SAMPLE 20.1 **Kinematics in 3-D—some basic questions:** The following questions are about the velocity and acceleration formulae for the non-constant rate circular motion about a fixed axis:

$$\begin{aligned}\vec{\boldsymbol{v}} &= \vec{\omega}\times\vec{\boldsymbol{r}}\\ \vec{\boldsymbol{a}} &= \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}+\vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}})\end{aligned}$$

1. In the formulae above, what is $\vec{\boldsymbol{r}}$? How is it different from $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R$ used in the formulae $\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$?
2. What is the difference between $\dot{\theta}$ and $\vec{\omega}$, and $\ddot{\theta}$ and $\dot{\vec{\omega}}$?
3. Are the parentheses around the $\vec{\omega}\times\vec{\boldsymbol{r}}$ term necessary in the acceleration formula?
4. Under what condition(s) can a particle have only tangential acceleration?

Solution

1. In the formulae for velocity and acceleration, $\vec{\boldsymbol{r}}$ refers to a vector from any point on the axis of rotation to the point of interest. Usually the origin of a coordinate system located on the axis of rotation is a convenient point to take as the base point for $\vec{\boldsymbol{r}}$. You can, however, choose any other point on the axis of rotation as the base point.

   The vector $\vec{\boldsymbol{r}}$ is different from $\vec{\boldsymbol{R}}$ in that $\vec{\boldsymbol{R}}$ is the position vector of the point of interest with respect to the center of the circular path that the point traces during its motion. See Fig. 5.2 of the text.

2. $\dot{\theta}$ and $\ddot{\theta}$ are the magnitudes of angular velocity and angular acceleration, respectively, in planar motion, *i.e.,* $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}} = \ddot{\theta}\hat{\boldsymbol{k}}$. We have introduced these notations to highlight the simple nature of planar circular motion. Of course, you are free to use $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}} = \alpha\hat{\boldsymbol{k}}$ if you wish.

3. Yes, the parentheses around $\vec{\omega}\times\vec{\boldsymbol{r}}$ in the acceleration formula are mandatory. The parentheses imply that this term has to be calculated before carrying out the cross product with $\vec{\omega}$ in the formula. Since the term in the parentheses is the velocity, you may also write the acceleration formula as

   $$\vec{\boldsymbol{a}} = \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}+\vec{\omega}\times\vec{\boldsymbol{v}}.$$

   Even if the formula is clear in your mind and you know which cross product to carry out first, it is a good idea to put the parentheses.

4. First of all let us identify the tangential and the normal (or radial) components of the acceleration:

   $$\vec{\boldsymbol{a}} = \overbrace{\dot{\vec{\omega}}\times\vec{\boldsymbol{r}}}^{\text{tangential}}+\overbrace{\vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}})}^{\text{radial/normal}}.$$

   Clearly, for a particle to have only tangential acceleration, the second term must be zero. For the second term to be zero we must have either $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$ or $\vec{\omega} = \vec{\boldsymbol{0}}$. But if $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$, then the tangential acceleration also becomes zero; the particle is on the axis of rotation and hence has no acceleration. Thus the condition that allows only tangential acceleration to survive is $\vec{\omega} = \vec{\boldsymbol{0}}$. Now remember that $\dot{\vec{\omega}}$ is *not* zero. Therefore, the condition we have found can be true only momentarily. This disappearance of the radial acceleration happens at start-up motions and in direction-reversing motions. ①

① In all start-up motions, the velocity is zero but the acceleration is not zero at the start up ($t = 0$). In direction-reversing motions, such as that of the washing machine drum during the wash-cycle, just at the moment when the direction of motion reverses, velocity becomes zero but the acceleration is non-zero.

SAMPLE 20.2 For a particle in circular motion, $\vec{\omega} = 2\,\mathrm{rad/s}\hat{\boldsymbol{k}}$, $\vec{\alpha} = 4\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{r}}_G = 3\,\mathrm{m}(\cos 30^\circ \hat{\boldsymbol{\imath}} + \sin 30^\circ \hat{\boldsymbol{\jmath}})$, where $\vec{\boldsymbol{r}}_G$ is the position vector of the particle. Find (i) $\vec{\boldsymbol{v}}$, (ii) the tangential acceleration $\vec{\boldsymbol{a}}_t$, and (iii) the radial acceleration $\vec{\boldsymbol{a}}_r$, and show the resulting vectors.

