is a point in the Newtonian frame on the fixed axis of rotation. Clearly, since points A and B are fixed in the body $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{B}$ is $\vec{\mathbf{0}}$. But, point A has some absolute velocity that is different from the absolute velocity of point B, as viewed from point O in the fixed frame. The relative velocity of points A and B, the difference in absolute velocity of the two points, is due to the difference in their positions relative to point O. Similarly, the relative acceleration of two points glued to one rigid body spinning about a fixed axis is

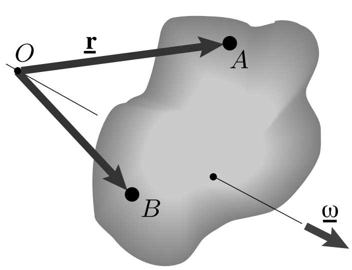


Figure 20.7: Two points, A and B on one body that has a fixed axis of rotation.

$$\vec{\boldsymbol{a}}_{\mathrm{B/A}} \equiv \vec{\boldsymbol{a}}_{\mathrm{B}} - \vec{\boldsymbol{a}}_{\mathrm{A}} = \vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}}_{\mathrm{B/A}}) + \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_{\mathrm{B/A}}. \qquad (20.14)$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. Like their junior 2D cousins, these kinematics results, 20.13 and 20.14, are useful for calculating angular momentum relative to the center-of-mass as well as for the understanding of the motions of machines with moving connected parts.

To repeat, for two points on one rigid body we have that

$$\dot{\vec{r}}_{B/A} = \vec{\omega} \times \vec{r}_{\mathrm{B/A}}. \tag{20.15}$$

Equation (20.15) is perhaps the most fundamental equation for those desiring an understanding of the motions of rigid bodies. Unless one desires to pursue matrix representations of rotation, equation (20.15) is *the* defining equation for $\vec{\omega}$. There is always exactly one vector $\vec{\omega}$ so that equation (20.15) is true for every pair of points on a rigid body.

Equation (20.15) is not so simple a defining equation as one would hope for such an intuitive concept as spinning. But, besides the pictorial definition with shadows, it's the simplest definition we have.

Relative velocity and acceleration using rotating frames

If we glued a coordinate system $x'y'$ to a rotating rigid body $\mathcal{C}$, we would have what is called a rotating frame as shown in fig. 20.8. The base vectors in this frame change in time the same way as did $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in section 15.1. That is

$$\frac{d}{dt}\hat{\boldsymbol{\imath}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}' \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\jmath}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}'.$$

If we now write the relative position of B to A in terms of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, we have

$$\vec{r}_{\mathrm{B/A}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'.$$

Since the coordinates x' and y' rotate with the body to which A and B are attached, they are constant with respect to that body,

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

So

$$\begin{aligned}
\frac{d}{dt}(\vec{r}_{\mathrm{B/A}}) &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\boldsymbol{\imath}}' + x'\frac{d}{dt}\hat{\boldsymbol{\imath}}' + \underbrace{\dot{y}'}_{0}\hat{\boldsymbol{\jmath}}' + y'\frac{d}{dt}\hat{\boldsymbol{\jmath}}' \\
&= x'(\vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\omega}_{\mathcal{C}} \times \underbrace{(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')}_{\vec{r}_{B/A}} \\
&= \vec{\omega}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}}.
\end{aligned}$$

If we now try to calculate the rate of change of $\vec{v}_{B/A}$,

$$\begin{aligned}
\frac{d}{dt}(\vec{v}_{\mathrm{B/A}}) &= \frac{d}{dt}\left(\vec{\omega}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}}\right) \\
&= \frac{d\vec{\omega}_{\mathcal{C}}}{dt} \times \vec{r}_{\mathrm{B/A}} + \vec{\omega}_{\mathcal{C}} \times \frac{d\vec{r}_{B/A}}{dt} \\
\vec{a}_{\mathrm{B/A}} &= \dot{\vec{\omega}}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}} + \vec{\omega}_{\mathcal{C}} \times (\vec{\omega}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}}).
\end{aligned}$$

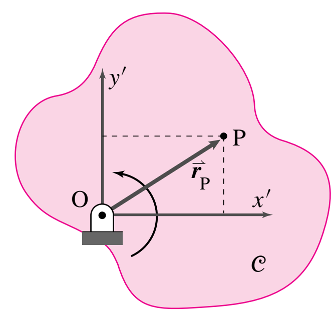


Figure 20.8: A rotating rigid body $\mathcal{C}$ with rotating frame $x'y'$ attached.

Mechanics

Now that we know the velocity and acceleration of every point in the system we are ready, in principle, to find $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ in terms of the angular velocity vector $\vec{\boldsymbol{\omega}}$, its rate of change $\dot{\vec{\boldsymbol{\omega}}}$, and the position of all the mass in the system. This we do in the next section.