19.3.1 Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.

a) The force of the plane on the particle is $mg\hat{k}$.

b) $\ddot{x} + \frac{k}{m}x = 0$

c) $\ddot{y} + \frac{k}{m}y = 0$

d) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{r}|$

e) $r =$ constant

f) $\dot{\theta} =$constant

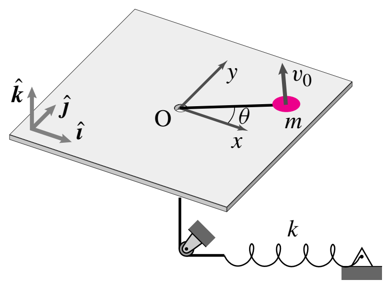
g) $r^2\dot{\theta} =$ constant.

h) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant

i) The trajectory is a straight line segment.

j) The trajectory is a circle.

k) The trajectory is not a closed curve.



Problem 19.1: Particle on a springy leash.

16.40

(a) The normal reaction is always $mg\hat{k}$

(b) $\ddot{x} + \frac{k}{m}x = 0$ always True.

(c) $\ddot{y} + \frac{k}{m}y = 0$

(d) $\ddot{r} + \frac{k}{m}r = 0$

(e) $\dot{r} = 0$ if particle is going in a circle with speed v_0 such that $kr = \frac{mv_0^2}{r}$
(Initial velocity $\vec{v}_0 \perp \vec{r}$)

(f) Same as (e)

(g) Same as (e) or (f)

(h) 2* Energy = $mv^2 + kr^2$ always.

(i) $\dot{\theta} = 0$ for straight line. Particle will oscillate over the origin if $\vec{v}_0 = c\vec{r}$

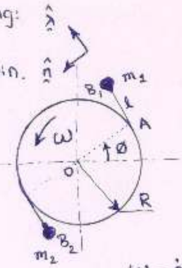
(j) Same as (e), (f) or (g)

(k) Never

19.3.2 "Yo-yo" mechanism of satellite de-spinning. A satellite, modeled as a uniform disk, is "de-spun" by the following mechanism. Before launch two equal length long strings are attached to the satellite at diametrically opposite points and then wound around the satellite with the same sense of rotation. At the end of the strings are placed 2 equal masses. At the start of de-spinning the two masses are released from their position wrapped against the satellite. Find the motion of the satellite as a function of time for the two cases: (a) the strings are wrapped in the same direction as the initial spin, and (b) the strings are wrapped in the opposite direction as the initial spin.

18.3.2 YO YO Mechanism of satellite de-spinning: $\hat{\boldsymbol{\lambda}}$

a) Strings wrapped in the direction of initial spin. $\hat{\mathbf{n}}$



Acceleration of mass 1,

$$\mathbf{a}_1 = \mathbf{a}_{o'} + \mathbf{a}_{rel} + \mathbf{a}_{cor}$$

(or)

$$\mathbf{r} = l\hat{\boldsymbol{\lambda}} - R\hat{\mathbf{n}}$$

$$\dot{\mathbf{r}} = \dot{l}\hat{\boldsymbol{\lambda}} + l\dot{\hat{\boldsymbol{\lambda}}} - R\dot{\hat{\mathbf{n}}}$$

$$\dot{\mathbf{r}} = \dot{l}\hat{\boldsymbol{\lambda}} + l\dot{\phi}\hat{\mathbf{n}} + R\dot{\phi}\hat{\boldsymbol{\lambda}}$$

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{k}}$$

$$l = R\theta - R\phi$$

$$l = R(\theta - \phi)$$

$$\dot{\hat{\boldsymbol{\lambda}}} = \dot{\phi}\hat{\mathbf{k}} \times \hat{\boldsymbol{\lambda}} = \dot{\phi}\hat{\mathbf{n}}$$

$$\dot{l} = R(\dot{\theta} - \dot{\phi})$$

$$\ddot{l} = R(\ddot{\theta} - \ddot{\phi})$$

$$\dot{\hat{\mathbf{n}}} = -\dot{\phi}\hat{\boldsymbol{\lambda}}$$

Acceleration of mass 1,

$$\mathbf{a}_1 = \ddot{\mathbf{r}} = \ddot{l}\hat{\boldsymbol{\lambda}} + \dot{l}\dot{\hat{\boldsymbol{\lambda}}} + \dot{l}\dot{\phi}\hat{\mathbf{n}} + l\ddot{\phi}\hat{\mathbf{n}} + l\dot{\phi}\dot{\hat{\mathbf{n}}} + R[\ddot{\phi}\hat{\boldsymbol{\lambda}} + \dot{\phi}\dot{\hat{\boldsymbol{\lambda}}}]$$

$$\Rightarrow \ddot{\mathbf{r}} = R(\ddot{\theta} - \cancel{\ddot{\phi}})\hat{\boldsymbol{\lambda}} + R(\dot{\theta} - \dot{\phi})\dot{\phi}\hat{\mathbf{n}} + R(\dot{\theta} - \dot{\phi})\dot{\phi}\hat{\mathbf{n}} + l\ddot{\phi}\hat{\mathbf{n}} + - l\dot{\phi}^2\hat{\boldsymbol{\lambda}} + \cancel{R\ddot{\phi}}\hat{\boldsymbol{\lambda}} + R\dot{\phi}^2\hat{\mathbf{n}}$$

$$\Rightarrow \ddot{\mathbf{r}} = (R\ddot{\theta} - l\dot{\phi}^2)\hat{\boldsymbol{\lambda}} + (l\ddot{\phi} + 2R\dot{\theta}\dot{\phi} - R\dot{\phi}^2)\hat{\mathbf{n}}$$

AMB/A

$$\sum \mathbf{M}_{/A} = \dot{\mathbf{H}}_{/A}$$

As $\sum \mathbf{M}_{/A} = 0$, $\dot{\mathbf{H}}_{/A} = 0 \Rightarrow \mathbf{r} \times m\mathbf{a} = \mathbf{0}$

$$\Rightarrow l\hat{\boldsymbol{\lambda}} \times m\left\{(R\ddot{\theta} - l\dot{\phi}^2)\hat{\boldsymbol{\lambda}} + (l\ddot{\phi} + 2R\dot{\theta}\dot{\phi} - R\dot{\phi}^2)\hat{\mathbf{n}}\right\} = \mathbf{0}$$

$\{\}\cdot\hat{\mathbf{k}}$

$$\Rightarrow m l^2\ddot{\phi} + 2mlR\dot{\theta}\dot{\phi} - mRl\dot{\phi}^2 = 0$$

$$\Rightarrow \boxed{\ddot{\phi} + \frac{2R}{l}\dot{\theta}\dot{\phi} - \frac{R}{l}\dot{\phi}^2 = 0} \quad \text{— (1)}$$

AMB/O $\quad \sum \mathbf{M}_{/O} = \dot{\mathbf{H}}_{/O}$

As $\sum \mathbf{M}_{/O} = \mathbf{0}$, $\dot{\mathbf{H}}_{/O} = \mathbf{0}$

$\Rightarrow$ Let disk mass be M

$$\frac{MR^2}{2}\ddot{\theta}\hat{\mathbf{k}} + \mathbf{r}_1 \times m_1\mathbf{a}_1 + \mathbf{r}_2 \times m_2\mathbf{a}_2 = \mathbf{0}$$

As $\mathbf{r}_2 = -\mathbf{r}_1$ and $\mathbf{a}_2 = -\mathbf{a}_1$ and $m_1 = m_2$

we have $\quad \frac{MR^2}{2}\ddot{\theta}\hat{\mathbf{k}} + 2m(\mathbf{r}_1 \times \mathbf{a}_1) = 0$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$\Rightarrow \frac{MR^2}{2}\ddot{\theta}\hat{\underset{\sim}{k}} + 2m\left(l\hat{\underset{\sim}{\lambda}} - R\hat{\underset{\sim}{n}}\right) \times \left[(R\ddot{\theta} - l\dot{\phi}^2)\hat{\underset{\sim}{\lambda}} + (l\ddot{\phi} + 2R\dot{\theta}\dot{\phi} - R\dot{\phi}^2)\hat{\underset{\sim}{n}}\right] = \underset{\sim}{0}$$

$\{\}\cdot\hat{\underset{\sim}{k}}$

$$\Rightarrow \frac{MR^2}{2}\ddot{\theta} + 2m\left[\left(l^2\ddot{\phi} + 2Rl\dot{\phi}\dot{\theta} - Rl\dot{\phi}^2\right) + R^2\ddot{\theta} - Rl\dot{\phi}^2\right] = 0$$

$$\Rightarrow \frac{MR^2\ddot{\theta}}{2} + 2mR^2\ddot{\theta} - 2mRl\dot{\phi}^2 = 0 \qquad [\text{From } 1]$$

$$\Rightarrow \boxed{(M+4m)R^2\ddot{\theta} - 4mRl\dot{\phi}^2 = 0} \quad — 2$$

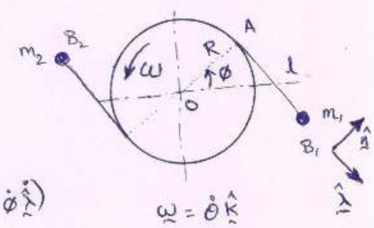
Equations 1 and 2 govern the motion of the disk when the strings are wrapped in the same direction as the initial spin.

b) Strings wrapped in the opposite direction of initial spin

$$\underset{\sim}{r}_{B_1} = l\hat{\underset{\sim}{\lambda}} + R\hat{\underset{\sim}{n}}$$

$$\dot{\underset{\sim}{r}}_{B_1} = \dot{l}\hat{\underset{\sim}{\lambda}} + l\dot{\hat{\underset{\sim}{\lambda}}} + R\dot{\hat{\underset{\sim}{n}}}$$

$$\dot{\underset{\sim}{r}}_{B_1} = \dot{l}\hat{\underset{\sim}{\lambda}} + l\dot{\phi}\hat{\underset{\sim}{n}} - R\dot{\phi}\hat{\underset{\sim}{\lambda}}$$



Acceleration,

$$\ddot{\underset{\sim}{r}}_{B_1} = \ddot{l}\hat{\underset{\sim}{\lambda}} + \dot{l}\dot{\hat{\underset{\sim}{\lambda}}} + \dot{l}\dot{\phi}\hat{\underset{\sim}{n}} + l\ddot{\phi}\hat{\underset{\sim}{n}} + l\dot{\phi}\dot{\hat{\underset{\sim}{n}}} - R(\ddot{\phi}\hat{\underset{\sim}{\lambda}} + \dot{\phi}\dot{\hat{\underset{\sim}{\lambda}}})$$

$$\ddot{\underset{\sim}{r}}_{B_1} = R(\cancel{\ddot{\phi}} - \ddot{\theta})\hat{\underset{\sim}{\lambda}} + R(\dot{\phi} - \dot{\theta})\dot{\phi}\hat{\underset{\sim}{n}} + R(\dot{\phi} - \dot{\theta})\dot{\phi}\hat{\underset{\sim}{n}} + l\ddot{\phi}\hat{\underset{\sim}{n}} + $$
$$- l\dot{\phi}^2\hat{\underset{\sim}{\lambda}} - R\cancel{\ddot{\phi}}\hat{\underset{\sim}{\lambda}} - R\dot{\phi}^2\hat{\underset{\sim}{n}}$$

$$\Rightarrow \ddot{\underset{\sim}{r}}_{B_1} = \left(-R\ddot{\theta} - l\dot{\phi}^2\right)\hat{\underset{\sim}{\lambda}} + \left(l\ddot{\phi} \mp 2R\dot{\phi}\dot{\theta} + R\dot{\phi}^2\right)\hat{\underset{\sim}{n}}$$

$\underset{\sim}{\omega} = \dot{\theta}\hat{\underset{\sim}{k}}$

$l = R\phi - R\theta$

$l = R(\phi - \theta)$

$\dot{\hat{\underset{\sim}{\lambda}}} = \dot{\phi}\hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{\lambda}} = \dot{\phi}\hat{\underset{\sim}{n}}$

$\dot{l} = R(\dot{\phi} - \dot{\theta})$

$\ddot{l} = R(\ddot{\phi} - \ddot{\theta})$

$\dot{\hat{\underset{\sim}{n}}} = -\dot{\phi}\hat{\underset{\sim}{\lambda}}$

AMB/A $\quad \Sigma \underset{\sim}{M}_{/A} = \dot{\underset{\sim}{H}}_{/A}$

As $\Sigma \underset{\sim}{M}_{/A} = \underset{\sim}{0}$, $\dot{\underset{\sim}{H}}_{/A} = \underset{\sim}{0}$

$$\Rightarrow \underset{\sim}{r} \times m\underset{\sim}{a} = \underset{\sim}{0} \Rightarrow l\hat{\underset{\sim}{\lambda}} \times m_1\left[(-R\ddot{\theta} - l\dot{\phi}^2)\hat{\underset{\sim}{\lambda}} + (l\ddot{\phi} - 2R\dot{\phi}\dot{\theta} + R\dot{\phi}^2)\hat{\underset{\sim}{n}}\right] = \underset{\sim}{0}$$

$\{\}\cdot\hat{\underset{\sim}{k}} \Rightarrow \quad m_1 l^2\ddot{\phi} - 2m_1 Rl\dot{\phi}\dot{\theta} + m_1 Rl\dot{\phi}^2 = 0$

$$\Rightarrow \boxed{\ddot{\phi} - \frac{2R}{l}\dot{\phi}\dot{\theta} + \frac{R}{l}\dot{\phi}^2 = 0} \quad — 3$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

AMB/O $\quad \sum \underset{\sim}{M}_{/O} = \dot{\underset{\sim}{H}}_{/O}$

$\Rightarrow \quad \underset{\sim}{r} \times m\underset{\sim}{a} = \underset{\sim}{0}$

$\Rightarrow \quad \frac{MR^2}{2}\ddot{\theta}\hat{\underset{\sim}{k}} + \underset{\sim}{r}_1 \times m_1\underset{\sim}{a}_1 + \underset{\sim}{r}_2 \times m_2\underset{\sim}{a}_2 = \underset{\sim}{0}$

As $\underset{\sim}{r}_1 = -\underset{\sim}{r}_2$, $\underset{\sim}{a}_1 = -a_2$, $m_1 = m_2 = m$

we have $\quad \frac{MR^2}{2}\ddot{\theta}\hat{\underset{\sim}{k}} + 2m(\underset{\sim}{r}_1 \times \underset{\sim}{a}_1) = \underset{\sim}{0}$

$\Rightarrow \quad \frac{MR^2}{2}\ddot{\theta}\hat{\underset{\sim}{k}} + 2m(l\hat{\underset{\sim}{\lambda}} + R\hat{\underset{\sim}{n}}) \times \left((-R\ddot{\theta} - l\dot{\phi}^2)\hat{\underset{\sim}{\lambda}} + (l\ddot{\phi} - 2R\dot{\phi}\dot{\theta} + R\dot{\phi}^2)\hat{\underset{\sim}{n}}\right) = \underset{\sim}{0}$

$\Rightarrow \quad \frac{MR^2}{2}\ddot{\theta}\hat{\underset{\sim}{k}} + 2m(l^2\ddot{\phi} - 2Rl\dot{\phi}\dot{\theta} + Rl\dot{\phi}^2)\hat{\underset{\sim}{k}} - 2m(-R^2\ddot{\theta} - Rl\dot{\phi}^2)\hat{\underset{\sim}{k}} = \underset{\sim}{0}$

$\{\ \}\cdot\hat{\underset{\sim}{k}}$

$\Rightarrow \quad \frac{MR^2}{2}\ddot{\theta} + 2mR^2\ddot{\theta} + 2mRl\dot{\phi}^2 = 0 \qquad$ [From 3]

$$\boxed{(M+4m)R^2\ddot{\theta} + 4mRl\dot{\phi}^2 = 0} \quad -4$$

Equations 3 and 4 govern the motion of the disk when the strings are wrapped in the opposite direction as the initial spin.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.3 A particle with mass m is held by two long springs each with stiffness k so that the springs are relaxed when the mass is at the origin. Assume the motion is planar. Assume that the particle displacement is much smaller than the lengths of the springs.

a) Write the equations of motion in cartesian components.

b) Write the equations of motion in polar coordinates.

c) Express the conservation of angular momentum in cartesian coordinates.

d) Express the conservation of angular momentum in polar coordinates.

e) Show that (a) implies (c) and (b) implies (d) even if you didn't note them *a-priori*.

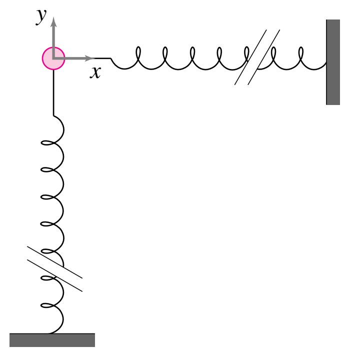
f) Express the conservation of energy in cartesian coordinates.

g) Express the conservation of energy in polar coordinates.

h) Show that (a) implies (f) and (b) implies (g) even if you didn't note them *a-priori*.

i) Find the general motion by solving the equations in (a). Describe all possible paths of the mass.

j) Can the mass move back and forth on a line which is not the x or y axis?



Problem 19.3

Answer:
(a) $-k(x\hat{\imath} + y\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$.
(b) $-kr\hat{e}_r = m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_t$.
(c) $\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0$.
(d) $\frac{d}{dt}(r^2\dot{\theta}) = 0$.
(e) dot equation in (a) with $(x\hat{\jmath} - y\hat{\imath})$ to get $(x\ddot{y} - y\ddot{x}) = 0$ which can be rewritten as (c); dot equation in (b) with $\hat{e}_t$ to get $(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$ which can be rewritten as (d).
(f) $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2) = const.$
(g) $\frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + \frac{1}{2}kr^2$=const.
(h) dot equation in (a) with $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ to get $m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) + k(\dot{x}x + \dot{y}y) = 0$ which can be rewritten as (f).
(i) $x = A_1\sin(\omega t + B_1)$, $y = A_2\sin(\omega t + B_2)$, where $\omega = \sqrt{\frac{k}{m}}$. The general motion is an ellipse.
(j) Yes. Consider $B_1 = B_2 = 0$, $A_1 = 1$, and $A_2 = 2$.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.42

(a) For small oscillations, the x component of force on the particle from vertical spring can be neglected. Hence, $\ddot{x} + \frac{k}{m} x = 0$

Similarly, $\ddot{y} + \frac{k}{m} y = 0$. $\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = -\frac{k}{m}(x\hat{\imath} + y\hat{\jmath})$

(b) $-kr\,\hat{e}_r = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right]$

(c) In 2-D, angular momentum about origin is $\vec{H}_o = (xL_y - yL_x)\hat{k}$, where $L_x = \vec{L}\cdot\hat{\imath}$ is the x-component of $\vec{L}$.

$$\sum M_o = 0 \Rightarrow \frac{d}{dt}\vec{H}_o = 0 \Rightarrow \frac{d}{dt}(xv_y - yv_x) = 0$$

or, $\boxed{\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0}$

(d) velocity in polar co-ord:

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{H}_o = r\hat{e}_r \times \vec{v} = r^2\dot{\theta}\,\hat{k}$$

Conservation $\Rightarrow$ $\boxed{\frac{d}{dt}(r^2\dot{\theta}) = 0}$

(e) From (c), $x\ddot{y} + \cancel{\dot{x}\dot{y}} - \cancel{\dot{y}\dot{x}} - y\ddot{x} = 0$

$\Rightarrow \boxed{x\ddot{y} = y\ddot{x}}$

From (a) $\ddot{y} = -\frac{k}{m} y$, $\frac{kx}{m} = -\ddot{x}$] multiply : $\boxed{x\ddot{y} = \ddot{x} y}$ (same)

From (d) $2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$

$\Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

which is obtained by finding $\{(b)\} \cdot \hat{e}_\theta$

(f) $\frac{d}{dt}\left(\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k(x^2 + y^2)\right) = 0$

(g) $\frac{d}{dt}\left[\frac{1}{2} m\left|(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)\right|^2 + \frac{1}{2} k r^2\right] = 0$

$\frac{1}{2} m[\dot{r}^2 + (r\dot{\theta})^2] + \frac{1}{2} k r^2 = \text{const}$

(h) In (a), dot both sides with $\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$

to get : $m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) + k(\dot{x}x + \dot{y}y) = 0$

i.e. $\frac{d}{dt}\left[m(\dot{x}^2 + \dot{y}^2) + k(x^2 + y^2)\right] = 0$

same as (f)

Now, dot both sides of (b) with $\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ to get

$$k r\dot{r} + m\left[\left(\ddot{r}\dot{r} - \dot{r}r\dot{\theta}^2\right) + \left(r^2\dot{\theta}\ddot{\theta} + 2r\dot{r}\dot{\theta}^2\right)\right] = 0$$

combine

or, $\dfrac{d}{dt}\left[(kr^2) + m(\dot{r})^2 + m(r\dot{\theta})^2\right] = 0$

Same as (g)

(i) $x = A_1 \sin(\omega t + B_1) \qquad y = A_2 \sin(\omega t + B_2)$

with $\omega = \sqrt{k/m}$. This describes an ellipse as you can reduce down to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (Check!)

(j) If $B_1 = B_2 = 0$, $y = cx$ which is a straight line.