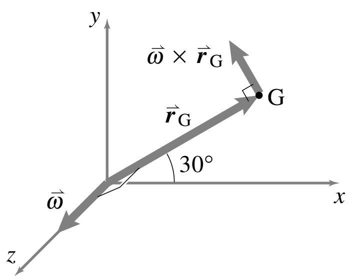


Figure 20.9: $\vec{\boldsymbol{v}} = \vec{\omega} \times \vec{\boldsymbol{r}}_G$.

Solution

$$\begin{aligned}
\vec{\omega} &= \omega\hat{\boldsymbol{k}} = 2\,\mathrm{rad/s}\hat{\boldsymbol{k}} \\
\vec{\alpha} &= \alpha\hat{\boldsymbol{k}} = 4\,\mathrm{rad/s^2}\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{r}}_G &= r_{G_x}\hat{\boldsymbol{\imath}} + r_{G_y}\hat{\boldsymbol{\jmath}} = 3\,\mathrm{m}(\cos 30^\circ \hat{\boldsymbol{\imath}} + \sin 30^\circ \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

(i) From the given formulae, the linear velocity

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\omega} \times \vec{\boldsymbol{r}}_G \\
&= \omega\hat{\boldsymbol{k}} \times (r_{G_x}\hat{\boldsymbol{\imath}} + r_{G_y}\hat{\boldsymbol{\jmath}}) = \omega r_{G_x}\hat{\boldsymbol{\jmath}} + \omega r_{G_y}(-\hat{\boldsymbol{\imath}}) \\
&= 6\,\mathrm{m/s}(\cos 30^\circ \hat{\boldsymbol{\jmath}} - \sin 30^\circ \hat{\boldsymbol{\imath}}) \\
&= 3\,\mathrm{m/s}(-\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}}).
\end{aligned}$$

The velocity vector $\vec{\boldsymbol{v}}$ is perpendicular to both $\vec{\omega}$ and $\vec{\boldsymbol{r}}_G$. These vectors are shown in Fig. 20.9. You should use the right hand rule to confirm the direction of $\vec{\boldsymbol{v}}$.

$$\boxed{\vec{\boldsymbol{v}} = 3\,\mathrm{m/s}(-\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})}$$

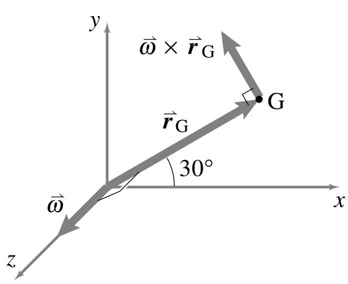


Figure 20.10: $\vec{\boldsymbol{a}}_t = \vec{\alpha} \times \vec{\boldsymbol{r}}_G$.

(ii) The tangential acceleration

$$\begin{aligned}
\vec{\boldsymbol{a}}_t &= \vec{\alpha} \times \vec{\boldsymbol{r}}_G \\
&= \alpha\hat{\boldsymbol{k}} \times (r_{G_x}\hat{\boldsymbol{\imath}} + r_{G_y}\hat{\boldsymbol{\jmath}}) = \alpha r_{G_x}\hat{\boldsymbol{\jmath}} - \alpha r_{G_y}\hat{\boldsymbol{\imath}} \\
&= 8\,\mathrm{m/s^2}(\cos 30^\circ \hat{\boldsymbol{\jmath}} - \sin 30^\circ \hat{\boldsymbol{\imath}}).
\end{aligned}$$

