19.3 Dynamics of multi-degree-of-freedom 2-D mechanisms

A typical machine has many parts. They may work in concert as a one degree-of-freedom system or they may be designed to move independently.

One degree-of-freedom machines. Some mechanisms are designed so the parts work together to do something, one thing, well. A car going straight down a road has dozens of moving parts: cylinders, connecting rods, crank shaft, cam shaft, transmission gears, drive shaft, differential, wheel axles, wheels and the car itself. All of these parts move, each in its own different way, approximately like a rigid object. But in the simplest description① they move as a one-degree-of freedom mechanism towards the end of fighting friction and moving the car down the road. If the crankshaft rotates a small amount, each part moves a corresponding amount; the amount of rotation of the crank shaft determines the position of all of the parts. And all of the velocities and accelerations of all of the bits of mass in the car can be determined by the rotation θ_{crank} and its derivatives ($dot\theta_{\text{crank}}$ and $ddot\theta_{\text{crank}}$). Thus a car, even with lots of pieces moving this way and that, might be well-modeled for some purposes as a one degree-of-freedom mechanism. For one-degree-of-freedom mechanisms the methods of Section 19.2 may be appropriate.

①**A car has one degree-of-freedom?** For studying the acceleration of a car going straight one might ignore the flexibility of the drive shaft and the bounce of the suspension, *etc.*thus leading to a 1 DOF model. But if you want to see how acceleration causes the car to tip back, or how the car oscillates when the engine lugs, then a 2, or more, DOF model would be appropriate.

Multi DOF systems In some modeling of machines it is important to keep track of multiple degrees of freedom (See fig. 18.51 on page 974 for examples of simple mechanisms). There are various reasons that a multi-DOF analysis is relevant:

- Some machines intentionally have various ways of moving that are controlled by various motors. Classic examples include:
  - Robots,
  - Animal bodies.
- Some systems intentionally have various degrees of freedom so as to respond smoothly to disturbances, for example the suspensions on the bottoms of cars and washing machines.
- Some systems have undesirable degrees of freedom due to the parts which are nominally rigid actually being elastic. Thus a machine which is intended to have one degree of freedom might have various undesirable vibration modes.

Mechanical analysis of multi-DOF systems. To solve problems with multiple degrees of freedom the basic strategy is, as described at the start of the chapter:

1. Draw FBDs of each object,
2. Pick configuration variables,
3. Write linear and angular momentum balance equations
4. Solve the equations for variables of interest (usually forces and second derivatives of the configuration variables).
5. Set up and solve the resulting differential equations (if you are trying to find the motion).

There are two basic approaches to these multi-object problems which, for lack of better language we call “brute-force” and “clever”.

A. In the **brute force** approach you write three times as many scalar balance equations as you have objects (limiting attention to 2D where each object has 3 DOFs and 3 independent momentum balance equations). That is, for example, for each free-body diagram you write linear momentum balance and angular momentum balance about the center of mass. Then you take this set of $3n$ equations and add and subtract them to solve for variables of interest.

   This approach is quite suitable for computers so most commercial general purpose dynamic simulators use a variant of this approach. For individual use with packaged software, the brute-force approach is generally both more reliable and more time consuming.

B. In the clever approach you write as many scalar momentum balance equations as you have unknowns. For example, if you have 2 degrees of freedom and you are concerned with motions and not reaction forces, you write 2 equations. You do this by finding momentum balance equations that do not include the variables you are not interested in. Usually this involves using angular momentum balance about hinge points, or linear momentum balance orthogonal to sliding contacts.

   The **clever** approach does not always work; the four-bar linkage is the classic problem case. However the desire to find such minimal sets of equations of motion it is historically important(2).

(2) Attempts to automate the clever approach, that is to quickly find minimal equations of motion, led to Lagrange equations which led to Hamilton’s equations which led to quantum mechanics (but we won’t be that clever here).

At this point in the subject there are no quick simple problems. All problems are involved, especially if taken from start all the way to plotting solutions to the differential equations. The examples that follow emphasize getting to the equations of motion. The skills for numerically solving the differential equations and plotting the solutions are the same as from the start of dynamics so are not discussed until the sample problems which show all the work from setup to solution.