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4 16.43) Solution

a) $\ddot{x} + \frac{k}{m}x = 0 \rightarrow -kx = m\ddot{x}$ ($\hat{\imath}$-direction)

$\ddot{y} + \frac{k}{m}y = 0 \rightarrow -ky = m\ddot{y}$ ($\hat{\jmath}$-direction)

combine: $-k(x\hat{\imath} + y\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$

b) $\Sigma\vec{F} = ma = m[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta]$

$\Sigma\vec{F} = -kr\hat{e}_r$

$$\boxed{-kr\hat{e}_r = m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta}$$

c) $\vec{H}^+ = \vec{H}^-$

$\vec{H}^+ = x\dot{y}^+ + y^+\dot{x}^+$ $\Rightarrow$ $x^+\dot{y}^+ + y^+\dot{x}^+ - (x^-\dot{y}^- + y^-\dot{x}^-) = 0$

$\vec{H}^- = x\dot{y}^- + y^-\dot{x}^-$ $\quad x^+\dot{y}^+ - x^-\dot{y}^- = \frac{d}{dt}x\dot{y}$

$\quad y^+\dot{x}^+ - y^-\dot{x}^- = \frac{d}{dt}y\dot{x}$

$$\boxed{0 = \frac{d}{dt}(x\dot{y} + y\dot{x})}$$

d) $\vec{H}^+ = r^+\hat{e}_r \times (r^+\dot{\theta}^+)\hat{e}_\theta = (r^+)^2\dot{\theta}^+\hat{k}$

$\vec{H}^- = (r^-)^2\dot{\theta}^-\hat{k}$ (likewise)

$\vec{H}^+ - \vec{H}^- = 0 = (r^+)^2\dot{\theta}^+ - (r^-)^2\dot{\theta}^- = \boxed{\frac{d}{dt}(r^2\dot{\theta}) = 0}$

e) $\{-k(x\hat{\imath} + y\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})\} \cdot (x\hat{\jmath} - y\hat{\imath})$

$= \underbrace{-kyx - kyx}_{0} = -my\ddot{x} + mx\ddot{y}$

$m(x\ddot{y} - y\ddot{x}) = 0 \Rightarrow \frac{d}{dt}(x\dot{y} + y\dot{x}) = 0$

$\{\text{equation in b}\} \cdot \hat{e}_\theta \Rightarrow 0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

f) $E_T = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}k\vec{r}^2 = \boxed{\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2) = \text{const.}}$

g) $\frac{1}{2}m\vec{v}^2 + \frac{1}{2}k\vec{r}^2 = \frac{1}{2}m\left[\frac{d}{dt}(r\hat{e}_r)\right]^2 + \frac{1}{2}kr^2$

$\frac{d}{dt}r\hat{e}_r = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

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$|\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta| = (\dot{r}^2 + (r\dot{\theta})^2)^{1/2}$

$$\boxed{E_T = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + \frac{1}{2}kr^2 = \text{constant}}$$

h) $\{-k(x\hat{\imath} + y\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})\} \cdot (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})$

$-kx\dot{x} - ky\dot{y} = mx\ddot{x} + my\ddot{y}$

$0 = m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + kx\dot{x} + ky\dot{y}$

$\int 0 = \text{const} = \int (m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + kx\dot{x} + ky\dot{y})dt$

$\text{const} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2)$

$\{\text{eq. from } b\} \cdot (r\dot{\theta}\hat{e}_\theta)$

$\Rightarrow 0 = mr^2\ddot{\theta}\dot{\theta} + m2\dot{r}\dot{\theta}r\dot{\theta}$

$0 = mr\dot{\theta}(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

i) $x = A\sin\left(\sqrt{\frac{k}{m}}t + B\right)$

$y = C\sin\left(\sqrt{\frac{k}{m}}t + D\right)$

This is the general form of an ellipse

j) Consider $B = D = 0$

$x = A\sin\left(\sqrt{\frac{k}{m}}t\right)$

$y = C\sin\left(\sqrt{\frac{k}{m}}t\right)$

say $\sin\left(\sqrt{\frac{k}{m}}t\right) = n$ for a given value of t

$x = An$

$y = Cn$

$y = \frac{1}{A}x$ ← linear equation

Yes this is possible

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

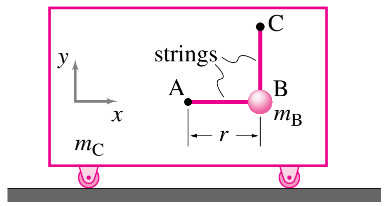
19.3.4 Cart and pendulum A mass $m_B = 6\,\text{kg}$ hangs by two strings from a cart with mass $m_C = 12\,\text{kg}$. Before string BC is cut string AB is horizontal. The length of string AB is $r = 1\,\text{m}$. At time $t = 0$ all masses are stationary and the string BC is cleanly and quietly cut. After some unknown time t_{vert} the string AB is vertical.

a) What is the net displacement of the cart x_C at $t = t_{\text{vert}}$?

b) What is the velocity of the cart v_C at $t = t_{\text{vert}}$?

c) What is the tension in the string at $t = t_{\text{vert}}$?

d) (**Optional**) What is $t = t_{\text{vert}}$? [You will either have to leave your answer in the form of an integral you cannot evaluate analytically or you will have to get part of your solution from a computer.]



Problem 19.4

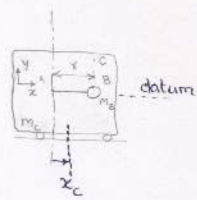
Answer:
(a) $0.33\,\text{m}\,\hat{\imath}$
(b) $\vec{v}_C = 1.82\,\text{m/s}\hat{\imath}$.
(c) $T \approx 240\,\text{N}$.

18.3.4: Cart and Pendulum

$m_B = 6\,\text{Kg}$

$m_c = 12\,\text{Kg}$

$r = 1\,\text{m}$



At $t = t_{vert}$, String AB is vertical

a) Net displacement of the cart at $t = t_{vert}$, $x_c = ?$

mass m_B, moves relative to the cart and cart is constrained to move horizontally.

$$\therefore (m_c + m_B)\, x_c = m_c(0) + m_B(r)$$

$$\Rightarrow x_c = \frac{m_B r}{m_c + m_B} \Rightarrow x_c = \frac{6(\text{Kg})\cdot 1(\text{m})}{(6+12)\text{Kg}} = \frac{6}{18}\,\text{m} = \frac{1}{3}\,\text{m}.$$

b) Velocity of cart $V_c = ?$

As the pendulum swings, the horizontal component of the momentum of the system is constant.

$$\therefore (m_c + m_B)\,0 = m_c V_c + m_B V_B$$

$\Rightarrow$ Velocity of mass m_B, $V_B = \dfrac{-m_c V_c}{m_B}$

At $t = 0$, Energy, $E_1 = 0$

At $t = t_{vert}$, Energy, $E_2 = \frac{1}{2} m_c V_c^2 + \frac{1}{2} m_B V_B^2 - m_B g r$

As $E_1 = E_2$, $\frac{1}{2} m_c V_c^2 + \frac{1}{2} m_B \left(-\frac{m_c V_c}{m_B}\right)^2 = m_B g r$

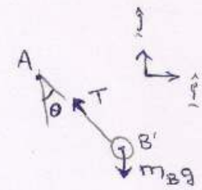
$$\Rightarrow V_c^2 = \frac{2 m_B^2 g r}{m_c(m_b + m_c)}$$

$$\Rightarrow V_c = \frac{m_b}{m_c}\sqrt{\frac{2gr}{1 + \left(\frac{m_b}{m_c}\right)}} \quad \therefore V_c = \frac{6}{12}\sqrt{\frac{2g(1)}{1+\frac{6}{12}}} = \sqrt{\frac{g}{3}} = 1.808\,\frac{m}{s}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

18.3.4

c) Tension in the string, $T = ?$ at $t = t_{vert}$.



$$\underset{\sim}{r}_{B'} = \underset{\sim}{r}_A + \underset{\sim}{r}_{B'/A}$$

$$\underset{\sim}{r}_{B'} = x\hat{\underset{\sim}{i}} + r\sin\theta\,\hat{\underset{\sim}{i}} - r\cos\theta\,\hat{\underset{\sim}{j}}$$

Velocity of mass, $\underset{\sim}{v}_B = \dot{\underset{\sim}{r}}_{B'}$

$$\underset{\sim}{v}_B = (\dot{x} + r\cos\theta\,\dot{\theta})\,\hat{\underset{\sim}{i}} + r\sin\theta\,\dot{\theta}\,\hat{\underset{\sim}{j}}$$

Acceleration, $\underset{\sim}{a}_B = (\ddot{x} + r\cos\theta\,\ddot{\theta} - r\sin\theta\,\dot{\theta}^2)\,\hat{\underset{\sim}{i}} + (r\sin\theta\,\ddot{\theta} + r\cos\theta\,\dot{\theta}^2)\,\hat{\underset{\sim}{j}}$

L.M.B. of mass

$$\Sigma \underset{\sim}{F} = m_B \underset{\sim}{a}_B$$

$$\{\underset{\sim}{T} - m_B g\,\hat{\underset{\sim}{j}} = m_B\,\underset{\sim}{a}_B\}$$

$$\{-T\sin\theta\,\hat{\underset{\sim}{i}} + T\cos\theta\,\hat{\underset{\sim}{j}} - m_B g\,\hat{\underset{\sim}{j}} = m_B\,\underset{\sim}{a}_B\}$$

$\{\}\cdot\hat{\underset{\sim}{i}} \Rightarrow -T\sin\theta = m_B(\ddot{x} + r\cos\theta\,\ddot{\theta} - r\sin\theta\,\dot{\theta}^2)$

$\{\}\cdot\hat{\underset{\sim}{j}} \Rightarrow T\cos\theta - m_B g = m_B(r\sin\theta\,\ddot{\theta} + r\cos\theta\,\dot{\theta}^2)$

Eliminating $\ddot{\theta}$ from both equations,

$$T = m_B g\cos\theta - m_B\ddot{x}\sin\theta + m_B r\dot{\theta}^2$$

At $t = t_{vert}$, from (b), $v_B = \dfrac{-m_c v_c}{m_B} = -\dfrac{12 \times 1.808}{6} = -3.616\ m/s$

$\therefore\ \underset{\sim}{\dot{\theta}} = \dfrac{\underset{\sim}{v}_B}{r} = -3.616\,\hat{\underset{\sim}{k}}$ rad/s

At $\theta = 0$, $T = m_B g + m_B r\dot{\theta}^2$

$= 6 \times 9.81 + 6 \cdot (1)(-3.616)^2$ N

$= 58.86 + 78.45$ N

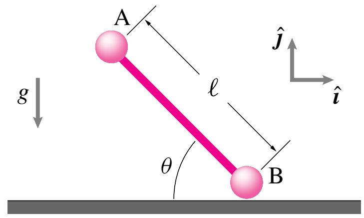
$T = 137.31$ N

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.5 A dumbbell slides on a floor. Two point masses m at A and B are connected by a massless rigid rod with length ℓ. Mass B slides on a frictionless floor so that it only moves horizontally. Assume this dumbbell is released from rest in the configuration shown. *[Hint: What is the acceleration of A relative to B?]*

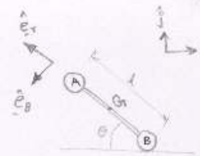
a) Find the acceleration of point B just after the dumbbell is released.

b) Find the velocity of point A just before it hits the floor



Problem 19.5

18.3.5



a) Acceleration of point B just after release:

$\hat{e}_r = -\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}$

Apply LMB on mass 'A': $\sum \underline{F} = m\underline{a}$

$\Rightarrow \{ -F_r \hat{e}_r - m_A g \hat{\jmath} = m_A(\underline{a}_B + \underline{a}_{A/B}) \}$.

$\{\}\cdot\hat{\jmath} \Rightarrow -F_r \sin\theta = m_A g - m_A l(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta)$ — ①

$\{\}\cdot\hat{\imath} \Rightarrow F_r\cos\theta = m_A l[-\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta] + m_A a_B$ — ②

eliminating $\ddot{\theta}$, $F_r = m_A l\dot{\theta}^2 + m_A a_B\cos\theta - m_A g\sin\theta$ — ③

Apply LMB on mass 'B': $\sum \underline{F} = m\underline{a}_B$

$\Rightarrow \{ F_r(\hat{e}_r) - m_B g\hat{\jmath} = m a_B \hat{\imath} \}$

$\{\}\cdot\hat{\jmath} \Rightarrow [m_A l\dot{\theta}^2 + m_A a_B\cos\theta - m_A g\sin\theta]\sin\theta - m_B g = 0$

$\Rightarrow \dot{\theta}^2 = \frac{g}{l}\sin\theta - \frac{a_B}{l}\cos\theta + \frac{m_B}{m_A}\frac{g}{l\sin\theta}$ — ④

From ②, ③ and ④, $a_B = \dfrac{l\dot{\theta}^2\cos\theta - g\sin\theta}{\left(\frac{m_B}{m_A} - \cos^2\theta\right)}$, For $m_B = m_A = 1$, $a_B = g\left[\sin\theta(\cos\theta - 1) + \frac{1}{\tan\theta}\right]$

b) Velocity of 'A' before it hits the floor.

As there are no dissipative forces, Loss in P.E = Gain in K.E.

$\underline{v}_G = \underline{v}_B + \underline{v}_{G/B} = v_B\hat{\imath} + \omega\hat{k}\times\frac{l}{2}\hat{e}_r = \left(v_B - \frac{l\omega}{2}\sin\theta\right)\hat{\imath} - \frac{l\omega}{2}\cos\theta\,\hat{\jmath}$

Loss in P.E $= (m_A + m_B)\,g\,\frac{l}{2}\sin\theta$

Gain in K.E $= \frac{1}{2}(m_A + m_B)v_G^2 + \frac{1}{2}I^G\omega^2$

$\Rightarrow v_B^2 - v_B l\omega\sin\theta + \frac{(l\omega)^2}{2} = gl\sin\theta$ — ⑤

As there are no forces in $\hat{\imath}$, momentum is conserved:

$m_B v_B + m_A(v_B - l\omega\sin\theta) = 0 \Rightarrow v_B = \dfrac{l\omega\sin\theta}{1 + \frac{m_B}{m_A}}$ — ⑥

Substituting ⑥ in ⑤ and noting that $m_A = m_B = m$

$$\omega^2 = \frac{g\sin\theta}{l\left[\frac{1}{2} - \frac{1}{4}\sin^2\theta\right]}$$

Velocity of mass. A, $\underline{v}_A = \underline{\omega}\times\underline{r} = \omega\hat{k}\times l(-\hat{\imath}) = -l\omega\hat{\jmath}$

$$\therefore \underline{v}_A = -\sqrt{\frac{4lg\sin\theta}{(1+\cos^2\theta)}}\,\hat{\jmath}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.6 After a winning goal one second before the clock ran out a psychologically stunned hockey player (modeled as a uniform rod) stands nearly vertical, stationary and rigid. The player's perfectly slippery skates start to pop out from under her as she falls. Her height is ℓ, her mass m, her tip from the vertical θ, and the gravitational constant is g.

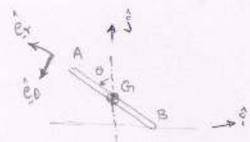
a) What is the path of her center of mass as she falls? (Show clearly with equations, sketches or words.)

b) What is her angular velocity just before she hits the ice, a millisecond before she sticks out her hands and breaks her fall (first assume her skates remain in contact the whole time and then check the assumption)?

c) Find a differential equation that only involves θ, its time derivatives, m, g, and ℓ. (This equation could be solved to find θ as a function of time. It is a non-linear equation and you are not being asked to solve it numerically or otherwise.)

18.3.6

a) Path of the center of mass



As there are no forces acting in $\hat{i}$, linear Momentum is conserved $m v_x = 0 \Rightarrow v_x = 0$

$\hat{e}_r = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\hat{e}_\theta = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j}$

$\left\{ \underline{v}_G = \underline{v}_B + \underline{v}_{G/B} = v_B\hat{i} + \omega\hat{\underline{k}} \times \frac{l}{2}\hat{\underline{e}}_r = v_B\hat{i} + \frac{\omega l}{2}\hat{e}_\theta \right\}$

$\{\}\cdot\hat{i} \Rightarrow v_B = \frac{\omega l}{2}\cos\theta \quad \Rightarrow \underline{v}_G = -\frac{\omega l}{2}\sin\theta\,\hat{j}$

$\therefore$ Center of mass moves on a straight line normal to ground.

b) Angular velocity just before hit, $\omega = ?$

Equating the energies at two instants,

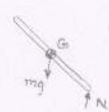
$E_1 = mg\frac{l}{2} \equiv E_2 = \frac{1}{2}mv_G^2 + \frac{1}{2}I^G\omega^2 + mgh$

$\Rightarrow mg\frac{l}{2} = \frac{1}{2}m\frac{\omega^2 l^2}{4}\sin^2\theta + \frac{1}{2}\frac{ml^2}{12}\omega^2 + mg\frac{l}{2}\cos\theta$

$\Rightarrow \omega^2 = \frac{12g(1-\cos\theta)}{l(1+3\sin^2\theta)}$ For $\theta = 90°$, $\omega = \sqrt{\frac{3g}{l}}$ //

c) Differential equation of motion:

Applying L.M.B on the object: $\Sigma\underline{F} = m\underline{a}_G$



$\underline{a}_G = \underline{a}_B + \underline{a}_{G/B} = a_B\hat{i} + \dot{\omega}\hat{\underline{k}} \times \frac{l}{2}\hat{e}_r + \underline{\omega}\times(\underline{\omega}\times\frac{l}{2}\hat{e}_r)$

$\left\{ \underline{a}_G = a_B\hat{i} - \frac{\dot{\omega} l}{2}\cos\theta\,\hat{i} + \omega^2\frac{l}{2}\sin\theta\,\hat{i} - \dot{\omega}\frac{l}{2}\sin\theta\,\hat{j} - \omega^2\frac{l}{2}\cos\theta\,\hat{j} \right\}$

$\{\}\cdot\hat{i} = 0 \Rightarrow a_B = \frac{\dot{\omega} l}{2}\cos\theta - \frac{\omega^2 l}{2}\sin\theta \Rightarrow \underline{a}_G = -\frac{\dot{\omega} l}{2}\sin\theta\,\hat{j} - \frac{\omega^2 l}{2}\cos\theta\,\hat{j}$

$\therefore N\hat{j} - mg\hat{j} = -ml\left[\frac{\dot{\omega}}{2}\sin\theta + \frac{\omega^2}{2}\cos\theta\right]\hat{j} \Rightarrow N = mg - \frac{ml}{2}(\dot{\omega}\sin\theta + \omega^2\cos\theta)$ //

Applying A.M.B about G: $\Sigma\underline{M}_{/G} = \dot{\underline{H}}_{/G}$

$\Rightarrow -\frac{l}{2}\hat{e}_r \times N\hat{j} = I^G\dot{\omega}\hat{k}$

$\Rightarrow \left\{ -\frac{l}{2}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times \left(mg - \frac{ml}{2}[\dot{\omega}\sin\theta + \omega^2\cos\theta]\right)\hat{j} = \frac{ml^2}{12}\dot{\omega}\hat{k} \right\}$

$\{\}\cdot\hat{k} \Rightarrow g\sin\theta - \frac{l}{2}\sin^2\theta\,\ddot{\theta} - \frac{l\dot{\theta}^2}{4}\sin 2\theta = \frac{l\ddot{\theta}}{6}$

$\Rightarrow 2(1+3\sin^2\theta)\ddot{\theta} + 3\sin 2\theta\,\dot{\theta}^2 - \frac{12g}{l}\sin\theta = 0$ is the governing differential equation. //

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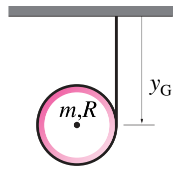
19.3.7 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find y_G at a later time t in terms of any or all of m, R, g, and t.

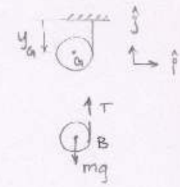
b) Does G move sideways as the hoop falls and unrolls?



Problem 19.7

18.3.7 : Falling hoop :

LMB of hoop : $\sum \underline{F} = m\underline{a}_G$



$\Rightarrow \{T\hat{j} - mg\hat{j} = m a_{Gx}\hat{i} + m a_{Gy}\hat{j}\}$

$\{\}\cdot\hat{i} \Rightarrow a_{Gx} = 0$

$\{\}\cdot\hat{j} \Rightarrow m a_{Gy} = T - mg$.

As $a_{Gx} = 0$, hoop does not move sideways while falling.

Apply AMB about B : $\sum \underline{M}_{/B} = \dot{\underline{H}}_{/B}$

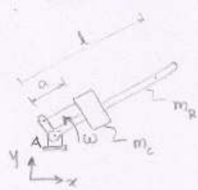
$\Rightarrow -R\hat{i} \times mg(-\hat{j}) = -R\hat{i} \times m\underline{a}_G + I^{cm}\dot{\omega}\hat{k} = -2mRa_G\hat{k} \Rightarrow \underline{a}_G = -\frac{g}{2}\hat{j}$

$\int a_G\,dt = V_G = -\frac{g}{2}t + c \Rightarrow y_G = \int V_G\,dt = -\frac{gt^2}{4} + ct + D$

At $t=0$, $\dot{y} = 0$, $y = -y_G$ $\therefore y = -y_G - \frac{gt^2}{4}$ //

18.3.9 :

~~(a)~~ Speed of the collar after it flies off the rod end:



Conservation of angular momentum at two instances

$\underline{H}^-_{/A} = \underline{H}^+_{/A}$

$\Rightarrow I^A_R\omega_1\hat{k} + m_c a^2\omega_1\hat{k} = I^A_R\omega_2\hat{k} + m_c l^2\omega_2\hat{k} \Rightarrow \omega_2 = \dfrac{\frac{1}{3}m_R l^2 + m_c a^2}{\frac{1}{3}m_R l^2 + m_c l^2}\cdot\omega_1$

Given, $\omega_1 = 1$ rad/s, $m_c = 3$ kg, $m_R = 1$ kg, $a = 1$m, $l = 3$m $\Rightarrow \omega_2 = 0.2$ rad/s

Using energy balance: $E_1 = E_2$

$\frac{1}{2}m_c a^2\omega_1^2 + \frac{1}{2}(\frac{1}{3}m_R l^2)\omega_1^2 = \frac{1}{2}m_c l^2\omega_2^2 + \frac{1}{2}(\frac{1}{3}m_R l^2)\omega_2^2 + \frac{1}{2}m_c v_c^2$

upon substitution, $V_2^2 = 1.6\ (m/s)^2$ (or) $V_2 = 1.265$ m/s

b) If $m_R = 0$, Path of the collar motion.

Applying LMB on collar : $N\hat{e}_\theta = m_c[r\dot{\omega}\hat{e}_\theta - \omega^2 r\hat{e}_r + \ddot{r}\hat{e}_r + 2\omega\dot{r}\hat{e}_\theta]$

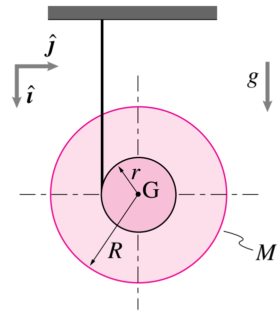
$\Rightarrow \ddot{r} - r\omega^2 = 0$

Applying AMB about A ; $\underline{0} = \dot{\underline{H}}_{/A} \Rightarrow \frac{ml^2}{3}\dot{\omega}\hat{k} + mr^2\dot{\omega}\hat{k} + 2mr\omega\dot{r}\hat{k} \Rightarrow \ddot{\theta} = \dfrac{-2m_c r\dot{r}}{\frac{m_r l^2}{3} + m_c r^2}\dot{\theta}$

$\Rightarrow \ddot{\theta} = -\frac{2\dot{r}}{r}\dot{\theta}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

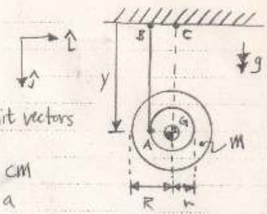
19.3.8 A model for a yo-yo consists of a thin disk of mass M and radius R and a light drum of radius r, rigidly attached to the disk, around which a light inextensible cable is wound. Assuming that the cable unravels without slipping on the drum, determine the acceleration a_G of the center of mass.



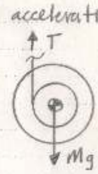
Problem 19.8

#7.19

Find a_G acceleration of CM.



FBD



Note: unit vectors

Note:
$\underline{v}_{cm} = \dot{y}\hat{\jmath}$
$\underline{a}_G = \ddot{y}\hat{\jmath}$

assume CM falls in a straight line

Power Balance / Conservation of Energy $\dot{W} = \dot{E}_K + \dot{E}_P + D$

work done gravity = kinetic energy of yo-yo

$$\frac{d}{dt}\left\{ mgy = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{zz}^{cm} \omega^2 + 0 + 0 \right\}$$

but $\underline{v}_{cm} = \underline{\omega} \times \underline{r}_{G/A}$ analogy to rolling $\underline{\omega} = \omega\hat{k}$

$v_{cm} = \omega r$ but $\dot{y} = v_{cm} \Rightarrow \dot{y} = \omega r$

$I_{zz}^{cm} = \frac{1}{2} M R^2$

$$\frac{d}{dt}\left\{ mgy = \frac{1}{2} m \dot{y}^2 + \frac{1}{4} M R^2 \frac{\dot{y}^2}{r^2} \right\}$$

putting this all together & differentiating

$$mg\dot{y} = \frac{1}{2} m\, 2\dot{y}\ddot{y} + \frac{1}{4}\frac{MR^2}{r^2} 2\dot{y}\ddot{y}$$

$$\boxed{\frac{g}{1+\frac{1}{2}\frac{R^2}{r^2}} = \ddot{y} = a_G}$$

what is a_G when $r << R$, e.g. $\frac{R}{r} = 10$?