Since $\vec{\omega}$ and $\vec{\alpha}$ are in the same direction, calculation of $\vec{\boldsymbol{a}}_t$ is similar to that of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}_t$ has to be in the same direction as $\vec{\boldsymbol{v}}$. This vector is shown in Fig. 20.10. Once again, just as in the case of $\vec{\boldsymbol{v}}$ we could easily check that $\vec{\boldsymbol{a}}_t$ is perpendicular to both $\vec{\alpha}$ and $\vec{\boldsymbol{r}}_G$.

$$\boxed{\vec{\boldsymbol{a}}_t = 4\,\mathrm{m/s^2}(3\,\mathrm{m/s}(-\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})}$$

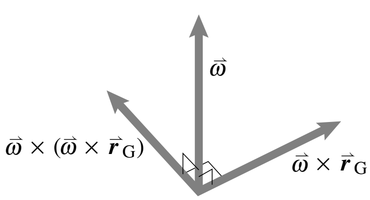


Figure 20.11: $\vec{\boldsymbol{a}}_r = \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_G)$

(iii) Finally, the radial acceleration

$$\begin{aligned}
\vec{\boldsymbol{a}}_r &= \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_G) \\
&= \omega\hat{\boldsymbol{k}} \times (\omega r_{G_x}\hat{\boldsymbol{\jmath}} - \omega r_{G_y}\hat{\boldsymbol{\imath}}) \\
&= \omega^2 r_{G_x}(-\hat{\boldsymbol{\imath}}) - \omega^2 r_{G_y}\hat{\boldsymbol{\jmath}} = -\omega^2\vec{\boldsymbol{r}}_G \\
&= 12\,\mathrm{m/s^2}(-\cos 30^\circ \hat{\boldsymbol{\imath}} - \sin 30^\circ \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

This cross product is illustrated in Fig. 20.11. Both from the illustration as well as the calculation you should be able to see that $\vec{\boldsymbol{a}}_r$ is in the direction of $-\vec{\boldsymbol{r}}_G$. In fact, you could show that $\vec{\boldsymbol{a}}_r = -\omega^2\vec{\boldsymbol{r}}_G$.

$$\boxed{\vec{\boldsymbol{a}}_r = 6\,\mathrm{m/s^2}(-\sqrt{3}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}})}$$

SAMPLE 20.3 **Simple 3-D circular motion.** A system with two point masses A and B is mounted on a rod OC which makes an angle $\theta = 45^\circ$ with the horizontal. The entire assembly rotates about the y-axis with constant angular speed $\omega = 3\,\mathrm{rad/s}$, maintaining the angle θ. Find the velocity of point A. What is the radius of the circular path that A describes? Assume that at the instant shown, AB is in the xy plane.

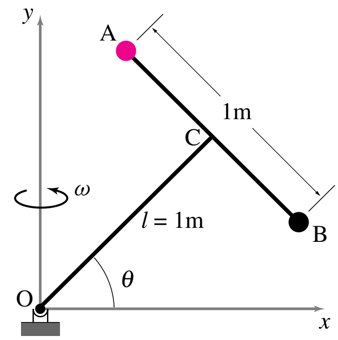


Figure 20.12

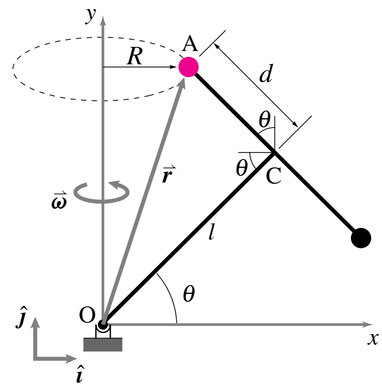


Figure 20.13

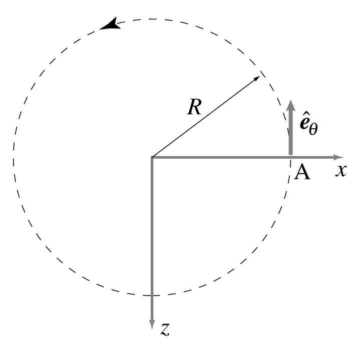


Figure 20.14: Circular trajectory of point A as seen by looking down along the y-axis. At the instant shown, $\hat{\boldsymbol{e}}_\theta = -\hat{\boldsymbol{k}}$.

