SAMPLE 19.15 Dynamics of sliding wedges. A wedge shaped body of mass m_2 sits on a frictionless ground. Another wedge shaped body of mass m_1 is gently placed on the inclined face of the stationary wedge. The top wedge starts to slide down. The coefficient of friction between the two wedges is μ. Find the sliding acceleration of the top wedge along the incline (*i.e.,* the relative acceleration of m_1 with respect to m_2).

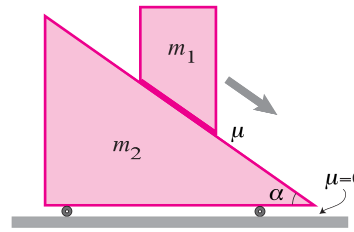

Figure 19.45

Solution The free-body diagrams of the two wedges are shown in fig. 19.46. Note that the friction force is μN since the wedges are sliding with respect to each other (if they were not sliding already then the friction force is an unknown force $F \leq \mu N$). Let the absolute acceleration of m_2 be $\vec{\boldsymbol{a}}_2 = a_2\hat{\boldsymbol{\imath}}$. Then, the absolute acceleration of m_1 is $\vec{\boldsymbol{a}}_1 = \vec{\boldsymbol{a}}_2 + \vec{\boldsymbol{a}}_{1/2} = a_2\hat{\boldsymbol{\imath}} + a_{\text{rel}}\hat{\boldsymbol{\lambda}}$. Now, we can write the linear momentum balance for m_1 and m_2 as follows.

$$\begin{aligned} N\hat{\boldsymbol{n}} - m_1 g\hat{\boldsymbol{\jmath}} - \mu N\hat{\boldsymbol{\lambda}} &= m_1(a_2\hat{\boldsymbol{\imath}} + a_{\text{rel}}\hat{\boldsymbol{\lambda}}) && (19.55)\\ (R - m_2 g)\hat{\boldsymbol{\jmath}} - N\hat{\boldsymbol{n}} + \mu N\hat{\boldsymbol{\lambda}} &= m_2 a_2\hat{\boldsymbol{\imath}} && (19.56)\end{aligned}$$

where $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\boldsymbol{\imath}} - \sin\alpha\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = \sin\alpha\hat{\boldsymbol{\imath}} + \cos\alpha\hat{\boldsymbol{\jmath}}$. Here, we have 4 independent scalar equations (from the two 2-D vector equations) in four unknowns N, R, a_2, and a_{rel}. Thus, we can certainly solve for them. We are, however, only interested in a_{rel}. So, we should try to find the answer with fewer calculations. Dotting eqn. (19.55) with $\hat{\boldsymbol{\lambda}}$, we have

$$m_1 a_{\text{rel}} = -m_1 a_2 \cos\alpha - \mu N + m_1 g \sin\alpha \qquad (19.57)$$

So, to find a_{rel}, we need a_2 and N. Dotting eqn. (19.55) with $\hat{\boldsymbol{n}}$, we have

$$m_1 a_2 \sin\alpha = N - m_1 g\cos\alpha, \qquad (19.58)$$

and dotting eqn. (19.56) with $\hat{\boldsymbol{\imath}}$, we have

$$m_2 a_2 = -N\sin\alpha + \mu N\cos\alpha. \qquad (19.59)$$

Solving eqn. (19.58) and (19.59) simultaneously, and using new variables (for convenience) $M = m_1/m_2$, $C = \cos\alpha$, and $S = \sin\alpha$, we get

$$a_2 = \frac{MC(\mu C - S)}{1 - MS(\mu C - S)}\, g, \quad N = m_1 g\,\frac{C}{1 - MS(\mu C - S)}$$

Substituting these expression in eqn. (19.57), we get

$$a_{\text{rel}} = gS - \frac{gC\big[MC(\mu C - S) - \mu\big]}{1 - MS(\mu C - S)}$$

$$a_{\text{rel}} = g\sin\alpha - \frac{g\cos\alpha\left[\frac{m_1}{m_2}\cos\alpha(\mu\cos\alpha - \sin\alpha) - \mu\right]}{1 - \frac{m_1}{m_2}\sin\alpha(\mu\cos\alpha - \sin\alpha)}$$

Note that when there is no friction ($\mu = 0$), the expression for a_{rel} reduces to

$$a_{\text{rel}} = gS + \frac{gMC^2S}{1 + MS^2}$$

and if we let $m_2 \to \infty$ (*i.e.,* m_2 represents fixed ramp) so that $M \to 0$, then $a_{\text{rel}} = g\sin\alpha$ which is the acceleration of a point mass down a frictionless ramp of slope $\tan\alpha$.