AMB $\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$ $\sum \underline{M}_{/A} = mgr\hat{k}$

$$\dot{\underline{H}}_{/A} = \underline{r}_{cm/A} \times \underline{a}_{cm} m_{tot} + \underbrace{\underline{\omega} \times [I^{cm}] \cdot \underline{\omega}}_{0 \text{ - 2D prob}} + [I^{cm}] \cdot \dot{\underline{\omega}}$$

$$= r\hat{\imath} \times (-\omega^2 r \hat{\imath} + \dot{\omega} r \hat{\jmath})\, m_{tot} + I_{zz}^{cm} \dot{\omega} \hat{k} \Rightarrow$$

$$\left\{ mgr\hat{k} = m\dot{\omega} r^2 \hat{k} + \frac{1}{2} MR^2 \dot{\omega} \hat{k} \right\} \cdot \hat{k}$$

since $\dot{y} = \omega r$, $\ddot{y} = \dot{\omega} r$

$$mgr = m\ddot{y} r + \frac{1}{2} MR^2 \frac{\ddot{y}}{r}$$

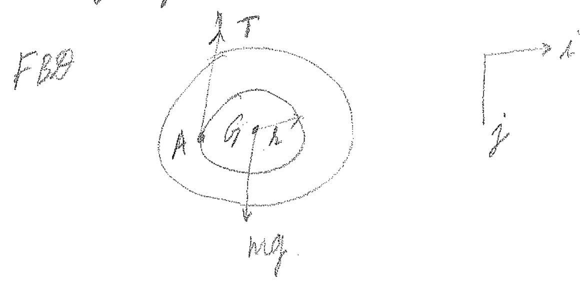
$$\boxed{\ddot{y} = \frac{g}{1+\frac{1}{2}\frac{R^2}{r^2}}}$$

same answer.

so, if $\frac{R}{r} = 10$, $\ddot{y} = \frac{g}{51}$, yo-yo falls very slowly which, besides coming back up, makes playing with a yo-yo fun! Prof. Ruina asks: "Does a yo-yo fall straight down or is the path curved?"

16.47 Yo-yo.

FBD



* Newton's IInd Law for vertical direction;

"LMB" $\quad mg - T = m a_G \quad$ — (i)

* Since the yoyo is rotating as it falls, we can also write the torque equation:

"AMB" $\quad \sum \vec{M}_G = I_{cm} \alpha \hat{k}$

$$-Tr\,\hat{k} = \frac{mR^2}{2}\alpha\,\hat{k} \quad \text{— (ii)}$$

* Observe that the no-slip constraint at pt. A is the same as that for rolling, i.e.

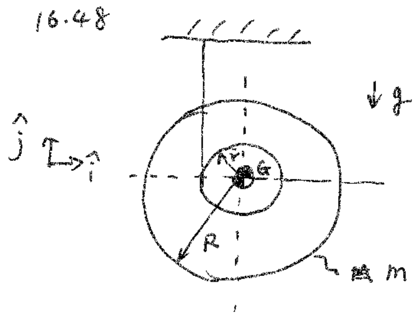
$$a_G = r\alpha \quad \text{— (iii)}$$

Solve simultaneously for a_G:

$$\boxed{\vec{a}_G = \frac{g}{1-\frac{R^2}{2r^2}}\,\hat{j}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.48



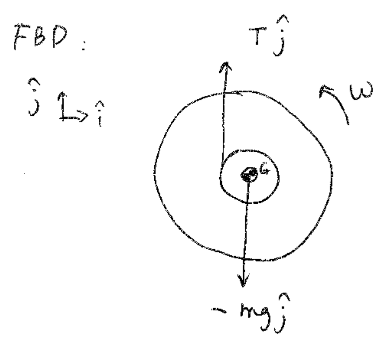
Note:

You can use the given coordinate system "$\hat{i}$, $\hat{j}$"

But you have to be very clear about sign conventions. To avoid confusion, we just use the normal system $\hat{j}$ $\hat{i}$

The cable unravels without slip, Find acceleration G.

FBD:



LMB:

$$T\hat{j} - mg\hat{j} = m\vec{a}_G$$

$$\Rightarrow \vec{a}_G = \left(\frac{T}{m} - g\right)\hat{j}$$

$\therefore$ The acceleration of G is in $\hat{j}$ direction.

Let $\vec{a}_G = -a_G\hat{j}$

then, $a_G = g - \frac{T}{m}$ ... ①

AMB about G:

$$-Tr\hat{k} = I^G\dot{\omega}\hat{k} \quad \dots ②$$

$\dot{\omega}$ is the angular acceleration and is counterclockwise.

No slip condition:

$$a_G = -\dot{\omega}r \quad \dots ③$$

②, ③ $\Rightarrow$ $T = -\frac{I^G\dot{\omega}}{r} = -\frac{I^G\left(-\frac{a_G}{r}\right)}{r} = \frac{I^G a_G}{r^2}$ ... ④

plug ④ in ①

$$\therefore a_G = g - \frac{I^G a_G}{mr^2}$$

$$\Rightarrow a_G = \frac{mr^2 g}{mr^2 + I^G}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$\because$ The drum of radius r is mass less. And assume the disk of radius R is uniform

$\Rightarrow \quad I^G = \frac{1}{2} m R^2$

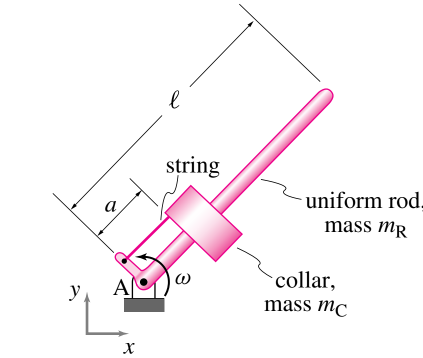
$$\therefore \quad \boxed{a_G = \frac{m r^2 g}{m r^2 + \frac{1}{2} m R^2} = \frac{2 r^2}{2 r^2 + R^2}\, g}$$

Of course G is ~~too~~ accelerating downwards, i.e. $\boxed{\vec{a}_G = -\frac{2r^2}{2r^2+R^2}\, g\hat{\jmath}}$

19.3.9 A uniform rod with mass m_R pivots without friction about point A in the xy-plane. A collar with mass m_C slides without friction on the rod after the string connecting it to point A is cut. There is no gravity. Before the string is cut, the rod has angular velocity ω_1.

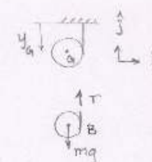
a) What is the speed of the collar after it flies off the end of the rod? Use the following values for the constants and initial conditions: $m_R = 1\,\mathrm{kg}$, $m_C = 3\,\mathrm{kg}$, $a = 1\,\mathrm{m}$, $\ell = 3\,\mathrm{m}$, and $\omega_1 = 1\,\mathrm{rad/s}$

b) Consider the special case $m_R = 0$. Sketch (approximately) the path of the motion of the collar from the time the string is cut until some time after it leaves the end of the rod.



Problem 19.9

18.3.7 : Falling hoop :



LMB of hoop : $\sum \underline{F} = m \underline{a}_G$

$\Rightarrow \{ T\hat{j} - mg\hat{j} = m a_{Gx}\hat{i} + m a_{Gy}\hat{j} \}$

$\{\}\cdot\hat{i} \Rightarrow a_{Gx} = 0$

$\{\}\cdot\hat{j} \Rightarrow m a_{Gy} = T - mg$.

As $a_{Gx} = 0$, hoop does not move side ways while falling

Apply AMB about B : $\sum \underline{M}_{/B} = \dot{\underline{H}}_{/B}$

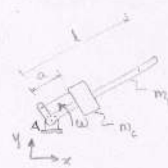
$\Rightarrow -R\hat{i} \times mg(-\hat{j}) = -R\hat{i} \times m\underline{a}_G + I^{cm}\dot{\omega}\hat{k} = -2mRa_G\hat{k} \Rightarrow \underline{a}_G = -\frac{g}{2}\hat{j}$

$\int a_G\,dt = V_G = -\frac{g}{2}t + c \Rightarrow y_G = \int V_G\,dt = -\frac{gt^2}{4} + ct + D$

At $t = 0$, $\dot{y} = 0$, & $y = -y_G$ $\therefore y = -y_G - \frac{gt^2}{4}$ //

18.3.9 :

~~(a)~~ Speed of the collar after it flies off the rod end:



Conservation of angular momentum at two instances

$\underline{H}^-_{/A} = \underline{H}^+_{/A}$

$\Rightarrow I_R^A \omega_1 \hat{k} + m_c a^2 \omega_1 \hat{k} = I_R^A \omega_2 \hat{k} + m_c l^2 \omega_2 \hat{k} \Rightarrow \omega_2 = \frac{\frac{1}{3}m_R l^2 + m_c a^2}{\frac{1}{3}m_R l^2 + m_c l^2}\cdot\omega_1$

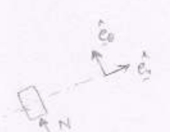
Given, $\omega_1 = 1$ rad/s, $m_c = 3$ kg, $m_R = 1$ kg, $a = 1$ m, $l = 3$ m $\Rightarrow \omega_2 = 0.2$ rad/s

Using energy balance: $E_1 = E_2$

$\frac{1}{2}m_c a^2\omega_1^2 + \frac{1}{2}(\frac{1}{3}m_R l^2)\omega_1^2 = \frac{1}{2}m_c l^2\omega_2^2 + \frac{1}{2}(\frac{1}{3}m_R l^2)\omega_2^2 + \frac{1}{2}m_c v_c^2$

upon substitution, $V_2^2 = 1.6\ (m/s)^2$ (or) $V_2 = 1.265$ m/s.

b) If $m_R = 0$, Path of the collar motion.



Applying LMB on Collar : $N\hat{e}_\theta = m_c[r\dot{\omega}\hat{e}_\theta - \omega^2 r\hat{e}_r + \ddot{r}\hat{e}_r + 2\dot{\omega}\dot{r}\hat{e}_\theta]$

$\Rightarrow \ddot{r} - r\omega^2 = 0$

Applying AMB about A ; $\underline{0} = \dot{\underline{H}}_{/A} \Rightarrow \frac{ml^2}{3}\dot{\omega}\hat{k} + mr^2\dot{\omega}\hat{k} + 2mr\omega\dot{r}\hat{k} \Rightarrow \ddot{\theta} = \frac{-2m_c r\dot{r}}{\frac{m_R l^2}{3} + m_c r^2}\dot{\theta}$

$\Rightarrow \ddot{\theta} = -\frac{2\dot{r}}{r}\dot{\theta}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.10 Assume the rod in the figure for problem **??** has polar moment of inertia $Iozz$. Assume it is free to rotate. The bead is free to slide on the rod. Assume that at $t = 0$ the angular velocity of the rod is 1 rad/s, that the radius of the bead is one meter and that the radial velocity of the bead, dR/dt, is zero.

a) Draw separate free-body diagrams of the bead and rod.

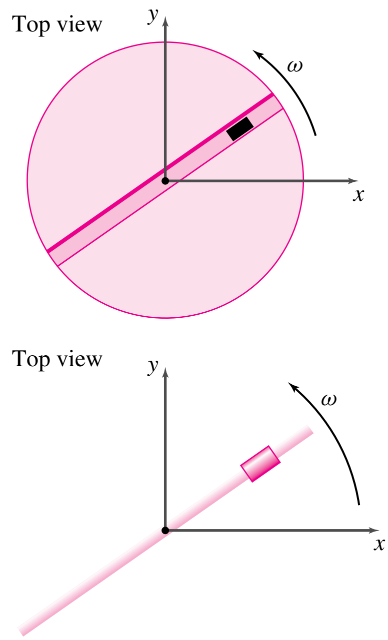
b) Write equations of motion for the system.

c) Use the equations of motion to show that angular momentum is conserved.

d) Find one equation of motion for the system using: (1) the equations of motion for the bead and rod and (2) conservation of angular momentum.

e) Write an expression for conservation of energy. Let the initial total energy of the system be, say, E_0.

f) As t goes to infinity does the bead's distance go to infinity? Its speed? The angular velocity of the turntable? The net angle of twist of the turntable?



Problem 19.10: **Coupled motion of bead and rod and turntable.**

Answer:
(b)

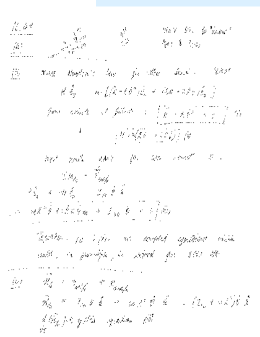
$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ (I_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} &= 0 \end{aligned}$$

(c) The second equation in part (b) can be rewritten in the form $\frac{d}{dt}\left[I_{zz} + mR^2)\dot{\theta}\right] = 0$. The quantity inside the derivative is angular momentum; thus, it is conserved and equal to a constant, say, $(H_{/o})_0$, which can be found in terms of the initial conditions.

(d) $\ddot{R} - R\left[\frac{(H_o)_0^2}{(I_{zz}^O + mR^2)^2}\right] = 0.$

(e) $E_0 = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}\omega(H_{/o})_0.$

(f) The bead's distance goes to infinity and its speed approaches a constant. The turntable's angular velocity goes to zero and its net angle of twist goes to a constant.



(d) Take $\dot{\theta} = \dfrac{H_{/o}}{I_{zz} + mR^2}$ from (c)

& plug into (i) to get:

$$\boxed{\ddot{R} - R\left(\frac{H_{/o}}{I_{zz} + mR^2}\right)^2 = 0}$$

(e) Total energy = rotational energy of (rod + bead)

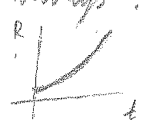
+ translational energy of bead

$$E_0 = \frac{1}{2}(I_{zz} + mR^2)\omega^2 + \frac{1}{2}m\dot{R}^2$$

or,

$$\boxed{E_0 = \frac{1}{2}H_{/o}\,\omega + \frac{1}{2}m\dot{R}^2}$$

(f) * From eq^n in (d), observe that $\ddot{R} > 0$ always. This implies that $R(t)$ is concave upward & increases initially



. In fact, $R \to \infty$ as $t \to \infty$.

However, note that $\ddot{R} \sim \dfrac{1}{R^3}$ for large R.

In particular, as $R \to \infty$, $\ddot{R} \to 0 \Rightarrow \dot{R} \to$ const.

$\therefore$ We deduce that distance of bead will approach ∞, & its speed will approach a constant

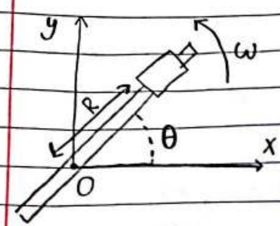
* Now consider $\vec{H}_{/O}$ = constant from (c)

$H_{/O} = (I_{zz} + mR^2)\dot{\theta} \sim mR^2\dot{\theta}$ at large R

Since $H_{/O}$ is constant & $R \to \infty$, the only possibility is $\dot{\theta} \to 0$ (i.e. $\omega \to 0$)

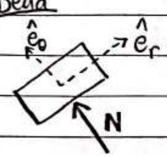
Therefore $\theta \to$ constant.

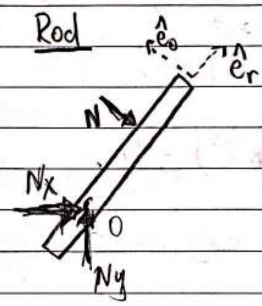
Ruina + Pratap #18.3.10, Solution (Walker Lee 5/6/19)



- Bead slides without friction on rod
- Rod is free to rotate

a. Draw FBDs for bead and rod





b. Write EOMs for the system

LMB for bead

$$\sum \vec{F} = m\vec{a} \rightarrow N\hat{e}_\theta = m\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_r + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta\right]$$

$$\{\ \}\cdot\hat{e}_r \rightarrow m(\ddot{R} - R\dot{\theta}^2) = 0 \rightarrow \boxed{\ddot{R} = R\dot{\theta}^2}$$

AMB for rod

$$\sum \vec{M}_{/O} = I_{zz}^{O}\ddot{\theta}\hat{k} \rightarrow \vec{r}_{bead/O} \times -N\hat{e}_\theta = I_{zz}^{O}\ddot{\theta}\hat{k}$$

$$\rightarrow R\hat{e}_r \times -N\hat{e}_\theta = I_{zz}^{O}\ddot{\theta}\hat{k}$$

From LMB for bead: $\{\}\cdot\hat{e}_\theta \rightarrow N = m(R\ddot{\theta} + 2\dot{R}\dot{\theta})$

$\rightarrow R\hat{e}_r \times -m(R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta = I^O_{zz}\ddot{\theta}\hat{k}$

$\rightarrow -mR^2\ddot{\theta} - 2mR\dot{R}\dot{\theta} = I^O_{zz}\ddot{\theta}\hat{k}$

$\rightarrow \boxed{(I^O_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} = 0}$

Could also get in 1 step from AMB/O of whole system

c. Show that angular momentum is conserved.

We can rewrite our answer to (b) as follows:

$$(I^O_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} = 0$$

$$\rightarrow (I^O_{zz} + mR^2)\frac{d}{dt}\dot{\theta} + \dot{\theta}\frac{d}{dt}(mR^2) = 0$$

Since $\frac{d}{dt}I^O_{zz} = 0$, we can freely insert I^O_{zz} into a time derivative without changing its value:

$$\rightarrow (I^O_{zz} + mR^2)\frac{d}{dt}\dot{\theta} + \dot{\theta}\frac{d}{dt}(I^O_{zz} + mR^2) = 0$$

Invoking the product rule:

$$\rightarrow \frac{d}{dt}\Big[\underbrace{(I^O_{zz} + mR^2)\dot{\theta}}_{\text{Angular momentum about O}}\Big] = 0$$

This shows that angular momentum about O doesn't change with time, which means angular momentum is conserved.

d. Combine your answers to (b) and (c) to form one EOM.

From (c): $\frac{d}{dt}\left[(I^o_{zz} + mR^2)\dot{\theta}\right] = 0 \rightarrow (I^o_{zz} + mR^2)\dot{\theta} = (H_{/o})_0$

$\rightarrow \dot{\theta} = \frac{(H_{/o})_0}{I^o_{zz} + mR^2}$

Substitute into (b):

$\ddot{R} - R\dot{\theta}^2 = 0$

$\rightarrow \boxed{\ddot{R} - R\left[\frac{(H_{/o})_0^2}{(I^o_{zz} + mR^2)^2}\right] = 0}$

e. Write an expression for conservation of energy.

$E_0 = \frac{1}{2}m\vec{v}\cdot\vec{v} + \frac{1}{2}I^G\vec{\omega}\cdot\vec{\omega}$

$= \frac{1}{2}m(\dot{R}\hat{e}_r \cdot \dot{R}\hat{e}_r) + \frac{1}{2}(I^o_{zz} + mR^2)(\dot{\theta}\hat{k}\cdot\dot{\theta}\hat{k})$

$= \frac{1}{2}m\dot{R}^2 + \frac{1}{2}(I^o_{zz} + mR^2)\dot{\theta}^2$

We can substitute our expression for angular momentum, $(H_{/o})_0 = (I^o_{zz} + mR^2)\dot{\theta}$:

$\boxed{E_0 = E(t) = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}\dot{\theta}(H_{/o})_0 = \text{constant}}$

f. As $t \rightarrow \infty$...

- Does $R \rightarrow \infty$? Does $\dot{R} \rightarrow \infty$?
- Does $\theta \rightarrow \infty$? Does $\dot{\theta} \rightarrow \infty$?

- From (e): $\dot{R}^2 + R^2\dot{\theta}^2 \sim$ constant $\rightarrow$ $\dot{R}$ cannot grow to ∞ $\rightarrow$ $\ddot{R}$ shrinks to 0
- From (d): $\ddot{R} \sim \frac{1}{R^3}$ $\rightarrow$ $\frac{1}{R^3}$ shrinks to 0 $\rightarrow$ $\boxed{R \rightarrow \infty}$ $\rightarrow$ $\boxed{\dot{R} \rightarrow \text{constant}}$
- From (c): $\dot{\theta}R^2 \sim C$ $\rightarrow$ $\boxed{\dot{\theta} \text{ shrinks to } 0}$ $\rightarrow$ $\boxed{\theta \rightarrow \text{constant}}$

const

as $1/t^2$, integrable

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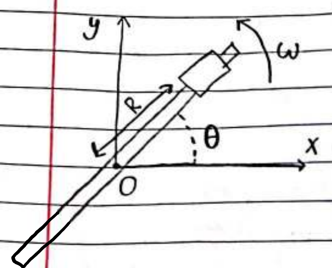
Revision: Duan Li

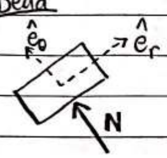
Ruina + Pratap #19.3.10, Solution (Walker Lee 5/6/19)



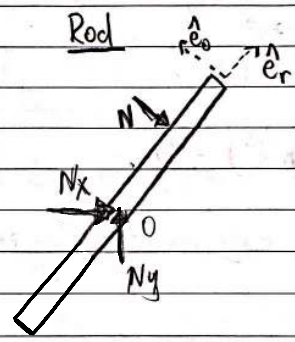
- Bead slides without friction on rod
- Rod is free to rotate

a. Draw FBDs for bead and rod

Bead



Rod



b. Write EOMs for the system 2 DoF (R , θ) → 2 EoMs

LMB for bead

unknown , so {LMB} · $\hat{e}_\theta$ doesn't give useful info

$$\sum \vec{F} = m\vec{a} \rightarrow N\hat{e}_\theta = m\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_r + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta\right]$$

$$\{\ \} \cdot \hat{e}_r \rightarrow m(\ddot{R} - R\dot{\theta}^2) = 0 \rightarrow \boxed{\ddot{R} = R\dot{\theta}^2}$$

AMB for whole system

$$\sum \vec{M}_{/0} = \dot{\vec{H}}_{/0} = \dot{\vec{H}}_{bead/0} + \dot{\vec{H}}_{rod/0}$$

$$\vec{0} = m\vec{r} \times \vec{a} + I_{zz}^{o}\,\ddot{\theta}\,\hat{k}$$

Note $\dot{\vec{H}}_{/0} \neq (\underbrace{I^{o}_{bead}}_{= mR^2} + I^{o}_{zz})\,\ddot{\theta}\,\hat{k}$

because the system is not a rigid body

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Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$\vec{0} = mR\hat{e}_r \times [(\ddot{R} - R\dot{\theta}^2)\hat{e}_r + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta] + I^o_{zz}\ddot{\theta}\hat{k}$$

$$= (mR^2\ddot{\theta} + 2mR\dot{R}\dot{\theta} + I^o_{zz}\ddot{\theta})\hat{k}$$

$$\rightarrow \boxed{(I^o_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} = 0}$$

c. Show that angular momentum is conserved.