Solution The angular velocity of the system is

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} = 3\,\mathrm{rad/s}\,\hat{\boldsymbol{\jmath}}.$$

Let $\vec{\boldsymbol{r}}_{\mathrm{A}}$ be the position vector of point A. Then the velocity of point A is

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{A}} \qquad (\text{ and } \vec{\boldsymbol{r}}_{\mathrm{A}} = \vec{\boldsymbol{r}}_{\mathrm{C}} + \vec{\boldsymbol{r}}_{\mathrm{A/C}}) \\
&= \omega\hat{\boldsymbol{\jmath}} \times (\underbrace{l\cos\theta\hat{\boldsymbol{\imath}} + l\sin\theta\hat{\boldsymbol{\jmath}}}_{\vec{\boldsymbol{r}}_{\mathrm{C}}} + \underbrace{d\cos\theta\hat{\boldsymbol{\jmath}} - d\sin\theta\hat{\boldsymbol{\imath}}}_{\vec{\boldsymbol{r}}_{\mathrm{A/C}}}) \\
&= \omega\hat{\boldsymbol{\jmath}} \times [(l\cos\theta - d\sin\theta)\hat{\boldsymbol{\imath}} + (l\sin\theta + d\cos\theta)\hat{\boldsymbol{\jmath}}] \\
&= -(\omega l\cos\theta - \omega d\sin\theta)\hat{\boldsymbol{k}} \\
&= -[3\,\mathrm{rad/s}(1\,\mathrm{m}\cdot\cos 45^\circ - 0.5\,\mathrm{m}\cdot\sin 45^\circ)] \\
&= -1.06\,\mathrm{m/s}\hat{\boldsymbol{k}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = -1.06\,\mathrm{m/s}\hat{\boldsymbol{k}}.}$$

We can find the radius of the circular path of A by geometry. However, we know that the velocity of A is also given by

$$\vec{\boldsymbol{v}} = \omega R\hat{\boldsymbol{e}}_\theta$$

where R is the radius of the circular path. At the instant of interest, $\hat{\boldsymbol{e}}_\theta = -\hat{\boldsymbol{k}}$ (see figure 20.14).

$$\text{Thus} \qquad \vec{\boldsymbol{v}} = -\omega R\hat{\boldsymbol{k}}.$$

Comparing with the answer obtained above, we get

$$\begin{aligned}
-1.06\,\mathrm{m/s}\hat{\boldsymbol{k}} &= -\omega R\hat{\boldsymbol{k}} \\
\Rightarrow \qquad R &= \frac{1.06\,\mathrm{m/s}}{3\,\mathrm{rad/s}} \\
&= 0.35\,\mathrm{m}.
\end{aligned}$$

$$\boxed{R = 0.35\,\mathrm{m}}$$

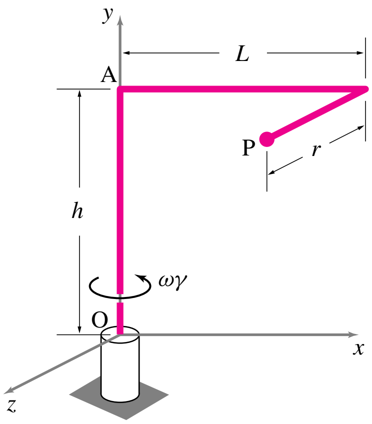


Figure 20.15

SAMPLE 20.4 **Velocity and acceleration in 3-D:** The rod shown in the figure rotates about the y-axis at angular speed 10 rad/s and accelerates at the rate of $2\,\mathrm{rad/s^2}$. The dimensions of the rod are $L = h = 2\,\mathrm{m}$ and $r = 1\,\mathrm{m}$. There is a small mass P glued to the rod at its free end. At the instant shown, the three segments of the rod are parallel to the three axes.