Example: Block sliding on sliding block: clever approach
Block 1 with mass m_1 rolls without friction on ideal massless wheels at A and B (see fig. 19.43). Block 2 with mass m_2 rolls down the tipped top of block 1 on

ideal massless rollers at C and D. The locations of G_1 relative to A and B, and of G_2 relative to C and D are known. How do blocks 1 and 2 move?

First look at the free-body diagram of the system and note that there are no unknown forces in the $\hat{\boldsymbol{\imath}}$ direction. So, for the system

$$\begin{aligned}\Big\{\sum \vec{\boldsymbol{F}}_{\mathrm{i}} &= \dot{\vec{\boldsymbol{L}}}\Big\}\cdot\hat{\boldsymbol{\imath}}\\ \Rightarrow\quad 0 &= m_1\ddot{x} + m_2(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}'\hat{\boldsymbol{\jmath}}')\cdot\hat{\boldsymbol{\imath}}\\ &= (m_1+m_2)\ddot{x} - \ddot{y}' m_2\cos\theta. \qquad (19.47)\end{aligned}$$

Looking at the free-body diagram of mass 2 note that there are no unknown forces in the $\hat{\boldsymbol{\jmath}}'$ direction, so

$$\begin{aligned}\Big\{\sum \vec{\boldsymbol{F}}_{\mathrm{i}} &= \dot{\vec{\boldsymbol{L}}}\Big\}\cdot\hat{\boldsymbol{\jmath}}'\\ \Rightarrow\quad m_2 g\sin\theta &= m_2(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}'\hat{\boldsymbol{\jmath}}')\cdot\hat{\boldsymbol{\jmath}}'\\ m_2 g\sin\theta &= m_2(-\cos\theta\,\ddot{x} + \ddot{y}'). \qquad (19.48)\end{aligned}$$

Eqns. 19.47 and 19.48 are a system of two equations in the two unknowns $\ddot{x}$ and $\ddot{y}'$

$$\begin{aligned}(m_1+m_2)\,\ddot{x} \quad -m_2\cos\theta\,\ddot{y}' &= 0\\ -m_2\cos\theta\,\ddot{x} \quad +m_2\,\ddot{y}' &= m_2 g\sin\theta\end{aligned}$$

which can be solved for $\ddot{x}$ and $\ddot{y}'$ by hand or on the computer. Finding $x(t)$ and $y'(t)$ is then easy because both $\ddot{x}$ and $\ddot{y}'$ are constants.

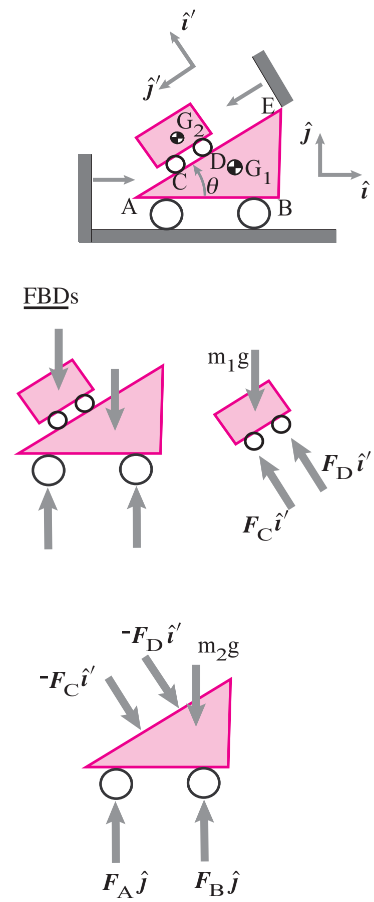

Figure 19.43: Block 2 rolls on block 1 which rolls on the ground. All rollers are ideal (frictionless and massless). The moving $\hat{\boldsymbol{\imath}}'\,\hat{\boldsymbol{\jmath}}'$ frame moves with the lower block and is oriented with the slope.

Now we look at the same example, but proceed in a more naive manner.

Example: Block sliding on sliding block: brute force approach

Now we look at the free-body diagrams of the two separate blocks. We will use 3 balance equations from each free-body diagram, taking account of the kinematic constraints.