We can rewrite our answer to (b) as follows:

$$(I^o_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} = 0$$

$$\rightarrow (I^o_{zz} + mR^2)\frac{d}{dt}\dot{\theta} + \dot{\theta}\frac{d}{dt}(mR^2) = 0$$

Since $\frac{d}{dt}I^o_{zz} = 0$, we can freely insert I^o_{zz} into a time derivative without changing its value:

$$\rightarrow (I^o_{zz} + mR^2)\frac{d}{dt}\dot{\theta} + \dot{\theta}\frac{d}{dt}(I^o_{zz} + mR^2) = 0$$

Invoking the product rule:

$$\rightarrow \frac{d}{dt}\Big[\underbrace{(I^o_{zz} + mR^2)\dot{\theta}}_{\text{Angular momentum about } O}\Big] = 0$$

This shows that angular momentum about O doesn't change with time, which means angular momentum is conserved.

d. Combine your answers to (b) and (c) to form one EOM.

From (c): $\frac{d}{dt}\left[(I^0_{zz} + mR^2)\dot{\theta}\right] = 0 \rightarrow (I^0_{zz} + mR^2)\dot{\theta} = (H_{/0})_0 = \text{const}$

$\rightarrow \dot{\theta} = \frac{(H_{/0})_0}{I^0_{zz} + mR^2}$

Given ICs $r_0 = 1\ \text{m}$, $\dot{\theta}_0 = 1\ \text{rad/s}$

$\rightarrow (H_{/0})_0 = (I^0_{zz} + m r_0^2)\dot{\theta}_0$

Substitute into (b):

$\ddot{R} - R\dot{\theta}^2 = 0$

$$\rightarrow \boxed{\ddot{R} - R\left[\frac{(H_{/0})_0^2}{(I^0_{zz} + mR^2)^2}\right] = 0}$$

e. Write an expression for conservation of energy.

$E = E_{bead} + E_{rod} = \frac{1}{2} m \vec{v}\cdot\vec{v} + \frac{1}{2} I^0_{zz} \vec{\omega}\cdot\vec{\omega}$

$= \frac{1}{2} m (\dot{R}\hat{e}_r + R\dot{\theta}\hat{e}_\theta)\cdot(\dot{R}\hat{e}_r + R\dot{\theta}\hat{e}_\theta) + \frac{1}{2} I^0_{zz} (\dot{\theta}\hat{k}\cdot\dot{\theta}\hat{k})$

$= \frac{1}{2} m\dot{R}^2 + \frac{1}{2}(I^0_{zz} + mR^2)\dot{\theta}^2$

$= E_0 = \frac{1}{2} m \dot{R}_0^2 + \frac{1}{2}(I^0_{zz} + mR_0^2)\dot{\theta}_0^2 = \frac{1}{2} m\dot{R}_0^2 + \frac{1}{2}\dot{\theta}_0 (H_{/0})_0$

$$\boxed{E_0 = E(t) = \frac{1}{2} m\dot{R}^2 + \frac{1}{2}(I^0_{zz} + mR^2)\dot{\theta}^2}$$

f. As $t \rightarrow \infty$...

- Does $R \rightarrow \infty$? Does $\dot{R} \rightarrow \infty$?
- Does $\theta \rightarrow \infty$? Does $\dot{\theta} \rightarrow \infty$?

From (e): $E = \frac{1}{2} m \dot{R}^2 + \frac{1}{2}(I_{zz}^{0} + mR^2)\dot{\theta}^2 = E_0 = \text{const}$

If $\dot{R} \to \infty$, Then $R = \int_0^{\infty} \dot{R}\, dt \to \infty$,

$R^2 \to +\infty$, $\dot{R}^2 \to +\infty$, $\dot{\theta}^2 > 0$, $E \to +\infty$ contradicts (e)

So $\dot{R}$ cannot $\to \infty$, which means $\ddot{R} \to 0$

From (d): $\ddot{R} - R\left[\frac{(H_{/O})_0}{(I_{zz}^{0} + mR^2)^2}\right] = 0$

$\because \ddot{R} \to 0$, $\therefore R\left[\frac{(H_{/O})_0}{(I_{zz}^{0} + mR^2)^2}\right] \to 0$ which requires $\boxed{R \to \infty}$

$\because \dot{R}$ cannot $\to 0$ to make $R \to \infty$, $\therefore \boxed{\dot{R} \to \text{const}} = c$

$\therefore R = \int_0^{\infty} \dot{R}\, dt \to ct \sim t$

From (c): $H_{/O} = (I_{zz}^{0} + mR^2)\dot{\theta} = \text{Const}$

$\because R \to \infty$, $\therefore \dot{\theta} \to \frac{\text{const}}{R^2} \sim \frac{1}{t^2}$, $\boxed{\dot{\theta} \to 0}$

$\therefore \theta = \int_0^{\infty} \dot{\theta}\, dt \sim \underbrace{\int_0^{t'} \dot{\theta}\, dt}_{\text{bounded}} + \underbrace{\int_{t'}^{\infty} \frac{1}{t^2}\, dt}_{= \frac{1}{t'} - \frac{1}{\infty} = \frac{1}{t'}\text{, bounded}}$

$t' > 0$

$\dot{\theta}(t=0) = 1\ \text{rad/s}$

$\dot{\theta}(t \to \infty) \sim \frac{1}{t^2}$

$\therefore \theta$ is bounded, $\boxed{\theta \to \text{const}}$

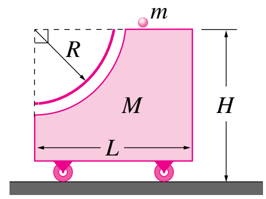
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.11 A primitive gun rides on a cart (mass M) and carries a cannon ball (of mass m) on a platform at a height H above the ground. The cannon ball is dropped through a frictionless tube shaped like a quarter circle of radius R.

a) If the system starts from rest, compute the horizontal speed (relative to the ground) that the cannon ball has as it leaves the bottom of the tube. Also find the cart's speed at the same instant.

b) Compute the velocity of the cannon ball relative to the cart.

c) If two balls are dropped simultaneously through the tube, what speed does the cart have when the balls reach the bottom? Is the same final speed also achieved if one ball is allowed to depart the system entirely before the second ball is released? Why/why not?



Problem 19.11

18.3.11: Velocities of ball (V_b) and cart (V_c)

As there are no horizontal forces, linear momentum is conserved

$\therefore (M+m)(0) = MV_c + mV_b \Rightarrow V_c = -\frac{m}{M}V_b$

Energy Balance : $mgH = \frac{1}{2}mV_b^2 + \frac{1}{2}MV_c^2 + mg(H-R)$

$\Rightarrow mgR = \frac{1}{2}mV_b^2 + \frac{1}{2}M\left(-\frac{m}{M}V_b\right)^2 \Rightarrow V_b = \sqrt{\frac{2gR}{1+\frac{m}{M}}} \Rightarrow V_c = -\frac{m}{M}\sqrt{\frac{2gR}{1+\frac{m}{M}}}$

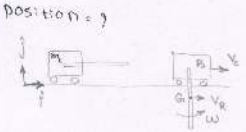
b) Velocity of ball relative to cart, $V_{b/c} = V_b - V_c = \sqrt{2gR\left(1+\frac{m}{M}\right)}$.

c) When 2 balls are dropped, $V_{c_2} = \frac{-(2m)}{M}\sqrt{\frac{2gR}{1+\frac{2m}{M}}}$ but when 1st ball is allowed to depart entirely before the release of 2nd ball, cart speed, $V_c = -\frac{2m}{M}\sqrt{\frac{2gR}{1+\frac{m}{M}}} > V_{c_2}$

18.3.15:

Angular speed of the rod in vertical position = ?

As the linear momentum is conserved along $\hat{i}$



$m_cV_c + m_RV_R = 0 \Rightarrow V_R = -\frac{m_c}{m_R}V_c$

$\underline{V}_G = \underline{V}_B + \underline{V}_{G/B} \Rightarrow \left\{V_R\hat{i} = V_c\hat{i} + \frac{\omega l}{2}\hat{i}\right\}\cdot\hat{i} \Rightarrow V_c = -\frac{\omega l}{2}\left(\frac{1}{1+\frac{m_c}{m_R}}\right)$

$\therefore V_R = \frac{m_c}{m_R}\left[\frac{\omega l/2}{1+\frac{m_c}{m_R}}\right]$

$m_R = 10\,lbm$
$l = 2\,ft$
$m_c = 10\,lbm$

Consider energy balance at two instants,

$m_R g\frac{l}{2} = \frac{1}{2}m_cV_c^2 + \frac{1}{2}m_RV_R^2 + \frac{1}{2}I^G\omega^2$

$\Rightarrow \omega^2\left\{\frac{m_c}{m_R}\frac{1}{\left(1+\frac{m_c}{m_R}\right)^2} + \left(\frac{m_c}{m_R}\right)^2\frac{1}{\left(1+\frac{m_c}{m_R}\right)^2} + \frac{1}{3}\right\} = \frac{4g}{l}$

$\Rightarrow \omega = \pm\sqrt{\frac{4g}{l\left[\frac{m_c/m_R}{1+m_c/m_R}+\frac{1}{3}\right]}} \quad \therefore \omega = -8.76\ \text{rad/sec}$

b) Displacement of the cart $(m_c+m_R)x_{cm} = m_c(0) + m_R\left(\frac{l}{2}\right)$

$\Rightarrow x_{cm} = \frac{m_R l}{2(m_c+m_R)} \Rightarrow x_{cm} = \frac{l}{4} = 0.5\,ft$

c) When the ~~cart~~ rod is horizontal on the left side of the cart

$x_{cm} = \frac{m_R l}{m_c+m_R} \Rightarrow x_{cm} = \frac{l}{2} = 1\,ft$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.12 Numerically simulate the coupled system in problem 19.3.10. Use the simulation to show that the net angle of the turntable or rod is finite.

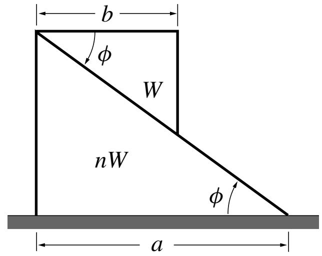
a) Write the equations of motion for the system from part (b) in problem 19.3.10 as a set of first order differential equations.

b) Numerically integrate the equations of motion.

Answer:
(a)

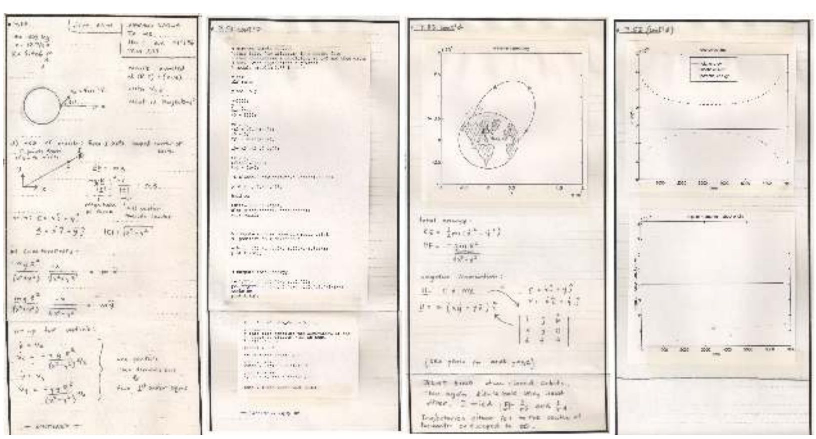
$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{2mRv\omega}{I_{zz} + mR^2} \\
\dot{R} &= v \\
\dot{v} &= \omega^2 R
\end{aligned}
$$

19.3.13 Two frictionless prisms of similar right triangular sections are placed on a frictionless horizontal plane. The top prism weighs W and the lower one nW. The prisms are held in the initial position shown and then released, so that the upper prism slides down along the lower one until it just touches the horizontal plane. The center of mass of a triangle is located at one-third of its height from the base. Compute the **velocities** of the two prisms at the moment just before the upper one reaches the bottom.



Problem 19.13

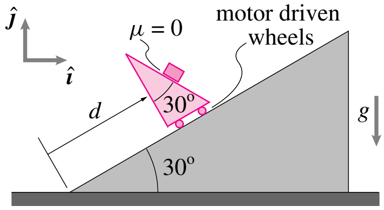
Answer:
$v_{2_x} = \sqrt{\frac{2g(a-b)\tan\phi}{(1+\frac{1}{n}+\left(\frac{n+1}{n}\right)^2\tan^2\phi)}}$, $v_{2_y} = v_{2_x}\frac{n+1}{n}\tan\phi$, $v_{1_x} = -\frac{v_{2_x}}{n}$, $v_{1_y} = 0$, where 2 refers to the top wedge and 1 refers to the lower wedge.



19.3.14 Mass slides on an accelerating cart. 2D. A cart is driven by a powerful motor to move along the 30° sloped ramp according to the formula: $d = d_0 + v_o t + a_0 t^2/2$ where d_0, v_0, and a_0 are given constants. The cart is held from tipping over. The cart itself has a 30° sloped upper surface on which rests a mass (given mass m). The surface on which the mass rests is frictionless. Initially the mass is at rest with regard to the cart.

a) What is the force of the cart on the mass? [in terms of $g, d_0, v_0, t, a_0, m, g, \hat{\imath}$, and $\hat{\jmath}$.]

b) For what values of d_0, v_0, t and a_0 is the acceleration of the mass exactly vertical (i.e., in the $\hat{\jmath}$ direction)?



Problem 19.14

Answer:
(a) $\vec{F} = m(g + a_0)\left[\frac{\sqrt{3}}{4}\hat{\imath} + \frac{3}{4}\hat{\jmath}\right]$.
(b) For $a_0 = -g$, the acceleration of the mass is exactly vertical; d_0, v_0, and t could be anything.

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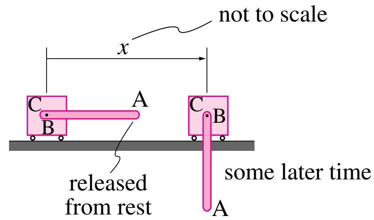
19.3.15 A thin rod AB of mass $W_{AB} = 10\,\text{lbm}$ and length $L_{AB} = 2\,\text{ft}$ is pinned to a cart C of mass $W_C = 10\,\text{lbm}$, the latter of which is free to move along a frictionless horizontal surface, as shown in the figure. The system is released from rest with the rod in the horizontal position.

a) Determine the angular speed of the rod as it passes through the vertical position (at some later time).

b) Determine the displacement x of the cart at the same instant.

c) After the rod passes through vertical, it is momentarily horizontal but on the left side of the cart. How far has the cart moved when this configuration is reached?

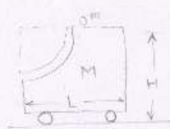


Problem 19.15

Answer:
(a) angular speed = 8.8 rad/s.
(b) displacement = 0.5 ft.

18.3.11: Velocities of ball (V_b) and cart (V_c)

As there are no horizontal forces, linear momentum is conserved



$(M+m)(0) = MV_c + mV_b \Rightarrow V_c = -\frac{m}{M}V_b$

Energy Balance : $mgH = \frac{1}{2}mV_b^2 + \frac{1}{2}MV_c^2 + mg(H-R)$

$\Rightarrow mgR = \frac{1}{2}mV_b^2 + \frac{1}{2}M\left(-\frac{m}{M}V_b\right)^2 \Rightarrow V_b = \sqrt{\frac{2gR}{1+\frac{m}{M}}} \Rightarrow V_c = -\frac{m}{M}\sqrt{\frac{2gR}{1+\frac{m}{M}}}$

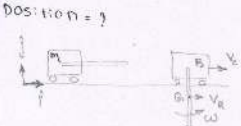
b) Velocity of ball relative to cart, $V_{b/c} = V_b - V_c = \sqrt{2gR\left(1+\frac{m}{M}\right)}$

c) When 2 balls are dropped, $V_{c_2} = \frac{-(2m)}{M}\sqrt{\frac{2gR}{1+\frac{2m}{M}}}$ but when 1st ball is allowed to depart entirely before the release of 2nd ball, Cart speed, $V_c = \frac{-2m}{M}\sqrt{\frac{2gR}{1+\frac{m}{M}}} > V_{c_2}$

18.3.15:

Angular speed of the rod in vertical position = ?

As the linear momentum is conserved along $\hat{i}$



$m_cV_c + m_RV_R = 0 \Rightarrow V_R = -\frac{m_c}{m_R}V_c$

$\underset{\sim}{V}_G = \underset{\sim}{V}_B + \underset{\sim}{V}_{G/B} \Rightarrow \left\{V_R\hat{i} = V_c\hat{i} + \frac{\omega l}{2}\hat{i}\right\}\cdot\hat{i} \Rightarrow V_c = -\frac{\omega l}{2}\left(\frac{1}{1+\frac{m_c}{m_R}}\right)$

$\therefore V_R = \frac{m_c}{m_R}\left[\frac{\omega l/2}{1+\frac{m_c}{m_R}}\right]$

$m_R = 10\,\text{lbm}$
$l = 2\,\text{ft}$
$m_c = 10\,\text{lbm}$

Consider energy balance at two instants,

$$m_Rg\frac{l}{2} = \frac{1}{2}m_cv_c^2 + \frac{1}{2}m_Rv_R^2 + \frac{1}{2}I^G\omega^2$$

$$\Rightarrow \omega^2\left\{\frac{m_c}{m_R}\frac{1}{\left(1+\frac{m_c}{m_R}\right)^2} + \left(\frac{m_c}{m_R}\right)^2\frac{1}{\left(1+\frac{m_c}{m_R}\right)^2} + \frac{1}{3}\right\} = \frac{4g}{l}$$

$$\Rightarrow \omega = \pm\sqrt{\frac{4g}{l\left[\frac{m_c/m_R}{1+m_c/m_R} + \frac{1}{3}\right]}} \qquad \therefore \omega = -8.76\,\text{rad/sec}$$

b) Displacement of the cart $(m_c+m_R)x_{cm} = m_c(0) + m_R\left(\frac{l}{2}\right)$

$$\Rightarrow x_{cm} = \frac{m_R l}{2(m_c+m_R)} \Rightarrow x_{cm} = \frac{l}{4} = 0.5\,\text{ft}$$

c) When the ~~cart~~ rod is horizontal on the left side of the cart

$$x_{cm} = \frac{m_R l}{m_c+m_R} \Rightarrow x_{cm} = \frac{l}{2} = 1\,\text{ft}$$

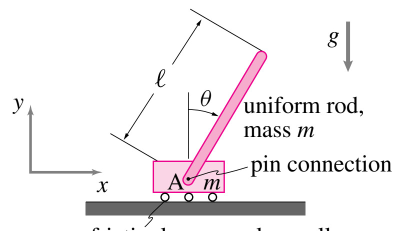
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.16 As shown in the figure, a block of mass m rolls without friction on a rigid surface and is at position x (measured from a fixed point). Attached to the block is a uniform rod of length ℓ which pivots about one end which is at the center of mass of the block. The rod and block have equal mass. The rod makes an angle θ with the vertical. Use the numbers below for the values of the constants and variables at the time of interest:

$$
\begin{aligned}
\ell &= 1\,\text{m} \\
m &= 2\,\text{kg} \\
\theta &= \pi/2 \\
d\theta/dt &= 1\,\text{rad/s} \\
d^2\theta/dt^2 &= 2\,\text{rad/s}^2 \\
x &= 1\,\text{m} \\
dx/dt &= 2\,\text{m/s} \\
d^2x/dt^2 &= 3\,\text{m/s}^2
\end{aligned}
$$

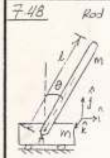
a) What is the kinetic energy of the system?

b) What is the linear momentum of the system (momentum is a vector)?



Problem 19.16

7.48 Rod and block (of equal mass) system. Given:



$\ell = 1\,m$; $m = 2\,kg$; $\theta = \pi/2$; $d\theta/dt = 1\,rad/s$; $d^2\theta/dt^2 = 2\,rad/s^2$; $x = 1\,m$; $\frac{dx}{dt} = 2\,m/s$; $\frac{d^2x}{dt^2} = 3\,m/s^2$

what is the kinetic energy?

$$E_K = \sum_{block, rod} \frac{1}{2} m v_{cm}^2 + \frac{1}{2}\underset{\sim}{\omega}\cdot[I^{cm}]\cdot\underset{\sim}{\omega}$$

for block: $\underset{\sim}{\omega} = \underset{\sim}{0}$; $E_K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

for rod: $E_K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\underset{\sim}{\omega}\cdot[I^{cm}]\cdot\underset{\sim}{\omega}$

where $|\underset{\sim}{v}_{cm}| = |\underset{\sim}{v}_A + \underset{\sim}{v}_{cm/A}| = \left|\frac{dx}{dt}\hat{\imath} + \underset{\sim}{\omega}\times\underset{\sim}{r}_{cm/A}\right| =$

$\left|\frac{dx}{dt}\hat{\imath} + \left((-1\,rad/s)\hat{k}\times\frac{\ell}{2}\left(\sin\pi/2\,\hat{\imath} + \cos\pi/2\,\hat{\jmath}\right)\right)\right| =$

$\left|\frac{dx}{dt}\hat{\imath} - \frac{\ell}{2}\left[\frac{1}{s}\right]\hat{\jmath}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{\ell}{2}\right)^2}$

thus $E_K(rod) = \frac{1}{2}m\left(\left(\frac{dx}{dt}\right)^2 + \frac{\ell^2}{4}\right) + \frac{1}{2}(-\dot\theta\hat{k})\cdot I^{cm}_{zz}(-\dot\theta\hat{k})$

$E_K(rod) = \frac{1}{2}m\left(\left(\frac{dx}{dt}\right)^2 + \frac{\ell^2}{4}\right) + \frac{1}{24}m\ell^2\dot\theta^2$

continued

7.48 continued p.8

$E_K(total) = E_K(block) + E_K(rod) =$

$\frac{1}{2}m\left(2\left(\frac{dx}{dt}\right)^2 + \frac{\ell^2}{4}\left[\frac{1}{s^2}\right] + \frac{\ell^2\dot\theta^2}{12}\right) \Rightarrow$ putting in #'s yields:

$E_K = 1\,kg\left(8\,m^2/s^2 + 0.25\,m^2/s^2 + 0.833\,m^2/s^2\right) =$

$$\boxed{E_K = 8.33\ \frac{kg\,m^2}{s^2}}$$

b) what is $\underset{\sim}{L}$? $\underset{\sim}{L} = \sum_{rod, block} m_i \underset{\sim}{v}_i$

$= m\left(\underset{\sim}{v}_{rod} + \underset{\sim}{v}_{block}\right)$ we already know (or have calculated) v_{cm} for the rod & block:

$$\underset{\sim}{L} = m\Big(\underbrace{\frac{dx}{dt}\hat{\imath} - \frac{\ell}{2}\left[\frac{1}{s}\right]\hat{\jmath}}_{v_{cm}\text{ for rod}} + \underbrace{\frac{dx}{dt}\hat{\imath}}_{v_{cm}\text{ for block}}\Big) =$$

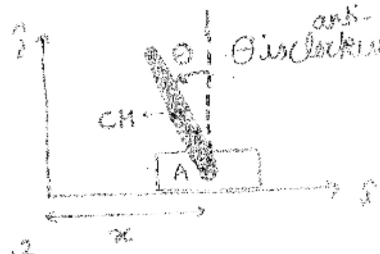
$$\underset{\sim}{L} = 2\,kg\left(4\,m/s\,\hat{\imath} - 1/2\,m/s\,\hat{\jmath}\right) = \boxed{(8\hat{\imath} - 1\hat{\jmath})\,kg\,m/s}$$

HW-26

16.55

Don't need FBD Here (no clamp)

NOTE θ is anti-clockwise



a) $K.E = K.E_{block} + K.E_{rod}$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{cm} \dot{\theta}^2 + \frac{1}{2} m V_{cm}^2$$

where $\vec{V}_{cm} = \vec{V}_A + \dot{\theta}\hat{k} \times \vec{r}_{cm/A}$

$$= \dot{x}\hat{\imath} + \dot{\theta}\hat{k} \times \left(-\frac{l}{2}\sin\theta\,\hat{\imath} + \frac{l}{2}\cos\theta\,\hat{\jmath}\right)$$

$$= \left(\dot{x} - \dot{\theta}\frac{l}{2}\cos\theta\right)\hat{\imath} - \dot{\theta}\frac{l}{2}\sin\theta\,\hat{\jmath} \quad \text{——— (1)}$$

$$V_{cm}^2 = \dot{x}^2 + \dot{\theta}^2\frac{l^2}{4} - 2\dot{x}\dot{\theta}\frac{l}{2}\cos\theta$$

$$K.E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\left(\frac{1}{12} m l^2\right)\dot{\theta}^2 + \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{\theta}^2 \frac{l^2}{4} - \frac{1}{2} m \dot{x}\dot{\theta} l \cos\theta$$

$$= m\dot{x}^2 + \frac{1}{6} m l^2 \dot{\theta}^2 - \frac{1}{2} m \dot{x}\dot{\theta} l \cos\theta$$

Put in values

$$= 2 \cdot 2^2 + \frac{1}{6} \cdot 2 \cdot 1^2 \cdot 1^2 - \frac{1}{2} 2 \cdot 2 \cdot 1 \cdot 1 \cos\left(\frac{\pi}{2}\right)$$

$$= \boxed{25/3}$$

b) linear momentum

$$\vec{p} = \underbrace{m\vec{V}_A}_{block} + \underbrace{m\vec{V}_{cm}}_{rod} = m\dot{x}\hat{\imath} + m\left(\dot{x} - \dot{\theta}\frac{l}{2}\cos\theta\right)\hat{\imath} - m\dot{\theta}\frac{l}{2}\sin\theta\,\hat{\jmath}$$

$$= m\left[\left(2\dot{x} - \dot{\theta}\frac{l}{2}\cos\theta\right)\hat{\imath} - \dot{\theta}\frac{l}{2}\sin\theta\,\hat{\jmath}\right]$$

put in values

$$= 2\left[(2 \cdot 2 - 0)\hat{\imath} - 1 \cdot \frac{1}{2}\sin\frac{\pi}{2}\,\hat{\jmath}\right] = \boxed{8\hat{\imath} - 1\hat{\jmath}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

note → in the problem θ is measured clockwise

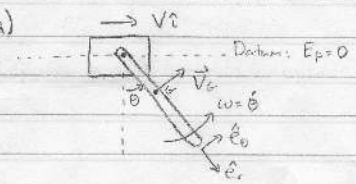
→ thats fine just change $\theta \to -\theta$ in all the equations.