1. Find the velocity of point P at the instant shown.
2. Find the acceleration of point P at the instant shown.

Solution We are given:

$$\vec{\omega} = \omega\hat{\jmath} = 10\,\mathrm{rad/s}\,\hat{\jmath} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\hat{\jmath} = 2\,\mathrm{rad/s^2}\,\hat{\jmath}.$$

1. The velocity of point P is

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

At the instant shown, the position vector of point P (the vector $\vec{r}_{P/O}$) seems to be a good choice for $\vec{r}$. (2) Thus,

$$\vec{r} \equiv \vec{r}_{P/O} = L\hat{\imath} + h\hat{\jmath} + r\hat{k}.$$

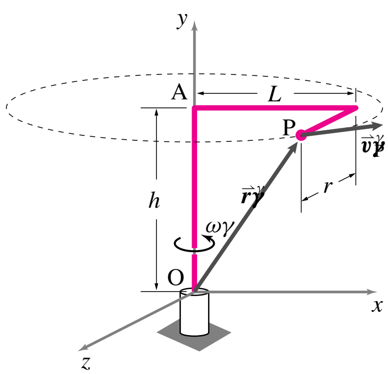
Therefore,

$$\begin{aligned}
\vec{v} &= \omega\hat{\jmath} \times (L\hat{\imath} + h\hat{\jmath} + r\hat{k}) \\
&= \omega(-L\hat{k} + r\hat{\imath}) \\
&= 10\,\mathrm{rad/s} \cdot (-2\,\mathrm{m}\hat{k} + 1\,\mathrm{m}\hat{\imath}) \\
&= (20\hat{\imath} - 10\hat{k})\,\mathrm{m/s}.
\end{aligned}$$

As a check, we look down the y-axis and draw a velocity vector at point P (tangent to the circular path at point P) without paying attention to the answer we got. From the top view in Fig. 20.16 we see that at least the signs of the components of $\vec{v}$ seem to be correct.

$$\boxed{\vec{v} = (20\hat{\imath} - 10\hat{k})\,\mathrm{m/s}}$$

(2) An even better choice, perhaps, is the vector $\vec{r}_{P/A}$. Remember, the only requirement on $\vec{r}$ is that it must start at *some* point on the axis of rotation and must end at the point of interest.



Path of point P seen from the top:

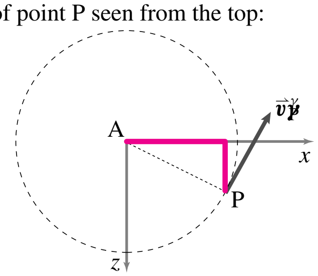


Figure 20.16: By drawing the velocity vector at point P (a vector tangent to the path) we see that $\vec{v}_P$ must have a positive x-component and a negative z-component.

2. The acceleration of point P is

$$\begin{aligned}
\vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\
&= \dot{\omega}\hat{\jmath} \times (L\hat{\imath} + h\hat{\jmath} + r\hat{k}) + \omega\hat{\jmath} \times \underbrace{\omega(-L\hat{k} + r\hat{\imath})}_{\vec{\omega}\times\vec{r}} \\
&= \dot{\omega}(-L\hat{k} + r\hat{\imath}) + \omega^2(-L\hat{\imath} - r\hat{k}) \\
&= 2\,\mathrm{rad/s^2}(-2\,\mathrm{m}\hat{k} + 1\,\mathrm{m}\hat{\imath}) - 100(\mathrm{rad/s})^2(2\,\mathrm{m}\hat{\imath} + 1\,\mathrm{m}\hat{k}) \\
&= -(98\hat{\imath} + 104\hat{k})\,\mathrm{m/s^2}.
\end{aligned}$$

We can check the sign of the components of $\vec{a}$ also. Note that the tangential acceleration, $\dot{\vec{\omega}} \times \vec{r}$, is much smaller than the centripetal acceleration, $\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Therefore, the total acceleration is almost in the same direction as the centripetal acceleration, that is, directed from point P to A. If you draw a vector from P to A, you should be able to see that it has negative components along both the x- and z-axes. Thus the answer we have got seems to be correct, at least in direction.

$$\boxed{\vec{a} = -(98\hat{\imath} + 104\hat{k})\,\mathrm{m/s^2}}$$