For the lower block we have

$$\mathrm{AMB}_{G_1} \quad\Rightarrow\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_1} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_1} \qquad (19.49)$$

$$\begin{aligned}\text{where}\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_1} &= \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{G}_1}\times(-F_D\hat{\boldsymbol{\imath}}')\\ &\quad+\vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{G}_1}\times(-F_C\hat{\boldsymbol{\imath}}')\\ &\quad+\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{G}_1}\times F_B\hat{\boldsymbol{\jmath}}\\ &\quad+\vec{\boldsymbol{r}}_{\mathrm{A}/\mathrm{G}_1}\times F_A\hat{\boldsymbol{\jmath}}\end{aligned}$$

$$\text{and}\quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_1} = \vec{\boldsymbol{0}} \qquad (\vec{\boldsymbol{\omega}}_1 = \vec{\boldsymbol{0}})$$

$$\text{and LMB} \quad\Rightarrow\quad \sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \dot{\vec{\boldsymbol{L}}} \qquad (19.50)$$

$$\text{where}\quad \sum \vec{\boldsymbol{F}}_{\mathrm{i}} = -F_D\hat{\boldsymbol{\imath}}' - F_C\hat{\boldsymbol{\imath}}' + F_B\hat{\boldsymbol{\jmath}} + F_A\hat{\boldsymbol{\jmath}} - m_1 g\hat{\boldsymbol{\jmath}}$$

$$\text{and}\quad \dot{\vec{\boldsymbol{L}}} = m_1\ddot{x}\hat{\boldsymbol{\imath}}$$

Similarly for the upper block:

$$\mathrm{AMB}_{G_2} \quad\Rightarrow\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_2} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_2} \qquad (19.51)$$

$$\begin{aligned}\text{where}\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_1} &= \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{G}_2}\times F_D\hat{\boldsymbol{\imath}}'\\ &\quad+\vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{G}_2}\times F_C\hat{\boldsymbol{\imath}}'\end{aligned}$$

$$\text{and}\quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_2} = \vec{\boldsymbol{0}} \qquad (\vec{\boldsymbol{\omega}}_1 = \vec{\boldsymbol{0}})$$

$$\text{and LMB} \quad\Rightarrow\quad \sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \dot{\vec{\boldsymbol{L}}} \qquad (19.52)$$

$$\text{where}\quad \sum \vec{\boldsymbol{F}}_{\mathrm{i}} = F_D\hat{\boldsymbol{\imath}}' + F_C\hat{\boldsymbol{\imath}}' - m_2 g\hat{\boldsymbol{\jmath}}'$$

$$\text{and}\quad \dot{\vec{\boldsymbol{L}}} = m_2(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}'\hat{\boldsymbol{\jmath}}')$$

Eqns. 19.49-19.52 can be written as scalar equations by dotting the LMB equations with $\hat{\imath}$ and $\hat{\jmath}$ and the AMB equations with $\hat{\boldsymbol{k}}$. All is known in these 6 equations but the six scalars: $F_A, F_B, F_C, F_D, \ddot{x}$, and $\ddot{y}'$. These could be set up as a matrix equation and solved on the computer, or you could try to find your way through by adding and subtracting equations. In any case you could solve for $\ddot{x}$ and $\ddot{y}'$ and thus have differential equations to solve to find the motions.

One quick inference one can make is from looking at the equations: no term in the coefficients of the unknowns depends on $x, \dot{x}, y$, or $\dot{y}'$. So all of the reactions F_A, F_B, F_C, and F_D as well as the accelerations $\ddot{x}$ and $\ddot{y}'$ are constants in time (until the upper mass hits the ground).

Example: **Block sliding on sliding block: even more brute force**
This "multi-body" problem can be solved in an even more naive and more brute force manner. The method is the same as shown in Section 14.1 on page 686 for a one dimensional problem.

We would use 6 configuration variables: the x and y coordinates of G_1 and G_2 and the rotations of the two bodies:

$$x_1, y_1, \theta_1, x_2, y_2, \quad \text{and} \quad \theta_2$$

That the two bodies don't rotate would be expressed indirectly by noting that the accelerations of points A and B on the lower mass must be in the $\hat{\imath}$ direction. These two equations would be added to the 6 linear and angular momentum balance equations. Similar constraint equations would be written for the interactions at C and D. Altogether there are now 6 configuration variables and 4 constraint forces. But there are 6 differential equations of motion and 4 constraint equations. Thus at one instant in time a set of 10 simultaneous equations needs to be solved. Then these are used to evaluate the right hand sides in the differential equations.