$\therefore$ T = same

$\vec{F} = 8\hat{\imath} + 1\hat{\jmath}$

Johannes Feng
HW problem 16.55
Due 12/4/08
TAM 2030
Section 2 - 1:25
TA: Rong Long

16.55) a)



At any time, $E_p = -W_{AB}\, g \frac{L_{AB}}{2} \cos\theta$

$E_k = \frac{1}{2} W_c V^2 + \frac{1}{2} W_{AB} \vec{V}_G \cdot \vec{V}_G + \frac{1}{2} I^G \omega^2$

3 unknowns: $\vec{V}_G, V, \omega$

Start with $\vec{V}_G = \vec{V} + \frac{L_{AB}}{2} \omega \hat{e}_\theta = \left(V + \frac{\omega L_{AB}}{2} \cos\theta\right)\hat{\imath} + \frac{\omega L_{AB}}{2} \sin\theta \hat{\jmath}$

Conservation of linear momentums $(W_{AB} \vec{V}_G + W_c \vec{V}) \cdot \hat{\imath} = 0$

$\cancel{W_{AB}}\left(V + \frac{\omega L_{AB}}{2}\cos\theta\right) + \cancel{W_c} V = 0$, because $W_{AB} = W_c$

$2V = \frac{-\omega L_{AB}}{2} \cos\theta \longrightarrow V = \frac{-\omega L_{AB}}{4} \cos\theta$

Therefore, $\vec{V}_G = \left(\frac{\omega L_{AB}}{4} \cos\theta\right)\hat{\imath} + \left(\frac{\omega L_{AB}}{2} \sin\theta \hat{\jmath}\right)$

Plug $\vec{V}_G$ and V into E_k: $E_k = \frac{1}{2} W_c \left(\frac{\omega^2 L_{AB}^2}{16} \cos^2\theta\right) + \frac{1}{2} W_{AB} \left(\frac{\omega^2 L_{AB}^2}{16} \cos^2\theta + \frac{\omega^2 L_{AB}^2}{4} \sin^2\theta\right) + \frac{1}{2}\left(\frac{W_{AB} L_{AB}^2}{12}\right)\omega^2$

$E_k = W L_{AB}^2 \omega^2 \left(\frac{1}{32}\cos^2\theta + \frac{1}{32}\cos^2\theta + \frac{1}{8}\sin^2\theta + \frac{1}{24}\right)$

When rod is in vertical position, $\theta = 0$, so $E_p = -\frac{1}{2} W g L_{AB}$

Conservation of Energy : $(E_p + \cancelto{0}{E_k})_{\theta = \frac{\pi}{2}} = (E_p + E_k)_{\theta = 0}$

$-\frac{1}{2} \cancel{W} g \cancel{L_{AB}} + \cancel{W} L_{AB}^{\cancel{2}} \omega^2 \left(\frac{1}{16} + \frac{1}{24}\right) = 0$

$\omega^2 \left(\frac{6}{96} + \frac{4}{96}\right) = \frac{1}{2} \frac{g}{L_{AB}}$

$\omega^2 \left(\frac{5}{48}\right) = \frac{1}{2} \frac{g}{L_{AB}}$

$\omega = -\sqrt{\frac{24}{5} \frac{g}{L_{AB}}}$ because rod is swinging clockwise

$\omega = -\sqrt{\left(\frac{24}{5}\right)\left(\frac{32}{2}\right)} = \boxed{-8.76 \frac{rad}{s}}$

b) $x = \int_0^T v\,dt$ -> Convert to angles instead of time

$\omega = \dot{\theta} = \frac{d\theta}{dt}$, $dt = \frac{d\theta}{\omega}$

$V = \frac{-\omega L_{AB}}{4} \cos\theta$

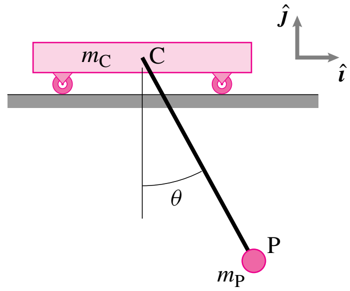
$t = 0 \rightarrow \theta = \frac{\pi}{2}$, $t = T \rightarrow \theta = 0$

$x = \int_{\pi/2}^{0} \frac{-L_{AB}}{4} \cos\theta\, d\theta = \frac{L_{AB}}{4} \int_0^{\pi/2} \cos\theta\, d\theta = \frac{L_{AB}}{4} \left[\sin\theta\right]_0^{\pi/2} = \frac{L_{AB}}{4} = \frac{2}{4} = \boxed{0.5 ft}$

c) $x = \int_{\pi/2}^{-\pi/2} \frac{-L_{AB}}{4} \cos\theta\, d\theta = \frac{L_{AB}}{4} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{L_{AB}}{2} \int_0^{\pi/2} \cos\theta\, d\theta = \frac{L_{AB}}{2} = \frac{2}{2} = \boxed{1 ft}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

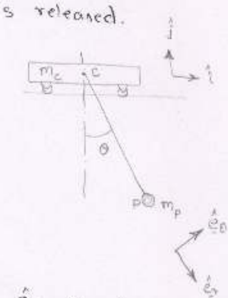
19.3.17 A pendulum of length ℓ hangs from a cart. The pendulum is massless except for a point mass of mass m_p at the end. The cart rolls without friction and has mass m_c. The cart is initially stationary and the pendulum is released from rest at an angle θ. What is the acceleration of the cart just after the mass is released? [Hints: $\mathbf{a}_P = \vec{\mathbf{a}}_C + \vec{\mathbf{a}}_P/C$. The answer is $\vec{\mathbf{a}}_C = (g/3)\mathbf{i}$ in the special case when $m_p = m_c$ and $\theta = \pi/4$.]



Problem 19.17

18.3.17

Acceleration of the cart just after the mass is released.



$\underline{r}_P = \underline{r}_C + \underline{r}_{P/C} = c\hat{i} + l\hat{e}_r$

$\underline{v}_P = \dot{c}\hat{i} + l\dot{\hat{e}}_r = \dot{c}\hat{i} + l(\omega\hat{k}\times\hat{e}_r) = \dot{c}\hat{i} + \omega l\hat{e}_\theta$

$\underline{a}_P = \ddot{c}\hat{i} + \omega l\dot{\hat{e}}_\theta + \dot{\omega} l\hat{e}_\theta = \ddot{c}\hat{i} + \dot{\omega} l\hat{e}_\theta - l\omega^2\hat{e}_r$

$\hat{e}_r = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

$\hat{e}_\theta = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Applying AMB about C : $\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$

=> $l\hat{e}_r \times m_p g(-\hat{j}) = l\hat{e}_r \times m\underline{a}_p + \overset{\to 0}{I^P}\dot{\omega}\hat{k}$

=> $\ddot{c}\cos\theta + l\dot{\omega} + g\sin\theta = 0$ — ①

Applying LMB for the system :

$\{-m_c g\hat{j} - m_p g\hat{j} = m_c\ddot{c}\hat{i} + m_p[\ddot{c}\hat{i} + \dot{\omega} l\hat{e}_\theta - \omega^2 l\hat{e}_r]\}\cdot\hat{i}$

=> $0 = m_c\ddot{c} + m_p\ddot{c} + m_p\dot{\omega} l\cos\theta - m_p\omega^2 l\sin\theta$ — ②

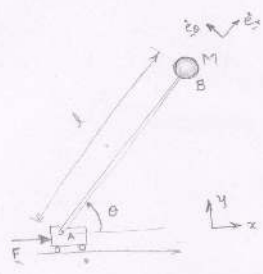
Solving for $\ddot{c}$ using ① and ②

$$\underline{a}_c = \ddot{c}\hat{i} = \frac{g\sin\theta\cos\theta + l\omega^2\tan\theta}{\left(\frac{m_c}{m_p}+1\right) - \cos^2\theta}\hat{i}$$

Check : $m_c = m_p$, $\theta = \pi/4$, $\ddot{c}\hat{i} = \frac{g}{3}\hat{i}$ //

18.3.18

Given $\underline{a}_A = a_A\hat{i}$



$\underline{a}_B = \underline{a}_A + \underline{a}_{B/A} = a_A\hat{i} + \ddot{\theta} l\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r$

Apply AMB about A

$\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$0\hat{k} = \underline{r}\times m\underline{a}_B$

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

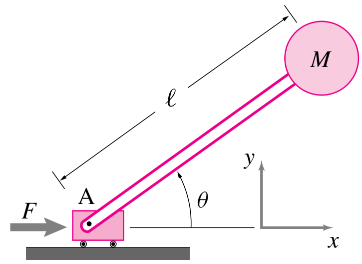
=> $l\hat{e}_r \times m(a_A\hat{i} + \ddot{\theta} l\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r) = \underline{0}$

=> $\{-a_A\sin\theta\,\hat{k} + \ddot{\theta} l\hat{k} = 0\}\cdot\hat{k}$

=> $l\ddot{\theta} = a_A\sin\theta$

(or) $\ddot{\theta} = \frac{a_A}{l}\sin\theta$

19.3.18 Due to the application of some unknown force F, the base of the pendulum A is accelerating with $\vec{\mathbf{a}}_A = a_A\hat{\imath}$. There is a frictionless hinge at A. The angle of the pendulum θ with the x axis and its rate of change $\dot{\theta}$ are assumed to be known. The length of the massless pendulum rod is ℓ. The mass of the pendulum bob M. There is no gravity. What is $\ddot{\theta}$? (Answer in terms of a_A, ℓ, M, θ and $\dot{\theta}$.)



Problem 19.18

18.3.17

Acceleration of the cart just after the mass is released.

$\underline{r}_p = \underline{r}_c + \underline{r}_{P/C} = c\,\hat{i} + l\,\hat{e}_r$

$\underline{v}_P = \dot{c}\,\hat{i} + l\,\dot{\hat{e}}_r = \dot{c}\,\hat{i} + l(\omega \underline{\hat{k}} \times \hat{e}_r) = \dot{c}\,\hat{i} + \omega l\,\hat{e}_\theta$

$\underline{a}_p = \ddot{c}\,\hat{i} + \omega l\,\dot{\hat{e}}_\theta + \dot{\omega} l\,\hat{e}_\theta = \ddot{c}\,\hat{i} + \dot{\omega} l\,\hat{e}_\theta - l\omega^2\,\hat{e}_r$

Applying AMB about C : $\Sigma \underline{M}_{/C} = \dot{\underline{H}}_{/C}$

$\Rightarrow \quad l\,\hat{e}_r \times m_p g(-\hat{j}) = l\,\hat{e}_r \times m\,\underline{a}_p + I\dot{\omega}\,\hat{\underline{k}}$

$\Rightarrow \quad \ddot{c}\cos\theta + l\dot{\omega} + g\sin\theta = 0 \quad$ —①

Applying LMB for the system :

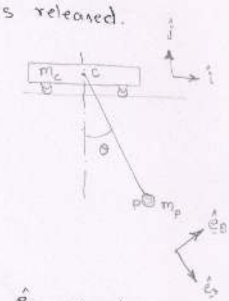
$\left\{ -m_c g\,\hat{j} - m_p g\,\hat{j} = m_c\ddot{c}\,\hat{i} + m_p[\ddot{c}\,\hat{i} + \dot{\omega} l\,\hat{e}_\theta - \omega^2 l\,\hat{e}_r] \right\} \cdot \hat{i}$

$\Rightarrow \quad 0 = m_c\ddot{c} + m_p\ddot{c} + m_p\dot{\omega} l\cos\theta - m_p\omega^2 l\sin\theta \quad$ —②

Solving for $\ddot{c}$ using ① and ②

$$\underline{a}_c = \ddot{c}\,\hat{i} = \frac{g\sin\theta\cos\theta + l\omega^2\tan\theta}{\left(\frac{m_c}{m_p}+1\right) - \cos^2\theta}\,\hat{i}$$

Check : $m_c = m_p$, $\theta = \pi/4$, $\ddot{c}\,\hat{i} = \frac{g}{3}\,\hat{i}$ //



$\hat{e}_r = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

$\hat{e}_\theta = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

18.3.18

Given $\underline{a}_A = a_A\,\hat{\underline{i}}$

$\underline{a}_B = \underline{a}_A + \underline{a}_{B/A} = a_A\,\hat{i} + \ddot{\theta} l\,\hat{e}_\theta - l\dot{\theta}^2\,\hat{e}_r$

Apply AMB about A

$\Sigma \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

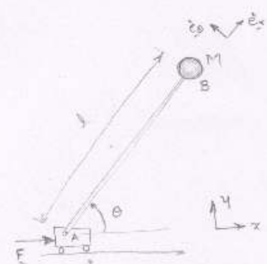
$0\,\hat{\underline{k}} = \underline{r} \times m\,\underline{a}_B$

$\Rightarrow \quad l\,\hat{e}_r \times m(a_A\,\hat{i} + \ddot{\theta} l\,\hat{e}_\theta - l\dot{\theta}^2\,\hat{e}_r) = \underline{0}$

$\Rightarrow \quad \{-a_A\sin\theta\,\hat{\underline{k}} + \ddot{\theta} l\,\hat{\underline{k}} = 0\} \cdot \hat{\underline{k}}$

$\Rightarrow \quad l\ddot{\theta} = a_A\sin\theta$

(or) $\quad \ddot{\theta} = \frac{a_A}{l}\sin\theta$

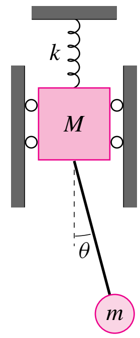


$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

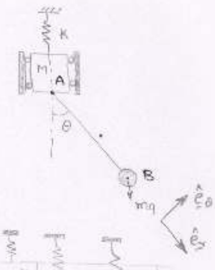
19.3.20 Using free-body diagrams and appropriate momentum balance equations, find differential equations that govern the angle θ and the vertical deflection y of the system shown. Be clear about your datum for y. Your equations should be in terms of θ, y and their time-derivatives, as well as M, m, ℓ, and g.



Problem 19.20

18.3.20 :

Equations of motion :



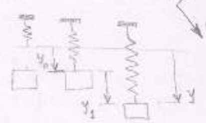
$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{a}_{B/A} = \ddot{y}(-\hat{j}) + \ddot{\theta}\hat{k} \times l\hat{e}_r + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times l\hat{e}_r)$

$\Rightarrow \underset{\sim}{a}_B = -\ddot{y}\hat{j} + \ddot{\theta} l \hat{e}_\theta - \dot{\theta}^2 l \hat{e}_r$

Apply AMB about A : $\Sigma \underset{\sim}{M}_{/A} = \underset{\sim}{\dot{H}}_{/A}$

$\Rightarrow \{ l\hat{e}_r \times mg(-\hat{j}) = l\hat{e}_r \times m\,\underset{\sim}{a}_B = \hat{e}_r \times -\ddot{y}\hat{j} + \ddot{\theta} l \hat{k} \} \cdot \hat{k}$

$\Rightarrow -g\sin\theta = -\ddot{y}\sin\theta + \ddot{\theta} l$



$\Rightarrow \ddot{\theta} = \frac{\ddot{y}_1}{l}\sin\theta - \frac{g}{l}\sin\theta$ —①

$y = y_0 + y_1$

$\dot{y} = \dot{y}_1$

$\ddot{y} = \ddot{y}_1$

Apply LMB for the system: $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

$\Rightarrow \{ Ky(+\hat{j}) - Mg\hat{j} - mg\hat{j} = M\ddot{y}(-\hat{j}) + m\ddot{y}(-\hat{j}) + m(\ddot{\theta} l \hat{e}_\theta - l\dot{\theta}^2 \hat{e}_r) \}$

$m\ddot{y} = mg - Ky$

$\{\}\cdot\hat{j}$ $\quad Ky_1 = (-(M+m)\ddot{y}_1) + ml\ddot{\theta}\sin\theta + ml\dot{\theta}^2\cos\theta$

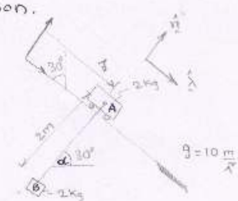
$= mg - Ky_0 - Ky_1$

$m\ddot{y}_1 = -Ky_1$

$\left(\frac{M}{m}+1\right)\ddot{y}_1 = -\frac{K}{m}y_1 + l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta$ —②

① and ② are the required equations of motion.

18.3.21 : Tension in the cable



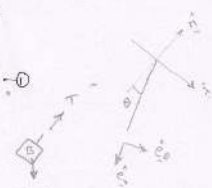
$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{a}_{B/A} = \ddot{s}\hat{\lambda} + l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r$

Applying LMB to the system: $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

$\{ m_B g(-\hat{j}) - m_A g\hat{j} = m_A\ddot{s}\hat{\lambda} + m_B\{\ddot{s}\hat{\lambda} + l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r\} - N\hat{\eta} \}$

$\{\}\cdot\hat{\lambda} \Rightarrow$

$(m_A+m_B)g\cos\alpha = (m_A+m_B)\ddot{s} + m_B l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$ —①



Applying LMB to the mass m_B, $\Sigma\underset{\sim}{F} = m\underset{\sim}{a}_B$

$\Rightarrow \{ m_B g(-\hat{j}) - T\hat{e}_r = m_B(\ddot{s}\hat{\lambda} + l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r) \}$

$\{\}\cdot\hat{e}_r \Rightarrow \quad T = m_B g\sin(\alpha+\theta) + m_B l\dot{\theta}^2 - m_B\ddot{s}\sin\theta$

$\hat{\lambda} = \sin\alpha\,\hat{i} - \cos\alpha\,\hat{j}$

a) $\theta = 0, \dot{\theta} = 0$ at $t = 0$ $\Rightarrow$ $T = m_B g\sin\alpha = 10\,N$

$\hat{\eta} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$

$\{\}\cdot\hat{e}_\theta \Rightarrow -m_B g(\hat{j}\cdot\hat{e}_\theta) = m_B\ddot{s}(\hat{\lambda}\cdot\hat{e}_\theta) + m_B l\ddot{\theta}$

$\hat{e}_r = -\cos(\alpha+\theta)\hat{i} - \sin(\alpha+\theta)\hat{j}$

$\hat{e}_\theta = \sin(\alpha+\theta)\hat{i} - \cos(\alpha+\theta)\hat{j}$

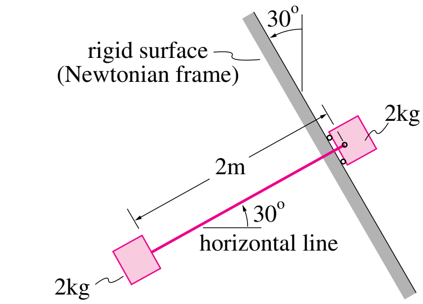
(or) $\quad \ddot{\theta} = \frac{g}{l}\cos(\alpha+\theta) - \frac{\ddot{s}}{l}\cos\theta$ —②

From ①+② $\quad \ddot{s} = g\cos\alpha - \frac{m_B l}{m_A+m_B}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$ —③

From ②+③ it is observed that even after 5 sec, tension remains same = 10N

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

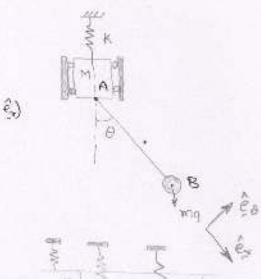
19.3.21 The two blocks shown are released from rest at $t = 0$. There is no friction and the cable is initially taut. (a) What is the tension in the cable immediately after release? (Use any reasonable value for the gravitational constant). (b) What is the tension after 5 s?