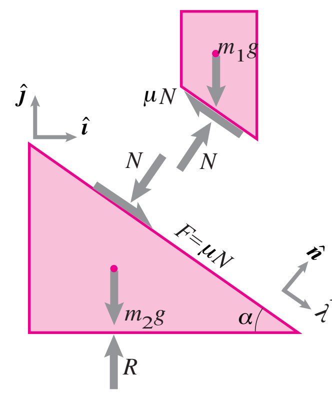

Figure 19.46

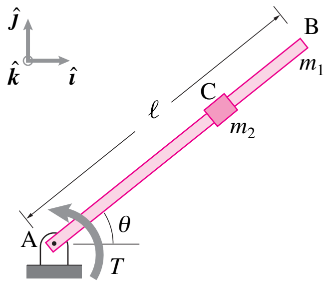

Figure 19.47

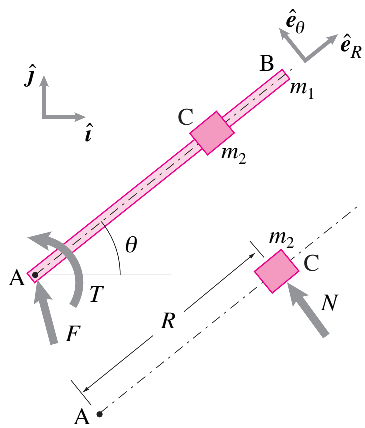

Figure 19.48

SAMPLE 19.16 Dynamics of a new gun. A new gun consists of a uniform rod AB of mass m_1 and a small collar C of mass m_2 that slides freely on the rod. A motor at A rotates the rod with constant torque T.

1. Find the equations of motion of the collar.
2. Show that if $T = 0$ then the equations of motion imply conservation of angular momentum about point A.

Solution

1. Let us denote the configuration of the collar with R, the radial distance from the fixed point A along the rod, and θ, the angular displacement of the rod. We need to find differential equations that determine R and θ as functions of time. The free-body diagram of the whole system (rod and collar together) and that of the collar is shown in fig. 19.48. We can write angular momentum balance for the whole system about point A so that the unknown reaction force F at A does not enter the equations. Noting that the acceleration of the collar is $\vec{a}_C = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, and letting $I_1 \equiv I_z^A z$ be the moment of inertia of the rod about A, we have,

$$\begin{aligned} \sum \vec{M}_A &= \dot{\vec{H}}_A \\ T\hat{k} &= I_1\ddot{\theta}\hat{k} + R\hat{e}_R \times m_2\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\right] \\ &= I_1\ddot{\theta}\hat{k} + m_2R^2\ddot{\theta}\hat{k} + 2m_2R\dot{R}\dot{\theta}\hat{k} \end{aligned}$$

Dotting this equation with $\hat{k}$, we have

$$\ddot{\theta} = \frac{T}{I_1 + m_2R^2} - \frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \qquad (19.60)$$

Thus we have obtained the equation of motion for θ. Now we consider the free-body diagram of the collar alone and write the linear momentum balance for it in the $\hat{e}_R$ direction, *i.e.*, $\hat{e}_R \cdot (\sum \vec{F} = m\vec{a})$, so that we do not have to care about the unknown normal reaction $\vec{N}$. So, we have,

$$\begin{aligned} 0 &= \hat{e}_R \cdot m_2\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\right] \\ &= m_2(\ddot{R} - R\dot{\theta}^2) \\ \Rightarrow \quad \ddot{R} &= R\dot{\theta}^2. \end{aligned} \qquad (19.61)$$

Thus we have the required equations of motion. Note that eqn. (19.60) and (19.61) are coupled nonlinear differential equations. So, to find $\theta(t)$ and $R(t)$ we need to solve them numerically.

$$\boxed{\ddot{\theta} = \frac{T}{I_1+m_2R^2} - \frac{2m_2R}{I_1+m_2R^2}\dot{R}\dot{\theta}, \qquad \ddot{R} = R\dot{\theta}^2}$$

2. Now we set $T = 0$ in our equations of motion. Note that the equation for R is independent of T. The equation for θ becomes

$$\ddot{\theta} = -\frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \quad \Rightarrow \quad (I_1 + m_2R^2)\ddot{\theta} + 2m_2R\dot{R}\dot{\theta} = 0$$

But the last expression is simply $\dot{H}_A$ for the system. Thus we have $\dot{H}_A = 0$ which implies that H_A = constant. That is conservation of angular momentum about point A.