These 10 equations are an impractical mess for solving one simple problem like this. But these equations lend themselves to easy automation and this is closest to the approach used by general purpose dynamics simulators.

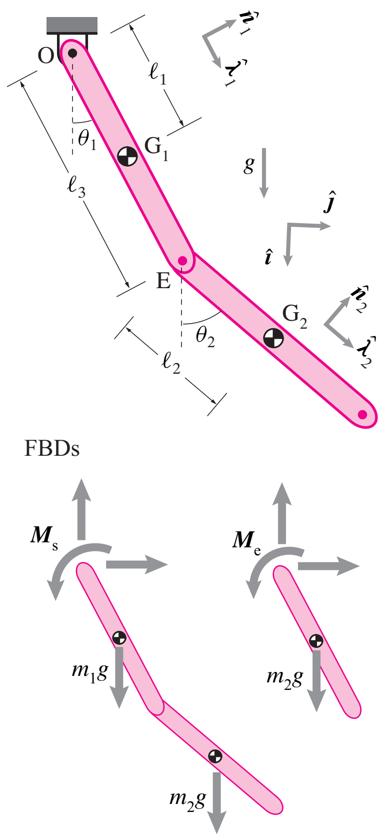

Figure 19.44: A two link robot arm. Free-body diagrams are shown of the whole system (including the motor torque at the shoulder M_s) and of the fore-arm (including the motor torque at the elbow M_e) .

The example above is particularly simple because block 1 moves in a straight line without rotating and block 2 moves in a straight line without rotating relative to block 1. Even for a system of just two objects the situation could be much more complex if the first object had a complex motion and the second a complex motion relative to the first. But because of the preponderance of hinges in the world, circular motion, and motion relative to circular motion, is the most complex motion that need be considered by many engineers. Here is a version of the most common example of that class.

Example: **A two link robot arm: finesse the finding of reactions**
A robot arm has two links. There are motors that apply known torque M_s at the shoulder (reacted by the base and a torque M_e at the elbow (reacted by the upper arm). Dimensions are as marked. This system has 2 degrees of freedom. So we need 2 configuration variables and 2 independent balance equations to find the motion.

The natural configuration variables are the angles of the upper arm relative to a fixed reference and the angle of the lower arm relative to a fixed reference. It would also be natural to instead use the angle of the lower arm relative to the upper arm. This leads to simpler equations in the end, but more work in set up.

Angular momentum balance for the system about the shoulder contains no unknown reaction forces, nor does angular momentum balance of the fore-arm about the hinge. So we base our work on these two equations:

$$\text{System AMB}_{/O} \quad \Rightarrow \quad \sum \vec{\boldsymbol{M}}_{/\mathrm{O}} \quad = \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \tag{19.53}$$

$$\text{Forearm AMB}_{/E} \quad \Rightarrow \quad \sum \vec{\boldsymbol{M}}_{/\mathrm{E}} \quad = \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}}. \tag{19.54}$$

The goal, equations of motion, is reached by evaluating the left and right sides of these equations in terms of known geometric and mass quantities as well as the configuration variables When we write $\sum \vec{\boldsymbol{M}}_{/\mathrm{O}}$ and $\sum \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ we implicitly mean for the whole system. And $\sum \vec{\boldsymbol{M}}_{/\mathrm{E}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}}$ apply just to the forearm.