Problem 19.21

18.3.20 :

Equations of motion :

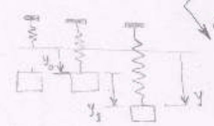


$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{a}_{B/A} = \ddot{y}(-\hat{j}) + \ddot{\theta}\hat{k} \times \ell\hat{e}_r + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times \ell\hat{e}_r)$$

$$\Rightarrow \underset{\sim}{a}_B = -\ddot{y}\hat{j} + \ddot{\theta}\ell\hat{e}_\theta - \dot{\theta}^2\ell\hat{e}_r$$

Apply AMB about A : $\sum \underset{\sim}{M}_{/A} = \dot{\underset{\sim}{H}}_{/A}$

$$\Rightarrow \left\{\ell\hat{e}_r \times mg(-\hat{j}) = \ell\hat{e}_r \times m\,\underset{\sim}{a}_B = \hat{e}_r \times -\ddot{y}\hat{j} + \ddot{\theta}\ell\hat{k}\right\}\cdot\hat{k}$$



$$\Rightarrow -g\sin\theta = -\ddot{y}\sin\theta + \ddot{\theta}\ell$$

$$\Rightarrow \ddot{\theta} = \frac{\ddot{y}_1}{\ell}\sin\theta - \frac{g}{\ell}\sin\theta \quad — ①$$

$y = y_0 + y_1$

$\dot{y} = \dot{y}_1$

$\ddot{y} = \ddot{y}_1$

Apply LMB for the system: $\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$$\Rightarrow \left\{Ky(+\hat{j}) - Mg\hat{j} - mg\hat{j} = M\ddot{y}(-\hat{j}) + m\ddot{y}(-\hat{j}) + m(\ddot{\theta}\ell\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r)\right\}$$

$m\ddot{y} = mg - Ky$

$= mg - Ky_0 - Ky_1$

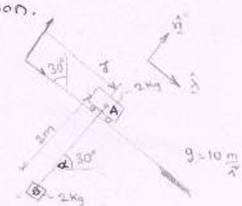
$m\ddot{y}_1 = -Ky_1$

$$\{\}\cdot\hat{j} \quad Ky_1 = (-(M+m)\ddot{y}_1) + m\ell\ddot{\theta}\sin\theta + m\ell\dot{\theta}^2\cos\theta$$

$$\left(\frac{M}{m} + 1\right)\ddot{y}_1 = -\frac{K}{m}y_1 + \ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta \quad — ②$$

① and ② are the required equations of motion.

18.3.21 : Tension in the cable



$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{a}_{B/A} = \ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r$$

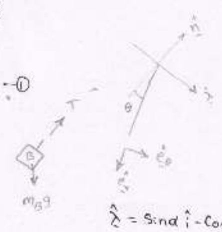
Applying LMB to the system: $\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$$\left\{m_B g(-\hat{j}) - m_A g\hat{j} = m_A\ddot{s}\hat{\lambda} + m_B[\ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r] - N\hat{n}\right\}$$

$$\{\}\cdot\hat{\lambda} \Rightarrow$$

$$(m_A+m_B)g\cos\alpha = (m_A+m_B)\ddot{s} + m_B\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) — ①$$

Applying LMB to the mass m_B, $\sum \underset{\sim}{F} = m\underset{\sim}{a}_B$



$$\Rightarrow \left\{m_B g(-\hat{j}) - T\hat{e}_r = m_B(\ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r)\right\}$$

$$\{\}\cdot\hat{e}_r \Rightarrow T = m_B g\sin(\alpha-\theta) + m_B\ell\dot{\theta}^2 - m_B\ddot{s}\sin\theta$$

a) $\theta = 0$, $\dot{\theta} = 0$ at $t = 0$ $\Rightarrow$ $T = m_B g\sin\alpha = 10\,\mathrm{N}$

$\hat{\lambda} = \sin\alpha\,\hat{i} - \cos\alpha\,\hat{j}$

$\hat{n} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$

$$\{\}\cdot\hat{e}_\theta \Rightarrow -m_B g(\hat{j}\cdot\hat{e}_\theta) = m_B\ddot{s}(\hat{\lambda}\cdot\hat{e}_\theta) + m_B\ell\ddot{\theta}$$

$\hat{e}_r = -\cos(\alpha+\theta)\hat{i} - \sin(\alpha+\theta)\hat{j}$

$\hat{e}_\theta = \sin(\alpha+\theta)\hat{i} - \cos(\alpha+\theta)\hat{j}$

(or) $$\ddot{\theta} = \frac{g}{\ell}\cos(\alpha+\theta) - \frac{\ddot{s}}{\ell}\cos\theta — ②$$

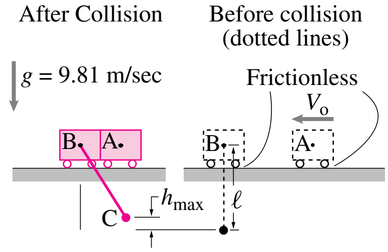
From ① & ②

$$\ddot{s} = g\cos\alpha - \frac{m_B\ell}{m_A+m_B}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) — ③$$

From ② & ③ it is observed that even after 5 sec, tension remains same = 10N

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

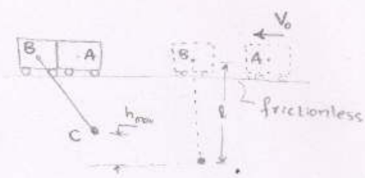
19.3.22 Carts A and B are free to move along a frictionless horizontal surface, and bob C is connected to cart B by a massless, inextensible cord of length ℓ, as shown in the figure. Cart A moves to the left at a constant speed $v_0 = 1\,\mathrm{m/s}$ and makes a perfectly plastic collision ($e = 0$) with cart B which, together with bob C, is stationary prior to impact. Find the maximum vertical position of bob C, h_{max}, after impact. The masses of the carts and pendulum bob are $m_A = m_B = m_C = 10\,\mathrm{kg}$.



Problem 19.22

18.3.22

Given, $V_o = 1\,m/s$, $e = 0$, $h_{max} = ?$



$\Rightarrow$ Final velocity is same for A and B.

At $h = h_{max}$, $\omega = 0 \Rightarrow V_c = V_B$.

Conserving linear momentum in $\hat{i}$ direction,

$$m_A V_o = (m_A + m_B) V_B + m_c V_c$$

$$\Rightarrow V_B = \frac{m_A}{m_A + m_B + m_c} V_o$$

Conservation of energy at two instants:

$$E_i = \frac{1}{2} m_A V_o^2$$

$$E_f = \left(\frac{1}{2}(m_A + m_B) + \frac{1}{2} m_c\right) V_B^2 + mgh$$

$$E_i = E_f$$

$$\Rightarrow \frac{1}{2} m_A V_o^2 = \frac{1}{2}\left[(m_A + m_B + m_c)\left(\frac{m_A}{m_A + m_B + m_c} V_o\right)^2\right] + mgh$$

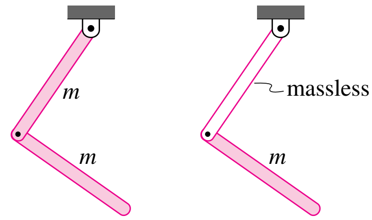
$$\Rightarrow m_A V_o^2 = \frac{m_A^2}{m_A + m_B + m_c} V_o^2 + 2mgh_{max}$$

$$\Rightarrow h_{max} = \frac{m_A (m_B + m_c) V_o^2}{m_c (m_A + m_B + m_c) 2g}$$

If $m_A = m_B = m_c = m = 10\,kg$, $V_o = 1\,m/s$

$$\Rightarrow h_{max} = \frac{V_o^2}{3g} = \frac{1^2}{3g} = 0.034\,m$$

19.3.23 A double pendulum is made of two uniform rigid rods, each of length ℓ. The first rod is massless. Find equations of motion for the second rod. Define any variables you use in your solution.



Problem 19.23

18.3.23: Obtain equations of motion:

Apply AMB about 'O': $\sum \underline{M}_{/O} = \dot{\underline{H}}_{/O}$

L.H.S: $\sum \underline{M}_{/O} = \underline{r}_{G_2/O} \times mg\hat{\imath} = \left[l_1\hat{\lambda}_1 + \frac{l_2}{2}\hat{\lambda}_2\right]\times mg\hat{\imath}$

$= mg\left[l_1\sin\phi_1 + \frac{l_2}{2}\sin\phi_2\right](-\hat{k})$

R.H.S: $\dot{\underline{H}}_{/O}$

$= \underline{r}_{G_2/O}\times m\,\underline{a}_{G_2} + I_2\ddot{\phi}_2\hat{k}$ $\quad\left[\underline{a}_{G_2} = \underline{a}_{A/O} + \underline{a}_{G_2/A}\right]$

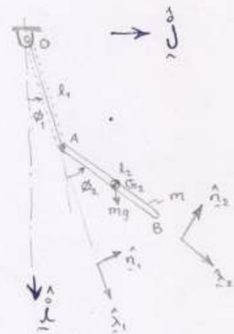
$= \left(l_1\hat{\lambda}_1 + \frac{l_2}{2}\hat{\lambda}_2\right)\times m\left[\left(\ddot{\phi}_1 l_1\hat{n}_1 - l_1\dot{\phi}_1^2\hat{\lambda}_1\right) + \left(\ddot{\phi}_2\frac{l_2}{2}\hat{n}_2 - \dot{\phi}_2^2\frac{l_2}{2}\hat{\lambda}_2\right)\right] + \frac{m l_2^2}{12}\ddot{\phi}_2\hat{k}$

$= m\Big[\ddot{\phi}_1 l_1^2\hat{k} - 0 + \ddot{\phi}_2\frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\hat{k} - \dot{\phi}_2^2\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1)\hat{k}$
$+ \ddot{\phi}_1\frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\hat{k} + \dot{\phi}_1^2\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1)\hat{k} + \ddot{\phi}_2\frac{l_2^2}{4}\hat{k}\Big] + \frac{m l_2^2}{12}\ddot{\phi}_2\hat{k}$

$= m\Big[\ddot{\phi}_1\left[l_1^2 + \frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\right] + \ddot{\phi}_2\left[\frac{l_2^2}{3} + \frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\right] +$
$\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1)(\dot{\phi}_1^2 - \dot{\phi}_2^2)\Big]\hat{k}$

$\left\{\sum \underline{M}_{/O} = \dot{\underline{H}}_{/O}\right\}\cdot\hat{k}$

$\Rightarrow \ddot{\phi}_1\left[l_1^2 + \frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\right] + \ddot{\phi}_2\left[\frac{l_2^2}{3} + \frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\right] +$
$\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1)(\dot{\phi}_1^2 - \dot{\phi}_2^2) = -g\left[l_1\sin\phi_1 + \frac{l_2}{2}\sin\phi_2\right]$ —(1)



$\hat{\lambda}_1 = \cos\phi_1\hat{\imath} + \sin\phi_1\hat{\jmath}$

$\hat{n}_1 = -\sin\phi_1\hat{\imath} + \cos\phi_1\hat{\jmath}$

$\hat{\lambda}_2 = \cos\phi_2\hat{\imath} + \sin\phi_2\hat{\jmath}$

$\hat{n}_2 = -\sin\phi_2\hat{\imath} + \cos\phi_2\hat{\jmath}$

$\hat{\lambda}_1\times\hat{n}_1 = \hat{k}$

$\hat{\lambda}_1\times\hat{\lambda}_2 = \sin(\phi_2-\phi_1)\hat{k}$

$\hat{\lambda}_1\times\hat{n}_2 = \cos(\phi_2-\phi_1)\hat{k}$

$\hat{\lambda}_2\times\hat{n}_2 = \hat{k}$

$\hat{\lambda}_2\times\hat{n}_1 = \cos(\phi_2-\phi_1)\hat{k}$

Apply AMB about 'A': $\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$\Rightarrow \underline{r}_{G_2/A}\times mg\hat{\imath} = \underline{r}_{G_2/A}\times m\,\underline{a}_{G_2/O} + I_2\ddot{\phi}_2\hat{k}$

$\Rightarrow \frac{l_2}{2}\hat{\lambda}_2\times mg\hat{\imath} = \frac{m l_2}{2}\hat{\lambda}_2\times\left[\ddot{\phi}_1 l_1\hat{n}_1 - l_1\dot{\phi}_1^2\hat{\lambda}_1 + \ddot{\phi}_2\frac{l_2}{2}\hat{n}_2 - \dot{\phi}_2^2\frac{l_2}{2}\hat{\lambda}_2\right] + \frac{m l_2^2}{12}\ddot{\phi}_2\hat{k}$

$\Rightarrow \left\{-\frac{mgl_2}{2}\sin\phi_2\hat{k} = m\ddot{\phi}_1\frac{l_1 l_2}{2}\cos(\phi_2-\phi_1)\hat{k} + m\frac{l_1 l_2}{2}\dot{\phi}_1^2\sin(\phi_2-\phi_1)\hat{k} + \frac{m l_2^2}{4}\ddot{\phi}_2\hat{k} + \frac{m l_2^2}{12}\ddot{\phi}_2\hat{k}\right\}$

$\{\ \}\cdot\hat{k} \Rightarrow \ddot{\phi}_1\frac{l_1 l_2}{2}\cos(\phi_2-\phi_1) + \ddot{\phi}_2\frac{l_2^2}{3} + \dot{\phi}_1^2\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1) = -\frac{g l_2}{2}\sin\phi_2$ —(2)

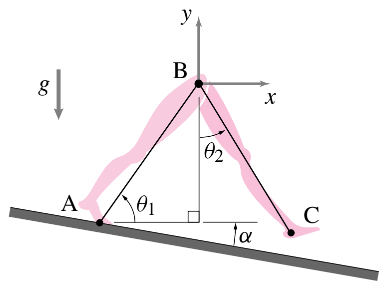
From (1) & (2)

$\ddot{\phi}_1 l_1^2 + \ddot{\phi}_2\frac{l_1 l_2}{2}\cos(\phi_2-\phi_1) - \dot{\phi}_2^2\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1) = -g l_1\sin\phi_2$

and $\ddot{\phi}_1\frac{l_1 l_2}{2}\cos(\phi_2-\phi_1) + \ddot{\phi}_2\frac{l_2^2}{3} + \dot{\phi}_1^2\frac{l_1 l_2}{2}\sin(\phi_2-\phi_1) = -\frac{g l_2}{2}\sin\phi_2$ are the governing differential equations of motion.

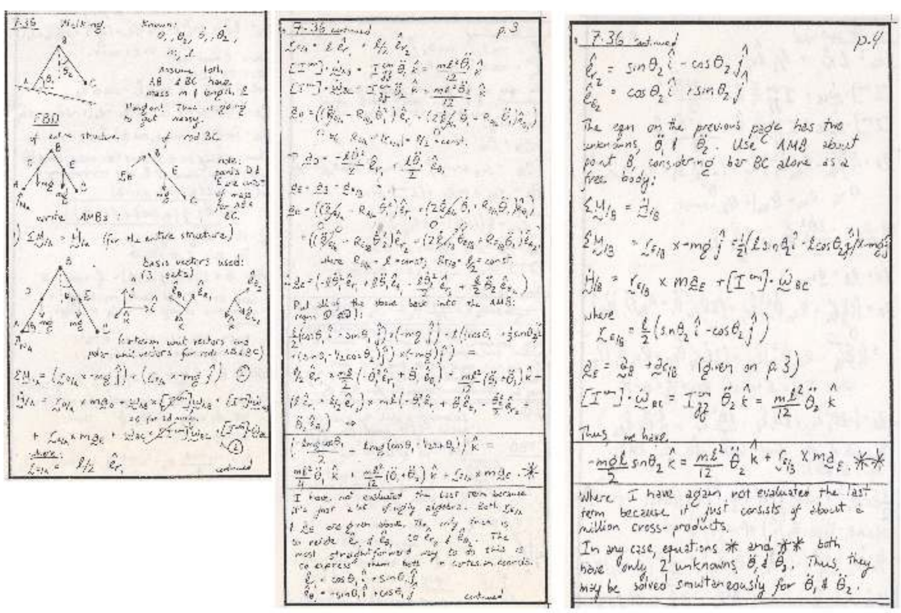
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.24 A model of walking involves two straight legs. During the part of the motion when one foot is on the ground, the system looks like the picture in the figure, confined to motion in the $x - y$ plane. Write two equations from which one could find $\ddot{\theta}_1$, and $\ddot{\theta}_2$ given $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ and all mass and length quantities.



Problem 19.24

$$\left[\begin{array}{lcll} \text{Hint:} & & & \\ \sum \vec{\boldsymbol{M}}_{/A} & = & \dot{\vec{\boldsymbol{H}}}_{/A} & \text{whole system} \\ \sum \vec{\boldsymbol{M}}_{/B} & = & \dot{\vec{\boldsymbol{H}}}_{/B} & \text{for bar BC} \end{array}\right]$$



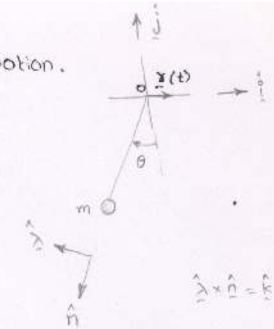
19.3.25 A pendulum is hanging from a moving support in the xy-plane. The support moves in a known way given by $\vec{r}(t) = r(t)\hat{\imath}$. For the following cases, find a differential equation whose solution would determine $\theta(t)$, measured clockwise from vertical; then find an expression for $d^2\theta/dt^2$ in terms of θ and $r(t)$:

a) with no gravity and assuming the pendulum rod is massless with a point mass of mass m at the end,

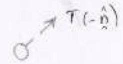
b) as above but with gravity,

c) assuming the pendulum is a uniform rod of mass m and length ℓ.

18.3.25 : Obtain differential equation of motion.



a) No gravity



Acceleration of the mass,

$$\underline{a} = \dot{\underline{\omega}}\times\underline{r} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + \ddot{\underline{r}}(t)$$
$$= -\ddot\theta\hat k\times l\hat n - \dot\theta^2 l\hat n + \ddot{\underline{r}}(t)$$
$$= l\ddot\theta\hat\lambda - l\dot\theta^2\hat n + \ddot r(t)\hat i$$

$\hat\lambda\times\hat n = \hat k$

$\hat\lambda = -\cos\theta\,\hat i + \sin\theta\,\hat j$

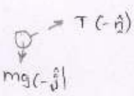
$\hat n = -\sin\theta\,\hat i - \cos\theta\,\hat j$

Applying LMB : $\sum \underline{F} = m\underline{a}$

$$\Rightarrow \left\{-T\hat n = m\left[l\ddot\theta\hat\lambda - l\dot\theta^2\hat n + \ddot r(t)\hat i\right]\right\}$$

$$\{\}\cdot\hat\lambda \Rightarrow m\left[\ddot\theta l + \ddot r(-\cos\theta)\right] = 0 \Rightarrow \ddot\theta = \frac{\ddot r(t)}{l}\cos\theta$$

b) With gravity



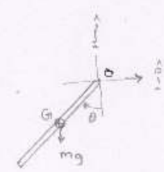
Applying LMB : $\sum \underline{F} = m\underline{a}$

$$\Rightarrow \left\{-T\hat n - mg\hat j = m\left[l\ddot\theta\hat\lambda - l\dot\theta^2\hat n + \ddot r(t)\hat i\right]\right\}$$

$$\{\}\cdot\hat\lambda \Rightarrow -mg\sin\theta = ml\ddot\theta - m\ddot r(t)\cos\theta$$

$$\Rightarrow \ddot\theta = \frac{\ddot r(t)}{l}\cos\theta - \frac{g}{l}\sin\theta$$

c) Uniform rod :



Applying AMB about 'O' : $\sum \underline{M}_{/O} = \dot{\underline{H}}_{/O}$

$$\Rightarrow \frac{l}{2}\hat n\times mg(-\hat j) = \underline{r}\times m\underline{a}_G + I^G\dot\omega\hat k$$
$$= \frac{l}{2}\hat n\times m\left[\frac{l}{2}\ddot\theta\hat\lambda - \frac{l}{2}\dot\theta^2\hat n + \ddot r(t)\hat i\right] + \frac{1}{12}ml^2(-\ddot\theta)\hat k$$

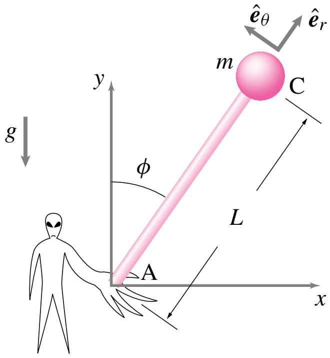
$$\Rightarrow -mg\frac{l}{2}(-\sin\theta) = \frac{ml^2}{4}\ddot\theta(-\hat k) + \frac{ml}{2}\ddot r(t)\cos\theta + \frac{1}{12}ml^2(-\ddot\theta)\hat k$$

$$\Rightarrow \ddot\theta = -\frac{3g}{2l}\sin\theta + \frac{3}{2l}\ddot r(t)\cos\theta$$

19.3.26 Robotics problem: balancing a broom stick by sideways motion. Try to balance a broom stick by moving your hand horizontally. Model your hand contact with the broom as a hinge. You can model the broom as a uniform stick or as a point mass at the end of a stick — your choice.

a) **Equation of motion.** Given the acceleration of your hand (horizontal only), the current tip, and the rate of tip of the broom, find the angular acceleration of the broom.

b) **Control?** Can you find a hand acceleration in terms of the tip and the tip rate that will make the broom balance upright?



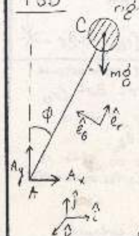
Problem 19.26

Answer:
(a) For point mass: $\ddot{\phi} = (g/L)\sin\phi - (a_{hand}/L)\cos\phi$.
(b) There are many correct solutions. Test your solution with a computer simulation. Show your result with appropriate plots.

7-44

Balancing a stick by sideways motion (I'll model the system as a point mass on a weightless rigid rod of length L).

FBD



Point A (the hand) has some (known) acceleration $\underset{\sim}{a}_A = a_A \hat{\imath}$.

Also known:

ϕ and $\underset{\sim}{\omega} = -\dot{\phi}\hat{k}$.

($\underset{\sim}{\omega} = -\dot{\phi}\hat{k}$ because increasing tipping (ϕ increasing, $\dot{\phi} > 0$) means the rod is tipping clockwise $\rightarrow \underset{\sim}{\omega} = -\dot{\phi}\hat{k}$ (check w/ R.H. Rule))

7-44 continued p. 5

a) Equation of motion: Take MMB about pt. A:

$\sum \underset{\sim}{M}_{/A} = \dot{\underset{\sim}{H}}_{/A}$ where:

$\sum \underset{\sim}{M}_{/A} = (\ell\sin\phi\,\hat{\imath} + \ell\cos\phi\,\hat{\jmath}) \times -mg\hat{\jmath} = -\ell mg\sin\phi\,\hat{k}$

$\dot{\underset{\sim}{H}}_{/A} = \underset{\sim}{r}_{C/A} \times m\underset{\sim}{a}_C + [I^{cm}]\dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times [I^{cm}]\underset{\sim}{\omega}$

$[I^{cm}] = \underset{\sim}{0}$ for point mass.

$\dot{\underset{\sim}{H}}_{/A} = \ell\hat{e}_r \times m\left((\ddot{R} - R\dot{\theta}^2)\hat{e}_r + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta + a_A\hat{\imath}\right)$

$\ddot{R} = 0$, $\dot{R} = 0$ b/c $R = \ell =$ const.

where I have used:

$\underset{\sim}{a}_C = \underset{\sim}{a}_A + \underset{\sim}{a}_{C/A}$;

noting that $\hat{\imath} = \sin\phi\,\hat{e}_r - \cos\phi\,\hat{e}_\theta$,

$\dot{\underset{\sim}{H}}_{/A} = \ell\hat{e}_r \times m(-\ell\dot{\phi}^2\hat{e}_r - \ell\ddot{\phi}\hat{e}_\theta + a_A(\sin\phi\,\hat{e}_r - \cos\phi\,\hat{e}_\theta))$

$\dot{\underset{\sim}{H}}_{/A} = (-m\ell^2\ddot{\phi} - m\ell a_A\cos\phi)\hat{k}$

Thus: $-\ell mg\sin\phi\,\hat{k} = (-m\ell^2\ddot{\phi} - m\ell a_A\cos\phi)\hat{k}$

Dotting the above w/ $\hat{k}$ and rearranging:

$m\ell^2\ddot{\phi} = -m\ell(a_A\cos\phi - g\sin\phi)$

$\Rightarrow \boxed{\ddot{\phi} = \frac{1}{\ell}(-a_A\cos\phi + g\sin\phi)}$

b) Here we go again.