SAMPLE 19.17 **Numerical solutions of new gun equations.** Consider Sample 19.16 again. Set up the equations of motion for numerical solution. Take $T = 1\,\mathrm{N{\cdot}m}$, $\ell = 1\,\mathrm{m}$, $m_2 = 1\,\mathrm{kg}$, and $m_1 = m_2/3$. Carry out numerical solutions for the following cases.

1. Let the system start from rest at $\theta = 0$ and $R = 0.1\,\mathrm{m}$. Find the solution from $t = 0$ to $t = 1\,\mathrm{s}$. Plot $R(t)$, $\theta(t)$ and $R(\theta)$ (in polar coordinates).
2. Find the solution till the collar leaves the rod. What is the speed of the collar at this instant?
3. Compute and plot the total energy of the system as a function of time. Also, plot the work done by the torque as a function of time and show that the work done is equal to the total energy of the system at each instant.
4. Vary torque T and carry out solutions for several values of T. Find the terminal value of θ_f (when the collar leaves the rod) for each T. Justify your observation about θ_f by plotting $\dot{R}/\dot{\theta}$ as a function of T.

Solution We first need to write the equations of motion, eqn. (19.60) and (19.61), as a set of first order ODEs. We can easily do so by introducing new variables $\omega \equiv \dot{\theta}$ and $v_R \equiv \dot{R}$, so that we have,

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{R} \\ \dot{v}_R \end{pmatrix} = \begin{pmatrix} \omega \\ \frac{T}{I_1 + m_2 R^2} - \frac{2 m_2 R}{I_1 + m_2 R^2} v_R \omega \\ v_R \\ R\omega^2 \end{pmatrix}$$

Given the values of all constants, we only need to specify the initial conditions for θ, ω, R, and v_R for solving these equations numerically.

1. We use the following pseudocode to carry out the numerical solution.

```
Set T = 1, L = 1, m2 = 1, m1 = m2/3
Let I1 = m1*L^2/3, I2 = m2*R^2,
ODEs = {thetadot = w,
        wdot = (T-2*m2*R*vR*w)/(I1+I2),
        Rdot = vR,
        vRdot = R*w^2}
IC = {theta = 0, w = 0, R = 0.1, vR = 0}
Solve ODEs with IC for t=0 to t=1
Plot t vs R, Plot t vs theta,
Polarplot theta vs R
```

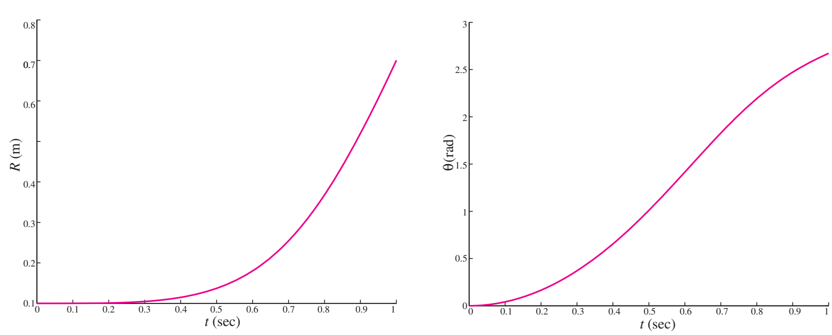

Figure 19.50:

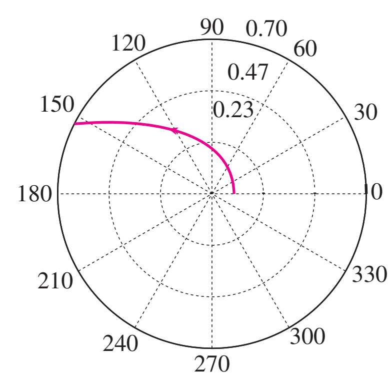

Figure 19.49

The $R(t)$ and $\theta(t)$ plots obtained from the numerical solution are shown in fig. 19.50 and the polar plot of $R(\theta)$ is shown in fig. 19.49.