At each step in the calculations below imagine the results can be substituted into the later steps. We don't do that here because the expressions grow in size. Further, if the angles and their rates of change are known, as they are when doing most dynamics problems, the intermediate calculations will result in numbers, and these do not become more numerous as the calculation proceeds.

$$\begin{aligned} &\hat{\boldsymbol{\lambda}}_1 = \cos\theta_1 \hat{\boldsymbol{\imath}} + \sin\theta_1 \hat{\boldsymbol{\jmath}} && \text{and} && \hat{\boldsymbol{\lambda}}_2 = \cos\theta_2 \hat{\boldsymbol{\imath}} + \sin\theta_2 \hat{\boldsymbol{\jmath}} \\ &\vec{\boldsymbol{r}}_{\mathrm{G_1/O}} = \ell_1 \hat{\boldsymbol{\lambda}}_1,\ \vec{\boldsymbol{r}}_{\mathrm{E/O}} = \ell_3 \hat{\boldsymbol{\lambda}}_1 && \text{and} && \vec{\boldsymbol{r}}_{\mathrm{G_2/E}} = \ell_2 \hat{\boldsymbol{\lambda}}_2 \\ & && \text{and} && \vec{\boldsymbol{r}}_{\mathrm{G_2/O}} = \vec{\boldsymbol{r}}_{\mathrm{E/O}} + \vec{\boldsymbol{r}}_{\mathrm{G_2/E}}. \end{aligned}$$

$$\begin{aligned} \vec{\boldsymbol{a}}_{\mathrm{G_1/O}} &= -\dot{\theta}_1^2 \vec{\boldsymbol{r}}_{\mathrm{G_1/O}} + \ddot{\theta}_1 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G_1/O}} && , \\ \vec{\boldsymbol{a}}_{\mathrm{E/O}} &= -\dot{\theta}_1^2 \vec{\boldsymbol{r}}_{\mathrm{E/O}} + \ddot{\theta}_1 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{E/O}} && , \\ \vec{\boldsymbol{a}}_{\mathrm{G_2/E}} &= -\dot{\theta}_2^2 \vec{\boldsymbol{r}}_{\mathrm{G_2/E}} + \ddot{\theta}_2 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G_2/E}} && \text{and} \quad \vec{\boldsymbol{a}}_{\mathrm{G_2/O}} = \vec{\boldsymbol{a}}_{\mathrm{E/O}} + \vec{\boldsymbol{a}}_{\mathrm{G_2/E}}. \end{aligned}$$

These terms are all we need to evaluate the 4 terms in Eqns. 19.53 and 19.54.

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{/\mathrm{O}} &= \vec{\boldsymbol{r}}_{\mathrm{G_1/O}} \times (m_1 g \hat{\boldsymbol{\imath}}) + \vec{\boldsymbol{r}}_{\mathrm{G_2/O}} \times (-m_2 g \hat{\boldsymbol{\jmath}}) + M_s \hat{\boldsymbol{k}}, \\ \sum \vec{\boldsymbol{M}}_{/\mathrm{E}} &= \vec{\boldsymbol{r}}_{\mathrm{G_2/E}} \times (m_2 g \hat{\boldsymbol{\imath}}) + M_e \hat{\boldsymbol{k}}, \end{aligned}$$

$$\begin{aligned} & \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = m_1 \vec{\boldsymbol{r}}_{\mathrm{G_1/O}} \times \vec{\boldsymbol{a}}_{\mathrm{G_1/O}} + \ddot{\theta}_1 I_1 \hat{\boldsymbol{k}} + m_2 \vec{\boldsymbol{r}}_{\mathrm{G_2/O}} \times \vec{\boldsymbol{a}}_{\mathrm{G_2/O}} + \ddot{\theta}_2 I_2 \hat{\boldsymbol{k}}, \\ \text{and} \quad & \dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}} = m_2 \vec{\boldsymbol{r}}_{\mathrm{G_2/E}} \times \vec{\boldsymbol{a}}_{\mathrm{G_2/O}} + \ddot{\theta}_2 I_2 \hat{\boldsymbol{k}}. \end{aligned}$$

Once these are substituted into Eqns. 19.53 and 19.54 one has 2 vector equations with only $\hat{\boldsymbol{k}}$ components. In other words we have two scalar equations in the two unknowns $\ddot{\theta}_1$ and $\ddot{\theta}_2$. One can go through the algebra and solve for them explicitly, but the expressions are quite complex, even when simplified. At given values of $\theta_1, \dot{\theta}_1, \theta_2$, and $\dot{\theta}_2$ however these are just two linear equations in two unknowns.

Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

Example: Four bar linkage.
It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ of any one of its joints. Why? Because finding all of the bar angles from one angle is a trigonometric mess. And differentiating that mess once (for velocities) and then once again (for accelerations) makes a huge mess.