Try $a_A = c_1\phi + c_2\dot{\phi}$

then $\ddot{\phi} + \frac{\cos\phi}{\ell}(c_1\phi + c_2\dot{\phi}) - \frac{g}{\ell}\sin\phi = 0$

Looks like it will work, resulting in convergence as long as c_1 & c_2 are large enough.

Let's take $\ell = 1\,\text{m}$, $g = 10\,\text{m/s}^2$, and try values of c_1 & c_2.

ODE: $\ddot{\phi} + \frac{c_2\cos\phi}{\ell}\dot{\phi} + \left(\frac{c_1\cos\phi}{\ell}\right)\phi - \frac{g}{\ell}\sin\phi = 0$

Reduce to first order system for MATLAB soln:

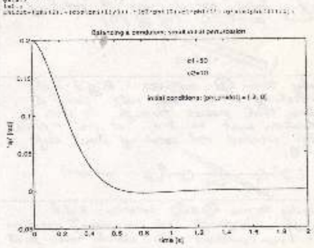
Let $\phi_1 = \phi$, let $\phi_2 = \dot{\phi}$. Then

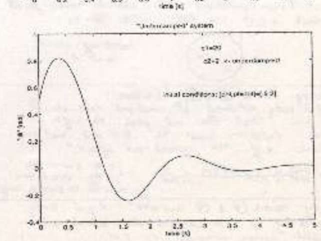
$\dot{\phi}_1 = \phi_2$

$\dot{\phi}_2 = -\frac{c_2\cos\phi_1}{\ell}\phi_2 - \frac{c_1\cos\phi_1}{\ell}\phi_1 + \frac{g}{\ell}\sin\phi_1$

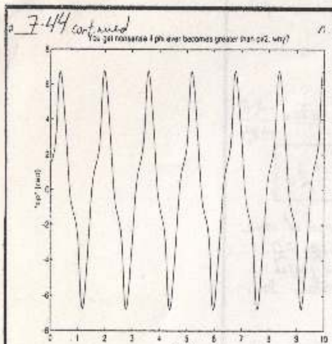
7-44 continued p. 6

Matlab soln.





7-44 continued p. 7



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.27 Balancing the broom again: vertical shaking works too. You can balance a broom by holding it at the bottom and applying appropriate torques, as in problem **??** or by moving your hand back and forth in an appropriate manner, as in problem 19.3.26. In this problem, you will try to balance the broom differently. The lesson to be learned here is more subtle, and you should probably just wonder at it rather than try to understand it in detail.

In the previous balancing schemes, you used knowledge of the state of the broom (θ and $\dot{\theta}$) to determine what corrective action to apply. Now balance the broom by moving it in a way that ignores what the broom is doing. In the language of robotics, what you have been doing before is 'closed-loop feedback' control. The new strategy, which is simultaneously more simple minded and more subtle, is an 'open loop' control. Imagine that your hand connection to the broom is a hinge.

a) **Picture and model.** Assume your hand oscillates sinusoidally up and down with some frequency and some amplitude. The broom is instantaneously at some angle from vertical. Draw a picture which defines all the variables you will use. Use any mass distribution that you like.

b) **FBD.** Draw a FBD of the broom.

c) **Momentum balance.** Write the equation of angular momentum balance about the point instantaneously coinciding with the hinge.

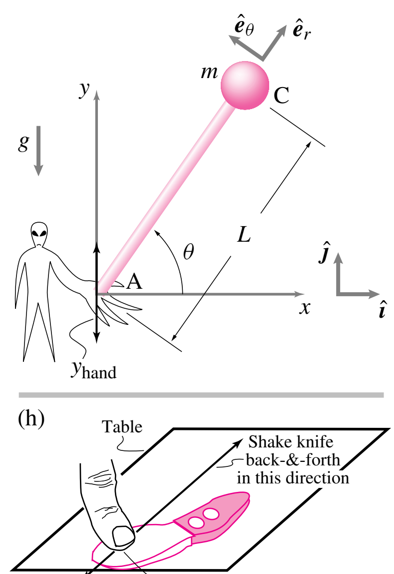
d) **Kinematics.** Use any geometry and kinematics that you need to evaluate the terms in the angular momentum balance equation in terms of the tip angle and its time derivatives and other known quantities (take the vertical motion of your hand as 'given'). [Hint: There are many approaches to this problem.]

e) **Equation of motion.** Using the angular momentum balance equation, write a governing differential equation for the tip angle.

f) **Simulation.** Taking the hand motion as given, simulate on the computer the system you have found.

g) **Stability?** Can you find an amplitude and frequency of shaking so that the broom stays upright if started from a near upright position? You probably *cannot* find linear equations to solve that will give you a control strategy. So, this problem might best be solved by guessing on the computer. Successful strategies require the hand acceleration to be quite a bit bigger than g, the gravitational constant. The stability obtained is like the stability of an undamped uninverted pendulum — oscillations persist. You improve the stability a little by including a little friction in the hinge.

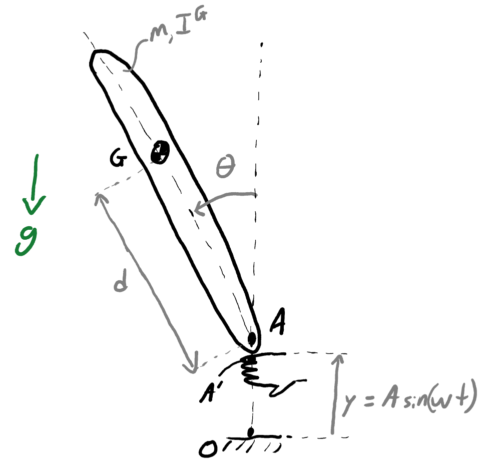
h) **Dinner table experiment for nerdy eaters.** If you put a table knife on a table and put your finger down on the tip of the blade, you can see that this experiment might work. Rapidly shake your hand back and forth, keeping the knife from sliding out from your finger but with the knife sliding rapidly on the table. Note that the knife aligns with the direction of shaking (use scratch-resistant surface). The knife is different from the broom in some important ways: there is no gravity trying to 'unalign' it and there is friction between the knife and table that is much more significant than the broom interaction with the air. Nonetheless, the experiment should convince you of the plausibility of the balancing mechanism. Because of the large accelerations required, you *cannot* do this experiment with a broom.

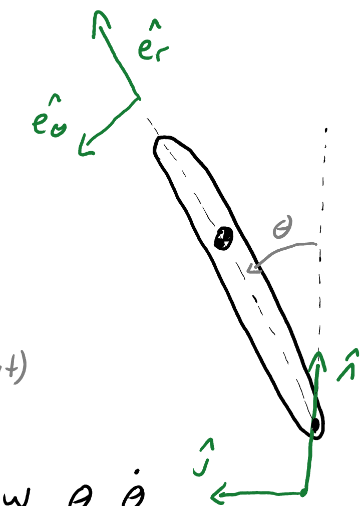


Problem 19.27: The sketch of the knife on the table goes with part (h).

Answer:
(e) $\ddot{\phi} - \left[\frac{\ddot{y}+g}{L}\right]\sin\phi = 0$, where $y(t)$ is the vertical displacement of your hand and $\phi(t)$ is the angle of the broom from the vertical.

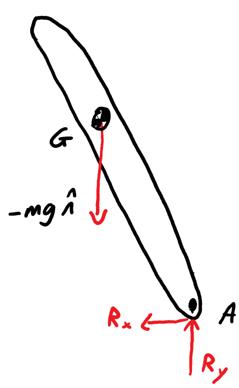
a)





Variables: $m,\ I^G,\ g,\ d,\ A,\ \omega,\ \theta,\ \dot\theta$

b) FBD:



$$\hat e_r = \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath$$
$$\hat e_\theta = -\sin\theta\,\hat\imath + \cos\theta\,\hat\jmath$$

c) AMB about point instantaneously coincident with hinge (A')

$$\sum \vec M_{/A'} = \dot{\vec H}_{/A'}$$

$$\boxed{\vec r_{G/A'} \times -mg\,\hat\imath = \vec r_{G/A'} \times m\vec a_{G/F} + I^G\ddot\theta\,\hat k} \quad (1)$$

d) Evaluate AMB, eqn c1) and kinematics

$$\vec{r}_{G/A'} = d\hat{e}_r$$

$$\vec{a}_{G/F} = \vec{a}_{G/A} + \vec{a}_{A/F}$$

$$\vec{a}_{G/A} = -d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\hat{e}_\theta$$

$$\vec{a}_{A/F} = \ddot{y}\hat{n} = -A\omega^2\sin(\omega t)\hat{n}$$

c1): $d\hat{e}_r \times -mg\hat{n} = d\hat{e}_r \times m\left(-d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\hat{e}_\theta - A\omega^2\sin(\omega t)\hat{n}\right) + I^G\ddot{\theta}\hat{k}$

$$\{\, dmg\sin\theta\,\hat{k} = md\left(d\ddot{\theta} + A\omega^2\sin(\omega t)\sin\theta\right)\hat{k} + I^G\ddot{\theta}\hat{k} \,\}$$

e) Write governing equation

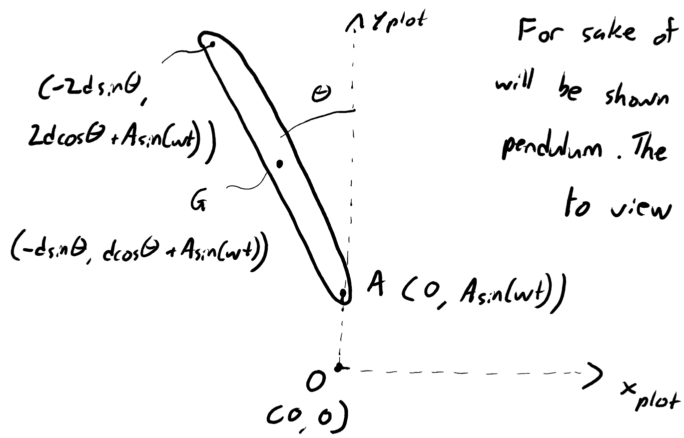
$\{\}\cdot\hat{k}$: $dmg\sin\theta = (md^2 + I^G)\ddot{\theta} + mdA\omega^2\sin(\omega t)\sin\theta$

$$\ddot{\theta} = \frac{dmg}{md^2 + I^G}\sin\theta - \frac{mdA\omega^2}{md^2 + I^G}\sin(\omega t)\sin\theta$$

$$\boxed{\ddot{\theta} = \underbrace{\frac{g - A\omega^2\sin(\omega t)}{d}}_{\text{Effective gravity due to acceleration of hinge}} \cdot \underbrace{\frac{1}{1 + \frac{I^G}{md^2}}}_{\text{Pendulum inertia term, as seen previously}} \cdot \sin\theta}$$

+ because inverted

f) Create simulation of inverted pendulum



For sake of plotting, point G will be shown half-way along the pendulum. The script can be run to view an animation

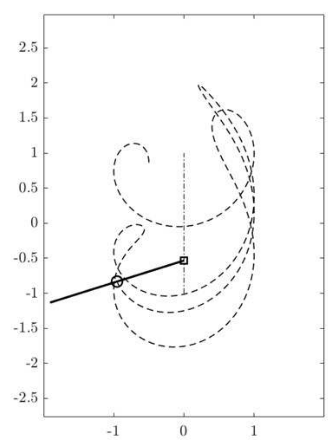
For the following parameters:

```
g = 10; m = 1; Ig = 1; d = 1; A = 1; w = 1; % arbitrarily selected
parameters
```

and the following initial conditions:

```
th0 = pi/6; w0 = 0;
```

this is the pendulum after 10 units of time. The solid line is the pendulum itself, the circle is point G, the square is point A, the thick dashed line is the path of point G, and the thin dashed line is the path of point A (sinusoidal vertical)



Pseudocode for MATLAB animations:

1) Obtain a "soln" object as the singular output of ode45. This struct contains information about the integrated time array (the *x* parameter) and the integrated state (each state variable is contained in each row of the *y* parameter).
2) Evaluate the state over a time array of your choosing (for example, you may choose to plot use an evenly space time domain, from *t0* to *t_end*). This can be done using ***deval(soln, t_plot, i)***, where *i* is the index of the variable in the state you are evaluating (ie. θ would be 1, and ω would be 2 in this problem). Other relevant arrays, like *x_G* and *y_G* can be evaluated as well.
3) Set up the initial plot window. You may choose to plot the *x_G* and *y_G* arrays you just computed to show the path of the points under your animation. Initialize the handles for each item you are animating (ie. the rod, each point), and store in variables.
4) Start a clock using *tic*. You may also define an *anim_speed*, which is the factor by which the animation speed is increased (or decreased) relative to the clock's time.
5) Set up the animation as a *while* loop, that continues until the criterion *toc < t_end/anim_speed* is satisfied. Each cycle in the loop, the animation time can be obtained as *t_cur = toc*anim_speed*, which polls and scales the clock time. Then, evaluate all relevant points using *deval(soln,t_cur,i)*.
6) Update the animation plot handles, by updating the *XData* and *YData* attributes of each handle. Once updated, use *drawnow* to refresh the plot axes.

These steps are enumerated in the attached code for reference.

g) Find a set of parameters for which the rod stabilizes in an upright position. Choose w sufficiently large so acceleration of point A >> gravity. I will do this using a "guess-and-check" method.

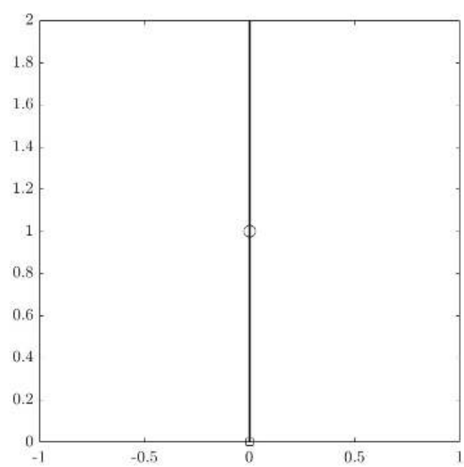
For the following parameters:

```
g = 0; m = 1; Ig = 1; d = 1; A = 0; w = 1; % no gravity or shaking
```

and the following initial conditions:

```
th0 = 0; w0 = 0;
```

this is the pendulum after 10 units of time. The solid line is the pendulum itself, the circle is point G, the square is point A.



As expected, the rod experiences no motion without gravity of shaking, and remains in its upright position.

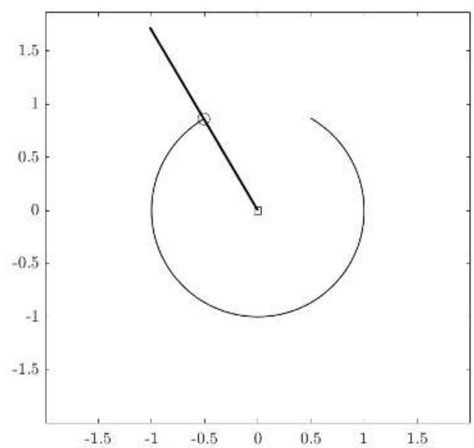
For the following parameters:

```
g = 10; m = 1; Ig = 1; d = 1; A = 0; w = 1; % no shaking only
```

and the following initial conditions:

```
th0 = pi/6; w0 = 0;
```

this is the pendulum after 10 units of time. Again, the solid line is the pendulum itself, the circle is point G, the square is point A. The thin black line now represents the motion of point G.



This acts like a "normal" pendulum acting under gravity, since the base is not undergoing any shaking.

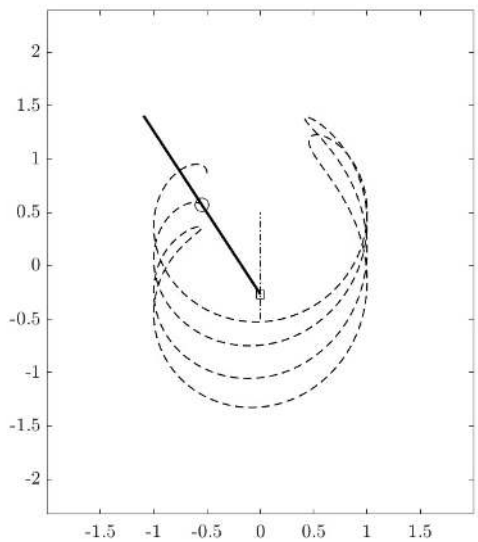
For the following parameters:

```
g = 10; m = 1; Ig = 1; d = 1; A = 0.5; w = 1; % large amplitude, low
frequency shaking
```

and the following initial conditions:

```
th0 = pi/6; w0 = 0;
```

this is the pendulum after 10 units of time. In addition to the previous lines, the thin dashed line is the path of point A.



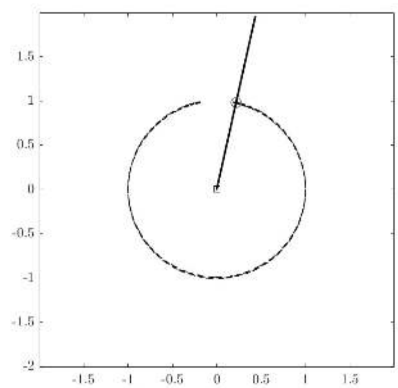
This acts largely like a normal pendulum under gravity, but with a time varying offset in the path due to the low frequency motion of point A.

We can then try increasing the frequency, and decreasing the amplitude of the oscillation. I will decrease the initial angular offset as well to more easily promote stabilization.

```
g = 10; m = 1; Ig = 1; d = 1; A = 0.01; w = 100; % higher frequency

th0 = pi/16; w0 = 0;
```

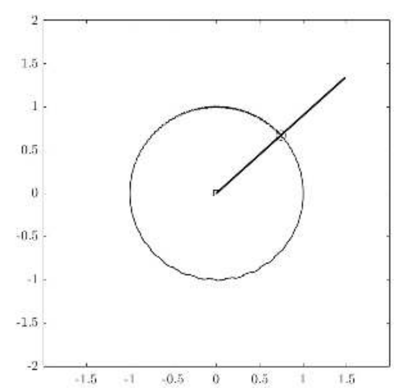
this is the pendulum after 10 units of time:



The rod still does not want to stabilize, so we can increase the frequency further.

```
g = 10; m = 1; Ig = 1; d = 1; A = 0.01; w = 500; % even higher
frequency

th0 = pi/16; w0 = 0;
```



Though the rod does not stabilize, it is seen that it now completes a complete rotation. This must mean that the frequency needed for stabilization is near.

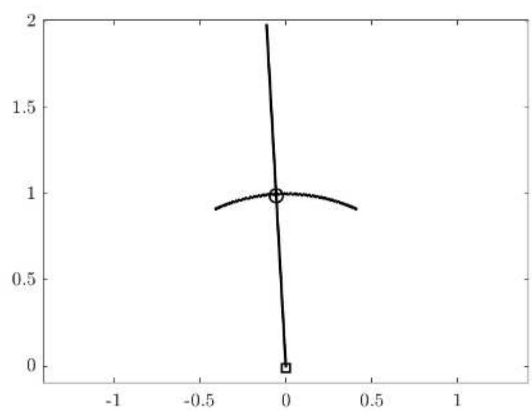
Lastly, for the following parameters:

```
g = 10; m = 1; Ig = 1; d = 1; A = 0.01; w = 1000;
```

and the following initial conditions:

```
th0 = pi/16; w0 = 0;
```

this is the pendulum after 10 units of time.



It can be seen that under these conditions, the rod remains upright, oscillating about θ=0! Stabilization therefore requires a high base oscillation frequency. The exact cutoff frequency needed could be obtained from a stability analysis.

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```
% % MAE 2030
% HW 14, Q75 part d Solution
% Solution by Michael Zakoworotny

clear all; clc; close all;
set(0,'DefaultAxesFontSize',14); % set plot font size

% Define parameters
% g = 0; m = 1; Ig = 1; d = 1; A = 1; w = 1; % arbitrarily selected parameters
% g = 0; m = 1; Ig = 1; d = 1; A = 0; w = 1; % no gravity or shaking
% g = 10; m = 1; Ig = 1; d = 1; A = 0; w = 1; % no shaking
% g = 10; m = 1; Ig = 1; d = 1; A = 0.5; w = 1; % large amplitude, low frequency ↙
shaking
% g = 10; m = 1; Ig = 1; d = 1; A = 0.01; w = 100; % higher frequency
% g = 10; m = 1; Ig = 1; d = 1; A = 0.01; w = 500; % even higher frequency
g = 10; m = 1; Ig = 1; d = 1; A = 0.01; w = 1000; % stable solution
p.g = g; p.m = m; p.Ig = Ig; p.d = d; p.A = A; p.w = w;

% Set initial conditions
th0 = pi/16; w0 = 0;
z0 = [th0, w0]';

% Time span
t0 = 0; tend = 10;
tspan = [t0, tend];

% 1) Solve with ode45
fun = @(t,z)myrhs(t,z,p);
tol = 1e-6;
opts = odeset('RelTol',tol,'AbsTol',tol);
soln = ode45(fun, tspan, z0, opts);

% 2) Get position vectors for point G over time of simulation
N = 1000;
t_ = linspace(t0, soln.x(end), N);
th_ = deval(soln, t_, 1);
y_A = A*sin(w*t_);
x_ = -d*sin(th_); % x coord to plot
y_ = d*cos(th_) + y_A; % y coord to plot

% 3) Set up plots
fig = figure; box on; hold on; axis equal;
% Scale window size to your computer screen
scrn = get(0,'ScreenSize');
set(gcf,'Position',[0.1*scrn(3), 0.1*scrn(4), 0.4*scrn(3), 0.7*scrn(4)]);
% Plot - path of point G, path of point A, the pendulum, point G itself,
% and point A. Store each plot handle in a variable
plot(x_, y_, '--k', 'LineWidth', 1);
plot([0,0],[min(y_A),max(y_A)], '-.k', 'LineWidth', 0.5);
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

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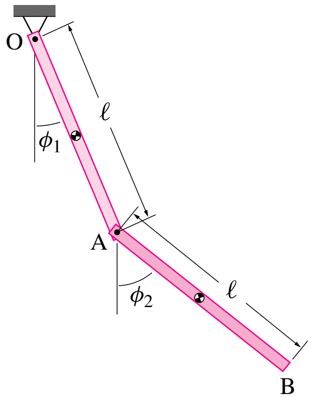
```
pend = plot([0, 2*x_(1)],[y_A(1), y_(1)+d*cos(th_(1))],'-k','LineWidth',2);
pt_G = plot(x_(1),y_(1),'ok','MarkerSize',10);
pt_A = plot(0,y_A(1),'sk','MarkerSize',8);
xlim([min(x_)-d,max(x_)+d]);
ylim([min(y_)-d,max(y_)+d]);

% 4) Start clock
anim_speed = 1; % animation speed
tic;
% 5) Create an animation by interpolating the angle theta at the given
% animation time (obtained using the tic and toc functions) in a loop
while toc < t_(end)/anim_speed
    % Scale current time by animation speed, and compute pos. of A and G
    t_cur = toc*anim_speed;
    th_cur = deval(soln,t_cur,1);
    y_A_cur = A*sin(w*t_cur);
    x_cur = -d*sin(th_cur);
    y_cur = d*cos(th_cur) + y_A_cur;

    % 6) Update pts in plot handles through the XData and YData attributes
    pend.XData = [0, 2*x_cur]; pend.YData = [y_A_cur, y_cur+d*cos(th_cur)];
    pt_G.XData = x_cur; pt_G.YData = y_cur;
    pt_A.YData = y_A_cur;

    drawnow;

    %Ends loop if figure is closed
    if ~ishghandle(fig)
        break
    end

end

% Return state derivative. Note that this in this problem, time is relevant
% in the solution
function zdot = myrhs(t,z,p)
    g = p.g; A = p.A; w = p.w; d = p.d; Ig = p.Ig; m = p.m;
    th = z(1);
    thdot = z(2);
    thddot = (g-A*w^2*sin(w*t))/d * 1/(1+Ig/(m*d^2)) * sin(th);

    zdot = [thdot, thddot]';
end
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.28 Double pendulum. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. The pendulum is released from rest at $\phi_1 = 0$ and $\phi_2 = \pi/2$. Just after release what are the values of $\ddot{\phi}_1$ and $\ddot{\phi}_2$? Answer in terms of other quantities.



Problem 19.28

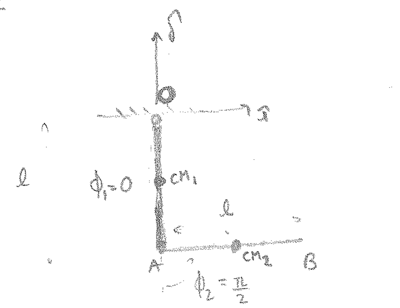
Answer:
$\ddot{\phi}_1 = 0,\ \ddot{\phi}_2 = -3g/2\ell.$

1

HW- 27

Q 16.67

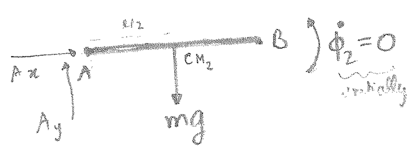
note $I_c = \frac{1}{12} m\ell^2$



initial configuration

rod 2 (just at t = 0)

FBD



$\dot{\phi}_2 = 0$ initially $\quad \ddot{\phi}_2 = ?$

by AMB about A (the moment of A_x, A_y will be 0)

$$\left(\frac{\ell}{2}\hat{\imath} \times -mg\hat{\jmath}\right) = \frac{1}{12}m\ell^2\,\ddot{\phi}_2\hat{k} + \frac{\ell}{2}\hat{\imath} \times (m\vec{a}_{CM_2}) \quad \text{— (A)}$$

now

$$\vec{a}_{CM_2} = \vec{a}_A - \dot{\phi}_2^{\,2}\,\vec{r}_{CM_2/A} + \ddot{\phi}_2\hat{k} \times \vec{r}_{CM_2/A} \quad \longrightarrow (1)$$

$$\vec{a}_A = -\dot{\phi}_1^{\,2}\,\vec{r}_{A/O} + \ddot{\phi}_1\hat{k} \times \vec{r}_{A/O} \quad \longrightarrow (2)$$

0 (initially)

$$\vec{a}_A = (\ddot{\phi}_1\hat{k}) \times (-\ell\hat{\jmath}) = \ddot{\phi}_1\ell\,\hat{\imath} \quad \longrightarrow (3)$$

$$\therefore \vec{a}_{CM_2} = \ddot{\phi}_1\ell + \ddot{\phi}_2\hat{k} \times \frac{\ell}{2}\hat{\imath} = \ddot{\phi}_1\ell\,\hat{\imath} + \ddot{\phi}_2\frac{\ell}{2}\hat{\jmath} \quad \longrightarrow (4)$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

plugging stuff into AMB/A for rod 2 (A)

$$-\frac{mg}{2} l \hat{k} = \frac{1}{12} m l^2 \ddot{\phi}_2 \hat{k} + \frac{ml}{2}\hat{\jmath} \times \left(\ddot{\phi}_1 l \hat{\imath} + \ddot{\phi}_2 \frac{l}{2}\hat{\jmath}\right)$$

$$= \frac{1}{12} m l^2 \ddot{\phi}_2 \hat{k} + \frac{ml^2}{4} \ddot{\phi}_2 \hat{k}$$

$$\therefore \{\}\cdot\hat{k} \qquad \ddot{\phi}_2 = -\frac{mg\frac{l}{2}}{\frac{ml^2}{4} + \frac{1}{12}ml^2} = -\frac{g}{l}\left(\frac{\frac{1}{2}}{\frac{1}{4}+\frac{1}{12}}\right)$$

$$\boxed{\ddot{\phi}_2 = -\frac{3}{2}\frac{g}{l}}$$

by LMB of rod 2

$$A_x\hat{\imath} + A_y\hat{\jmath} - mg\hat{\jmath} = m\left(\ddot{\phi}_1 l \hat{\imath} + \ddot{\phi}_2 \frac{l}{2}\hat{\jmath}\right)$$

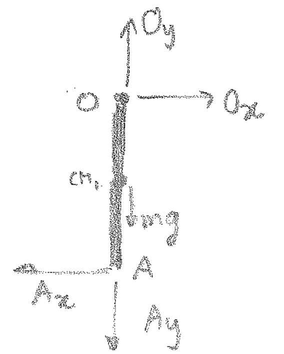
$$\therefore A_x = m\ddot{\phi}_1 l$$

$$A_y = mg + \frac{m}{2}\ddot{\phi}_2 l$$

AMB/O of rod 1

$-mg\hat{\jmath}$ and A_y give zero moment

FBD



$$\therefore \vec{r}_{A/O} \times -A_x\hat{\imath} = \frac{1}{12}ml^2\ddot{\phi}_1\hat{k} + \left(-\frac{l}{2}\hat{\jmath}\right)\times m(\vec{a}_{cm_1})$$

$$-lA_x\hat{k} = \frac{1}{12}ml^2\ddot{\phi}_1\hat{k} - \frac{l}{2}\hat{\jmath}\times m\left(-\dot{\phi}_1^2\vec{r}_{cm_1/O} + \ddot{\phi}_1\hat{k}\times\vec{r}_{cm_1/O}\right)$$

from above

0 initially

$$-l^2 m\ddot{\phi}_1\hat{k} = \frac{1}{12}ml^2\ddot{\phi}_1\hat{k} - \frac{l}{2}\hat{\jmath}\times m\left(\ddot{\phi}_1\frac{l}{2}\hat{\imath}\right)$$

$$-ml^2\ddot{\phi}_1\hat{k} = \frac{1}{12}ml^2\ddot{\phi}_1\hat{k} + \frac{ml^2}{4}\ddot{\phi}_1\hat{k} = \left(\frac{1}{12}+\frac{1}{4}\right)ml^2\ddot{\phi}_1\hat{k} = \frac{1}{3}ml^2\ddot{\phi}$$

$\{\}\cdot\hat{k}$

$$-ml^2\ddot{\phi}_1 = \frac{1}{3}ml^2\ddot{\phi}_1 \Rightarrow \ddot{\phi}_1\left(\frac{1}{3}ml^2 + ml^2\right) = 0$$

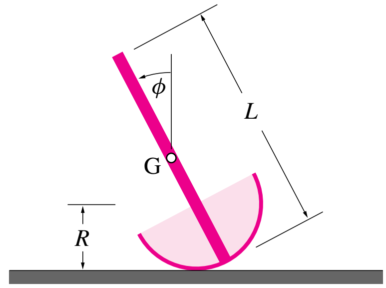
$$\Rightarrow \boxed{\ddot{\phi}_1 = 0}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.29 A rocker. A standing dummy is modeled as having massless rigid circular feet of radius R rigidly attached to their uniform rigid body of length L and mass m. The feet do not slip on the floor.

a) Given the tip angle ϕ, the tip rate $\dot{\phi}$ and the values of the various parameters (m, R, L, g) find $\ddot{\phi}$. [You may assume ϕ and $\dot{\phi}$ are small.]

b) Using the result of (a) or any other clear reasoning find the conditions on the parameters (m, R, L, g) that make vertical passive dynamic standing stable. [Stable means that if the person is slightly perturbed from vertically up that their resulting motion will be such that they remain nearly vertically up for all future time.]

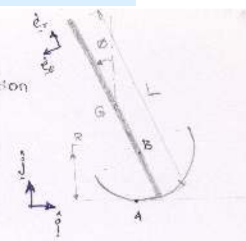


Problem 19.29

Answer:
(a) $\ddot{\phi} + \left[3(R - \frac{L}{2})\frac{g}{L^2}\right]\phi = 0$.
(b) This first-order approximation show that stable standing requires $R > \frac{L}{2}$. $R = \frac{L}{2}$ is neutrally stable (like a wheel). (Someone questions this answer, so don't trust it too much.)

18.3.29:

① Obtain equation of motion and condition for stable motion.



$$\underline{r}_{G/A} = \underline{r}_{B/A} + \underline{r}_{G/B}$$
$$= R\hat{j} + \left(\frac{L}{2} - R\right)\hat{e}_r$$

$$\hat{e}_r = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$
$$\hat{e}_\theta = -\cos\phi\,\hat{i} - \sin\phi\,\hat{j}$$

$$\underline{a}_{G/A} = (\dot{\omega}\hat{k} \times \underline{r}_{G/A}) + \omega\hat{k} \times (\omega\hat{k} \times \underline{r}_{G/A})$$
$$= \dot{\omega}\hat{k} \times \left(R\hat{j} + \left(\tfrac{L}{2} - R\right)\hat{e}_r\right) + \omega\hat{k} \times \left[\omega\hat{k} \times \left(R\hat{j} + \left(\tfrac{L}{2} - R\right)\hat{e}_r\right)\right]$$
$$= \dot{\omega}\hat{k} \times R\hat{j} + \dot{\omega}\left(\tfrac{L}{2} - R\right)(\hat{k} \times \hat{e}_r) + \omega\hat{k} \times \left[-\omega R\hat{i} + \omega\left(\tfrac{L}{2} - R\right)\hat{e}_\theta\right]$$
$$\Rightarrow \underline{a}_G = -\dot{\omega}R\hat{i} - \omega^2 R\hat{j} + \dot{\omega}\left(\tfrac{L}{2} - R\right)\hat{e}_\theta + \omega^2\left(\tfrac{L}{2} - R\right)\hat{e}_r$$

$$\begin{bmatrix}\omega = \dot{\phi}\\ \dot{\omega} = \ddot{\phi}\end{bmatrix}$$

Applying AMB about 'A': $\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$$\Rightarrow -\left(R\hat{j} + \left(\tfrac{L}{2} - R\right)\hat{e}_r\right) \times mg\hat{j} = \left\{R\hat{j} + \left(\tfrac{L}{2} - R\right)\hat{e}_r\right\} \times m\,\underline{a}_G + \frac{1}{12}mL^2\ddot{\phi}\hat{k}$$

$$\Rightarrow \Big\{ mg\left(\tfrac{L}{2} - R\right)\sin\phi\,\hat{k} = m\dot{\omega}R^2\hat{k} + m\dot{\omega}R\left(\tfrac{L}{2} - R\right)\cos\phi\,\hat{k} + \dot{\omega}R\left(\tfrac{L}{2} - R\right)m\cos\phi\,\hat{k} + m\dot{\omega}\left(\tfrac{L}{2} - R\right)^2\hat{k} + \frac{mL^2}{12}\ddot{\phi}\hat{k}\Big\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow mg\left(\tfrac{L}{2} - R\right)\sin\phi = 2m\ddot{\phi}R\left(\tfrac{L}{2} - R\right)\cos\phi + m\ddot{\phi}\left[\left(\tfrac{L}{2} - R\right)^2 + R^2 + \frac{L^2}{12}\right]$$

Assume, ϕ is small, $\Rightarrow \sin\phi = \phi$ and $\cos\phi = 1$

$$\Rightarrow mg\left(\tfrac{L}{2} - R\right)\phi = m\dot{\omega}\left\{2R\left(\tfrac{L}{2} - R\right) + R^2 + \left(\tfrac{L}{2} - R\right)^2 + \frac{L^2}{12}\right\} = m\dot{\omega}\left[\left(\tfrac{L}{2}\right)^2 + \left(\tfrac{L^2}{12}\right)\right] = m\dot{\omega}\frac{L^2}{3}$$

$$\Rightarrow \ddot{\phi} = \frac{3g}{L^2}\left(\frac{L}{2} - R\right)\phi$$

b) For the rocker to be stable, $\frac{3g}{L^2}\left(\frac{L}{2} - R\right) < 0$

$$\Rightarrow \left(R - \frac{L}{2}\right) > 0$$
$$\Rightarrow R > \frac{L}{2}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.3.30 Consider a rigid spoked wheel with no rim. Assume that when it rolls a spoke hits the ground and doesn't bounce. The body just swings around the contact point until the next spoke hits the ground. The uniform spokes have length R. Assume that the mass of the wheel is m, and that the polar moment of inertia about its center is I (use $I = mR^2/2$ if you want to get a better sense of the solution). Assume that just before collision number n, the angular velocity of the wheel is ω_{n-}, the kinetic energy is T_{n-}, the potential energy (you must clearly define your datum) is U_n-. Just after collision n the angular velocity of the wheel is ω_{n+}. The Kinetic Energy is T_{N+}, the potential energy (you must clearly define your datum) is U_{n+}. The wheel has k spokes (pick $k = 4$ if you have trouble with abstraction). This problem is not easy. It can be answered at a variety of levels. The deeper you get into it the more you will learn.

a) What is the relation between ω_{n-} and T_{n-}?

b) What is the relation between ω_{n-} and ω_{n+}?

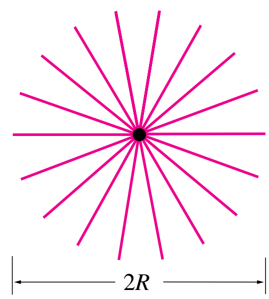
c) Assume 'rolling' on level ground. What is the relation between ω_n^+ and ω_{n+1}^+ (the angular velocities just after two successive collisions?

d) Assume rolling down hill at slope θ. What is the relation between ω_{n+} and ω_{n+1}?

e) Can it be true that $\omega_{n+} = \omega_{(n+1)+}$? About how fast is the wheel going in this situation?

f) As the number of spokes m goes to infinity, in what senses does this wheel become like an ordinary wheel?

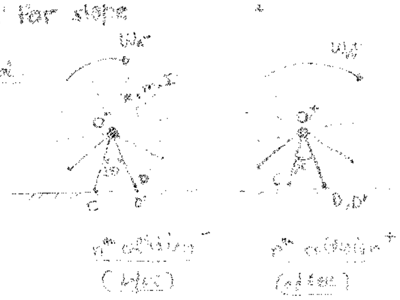
k evenly spaced spokes



Problem 19.30

NOTE I use 'α' for slope

16-69 rimless wheel



- In the figure D' is point on ground where spoke D will collide & stick

- Notice Impulse $\vec I$ is not perpendicular to floor

- Datum for P.E. we can choose to be ground.

- Angle bet spokes $2\theta = \frac{2\pi}{k}$

- ω is negative (clockwise)

FBD during collision

Impulse $= I_x\hat\imath + I_y\hat\jmath$

- LMB balance about D'

before

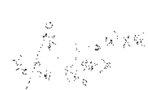
$$\vec H_{/D'}^{-} = I\vec\omega_n^- + \vec r_{G/D'} \times m\vec v_G^-$$

$$= I\vec\omega_n^- + \vec r_{G/D'} \times m(\vec\omega_n^- \times \vec r_{G/C})$$

$$= -I\omega_n^-\hat k + (-R\sin\theta\,\hat\imath + R\cos\theta\,\hat\jmath) \times m(-\omega_n^-\hat k \times (R\sin\theta\,\hat\imath + R\cos\theta\,\hat\jmath))$$

$$= -I\omega_n^-\hat k - mR^2\cos 2\theta\,\omega_n^-\hat k$$

$$= -(I + mR^2\cos 2\theta)\,\omega_n^-\hat k$$



after

$$\vec{H}^+_{/O'} = I\vec{\omega}^+_n\hat{k} + \vec{r}_{O/O'} \times m(\underbrace{\vec{\omega}^+_n \times \vec{r}_{O/O'}}_{\vec{v}_{O^+}})\hat{k}$$

= or simply $I_{O'}\vec{\omega}$

$$= (I + mR^2)\,\omega_n^+\,\hat{k}$$

by AMB/O' (because angular impulse about O' = 0)

$$\vec{H}^+ = \vec{H}^-$$

$$\therefore \boxed{\omega_n^+ = \frac{(I + mR^2\cos 2\theta)}{(I + mR^2)}\,\omega_n^-}$$

a) $T_{n^-} = \frac{1}{2} I_c\,\omega_n^-$ } C is a Hinge point b/c collision

$$\boxed{T_{n^-} = \frac{1}{2}(I + mR^2)\,\omega_n^-}$$

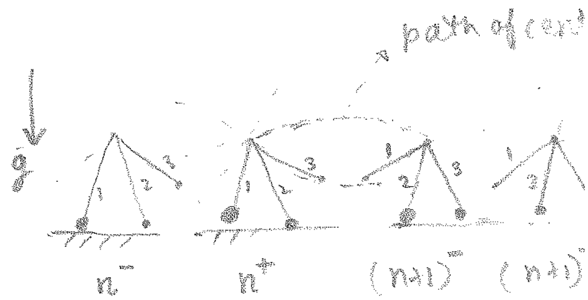
b)

$$\boxed{\omega_n^+ = \left(\frac{I + mR^2\cos 2\theta}{I + mR^2}\right)\omega_n^- = \left(1 - \frac{2\sin^2\theta}{\left(\frac{I}{mR^2} + 1\right)}\right)\omega_n^-}$$

using $\cos 2\theta = 1 - 2\sin^2\theta$

c) notice

$$T_{n^+} + U_{n^+} = T_{(n+1)^-} + U_{(n+1)^-}$$



- using energy conservation bet two collisions. (during collisions its not conserved)
- for level ground $U_{n^+} = U_{(n+1)^-}$ = because height of centre is same

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- $\therefore \quad T_{n^+} = T_{(n+1)^-}$

using part (a)

$$\omega_{n^+} = \omega_{(n+1)^-}$$

further using part (b)

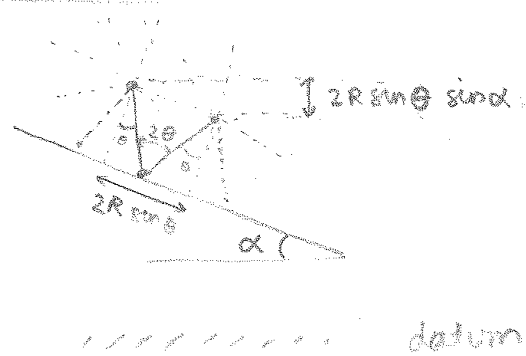
$$\omega_{n^+} = \omega_{(n+1)^-} = \frac{\omega_{(n+1)^+}}{\left(1 - \frac{2\sin^2\theta}{1+I/mR^2}\right)}$$

$$\therefore \boxed{\omega_{(n+1)^+} = \left(1 - \frac{2\sin^2\theta}{1+I/mR^2}\right)\omega_{n^+}}$$

d) again we can use energy cons. bet. @ n^+ and $(n+1)^-$ but now with datum at base of hill

$$U_{n^+} \neq U_{(n+1)^-}$$

$$\therefore T_{(n+1)^-} = T_{n^+} + U_{n^+} - U_{(n+1)^-}$$



$$T_{(n+1)^-} = T_{n^+} + mg(2R\sin\theta\sin\alpha)$$

using part (a)

$$\omega_{(n+1)^-} = \omega_{n^+} + \frac{mg(2R\sin\theta\sin\alpha)}{\frac{1}{2}(I+mR^2)}$$

using part (b)

$$\boxed{\omega_{(n+1)^+} = \left(1 - \frac{2\sin^2\theta}{1+I/mR^2}\right)\left(\omega_{n^+} + \frac{4mgR}{I+mR^2}\sin\theta\sin\alpha\right)}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

e) 1) from part c we can see [for level ground]

$$\omega_{(n+1)^+} = \omega_{n^+} \Rightarrow \text{only if } \frac{2\sin^2\theta}{1+\frac{I}{mR^2}} = 0$$

$$\Rightarrow \text{ie } \theta = 0, \pi, 2\pi$$

$\theta = 0$: when there are [illegible] no. of spokes

$\theta = \pi$: not possible, it will make the wheel like a rod (wheel won't roll any more

[for level ground]

$$\boxed{\omega^+_{(n+1)} = \omega^+_n \text{ if } \theta = 0}$$

ie number of spokes are ∞ → Valid for any ω

e) 2) from part d we see (for slope α)

$$\omega^+_{n+1} = \omega^+_n + \underbrace{\frac{4mgR\sin\theta\sin\alpha}{I+mR^2} - \frac{2\sin^2\theta}{1+I/mR^2}\omega^+_n - \frac{8mgR\sin^3\theta\sin\alpha}{(I+mR^2)\left(1+\frac{I}{mR^2}\right)}}_{\text{term II}}$$

$$\omega^+_{n+1} = \omega^+_n \text{ if term II} = 0$$

ie
$$\omega^+_n = \frac{\left(\frac{4mgR\sin\theta\sin\alpha}{I+mR^2}\right)\left(1 - \frac{2\sin^2\theta}{1+\frac{I}{mR^2}}\right)}{\left(\frac{2\sin^2\theta}{1+I/mR^2}\right)}$$

$$\boxed{\omega^+_n = \frac{4g\sin\theta\sin\alpha}{R\left(1+\frac{I}{mR^2}\right)}\left(\frac{1+I/mR^2}{2\sin^2\theta} - 1\right)}$$

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b) when spokes $\to \infty$

$\theta \to 0$

$\omega_n^+ \to \omega_n^-$

$\therefore$ wheel rolls with constant ω on ground

(accelerates on slope)

SIMPLIFICATIONS $I = \frac{1}{2} m R^2 \quad \therefore \quad \frac{I}{mR^2} = \frac{1}{2}$

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then,

- $\omega_n^+ = \left(1 - \frac{4}{3}\sin^2\theta\right)\omega_n^-$

and it will continue rolling ie $(\omega_{n+1}^+ = \omega_n^+)$

$\to$ never on level ground (unless $\theta = 0$ ie its a disc)

$\to$ on slope α if

$$\omega_n^+ = \frac{8}{3} g/R \, \sin\theta \sin\alpha \left(\frac{3}{4\sin^2\theta} - 1\right)$$

$$= g/R \, \sin\theta \sin\alpha \left(\frac{2}{\sin^2\theta} - \frac{8}{3}\right)$$

$$= \omega_{n+1}^+ = \omega_{n+m}^+$$

ie at start of slope we want $\omega = g/R \, \sin\theta \sin\alpha \left(\frac{2}{\sin^2\theta} - \frac{8}{3}\right)$

$\omega > 0$ otherwise it wont roll down

$\therefore \quad \frac{2}{\sin^2\theta} > \frac{8}{3}$ ie $\sin\theta < \frac{\sqrt{3}}{2}$ $\quad \boxed{\theta < 60}$

we need this otherwise it will stop after one rub on any slope

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