2. We do not know a priori the value of t at which the collar leaves the rod. So, we have to carry out the solution for some assumed t_f which gives us $R(t_f) > \ell$ so that we know the collar has gone past the end of the rod. We then plot $R(t)$, including the unreal value of $R(t_f) > \ell$, and find the time t at which $R(t) = \ell$, either by zooming into the graph or by interpolation (although, there are various sophisticated algorithms to find this t). Following the method of zooming into the graph (see fig. 19.51) we find the terminal value of t to be 1.147 s. We carry out the numerical solution again from $t = 0$ to $t_f = 0.147$ s and find that $R(t_f) = 1$ m, $v_R(t_f) =$ 2.13 m/s, and $\omega(t_f) = 1.03$ rad/s, so that $v_f = \sqrt{\dot{R}^2 + (R\dot{\theta})^2} = 2.37$ m/s. This is the terminal speed of the collar.

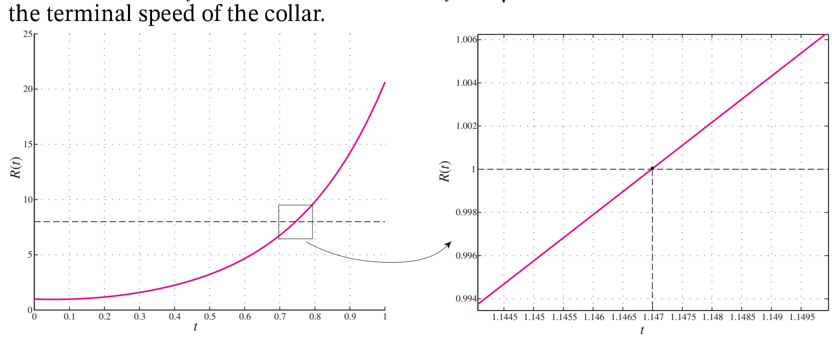

Figure 19.51: Finding the time t at which the collar leaves the rod from the graph of $R(t)$.

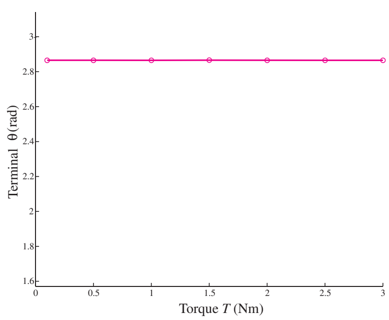

Figure 19.52: Angle θ at which the collar leaves the rod vs torque T.

3. The work done by the torque is $W = T\theta$ at any instant. The system only possesses kinetic energy. So the energy of the system at any instant is $E = E_1 + E_2$ where $E_1 = \frac{1}{2}I_1\omega^2$ and $E_2 = \frac{1}{2}m_2(\dot{R}^2 + R^2\omega^2)$. Computing these quantities for the solution obtained above, we plot W and E vs t as shown in fig. 19.53. Clearly, $W = E$.

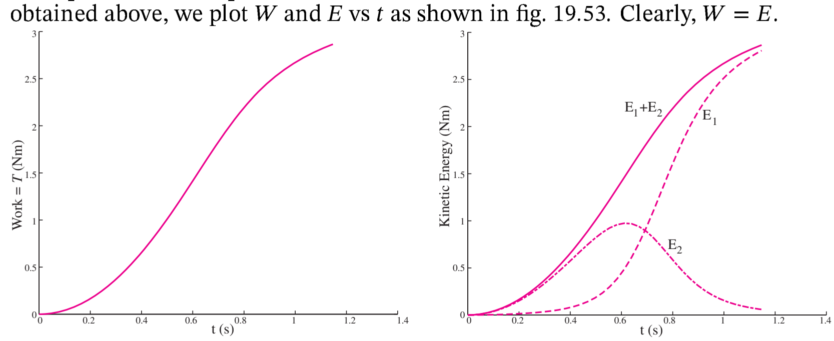

Figure 19.53: Work done by the torque and the kinetic energy of the system.

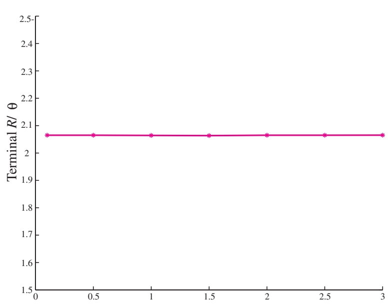

Figure 19.54: The ratio of terminal $\dot{R}/\dot{\theta}$ vs torque T.

4. Now we take several values of T (0.1, 0.5, 1, 1.5, 2, 2.5, and 3) and carry out the numerical solutions for each T. We note the terminal values of $\theta, \omega(= \dot{\theta})$, and $v_R(= \dot{R})$. By plotting the terminal value of θ against T (fig. 19.52), we see that the collar leaves the rod at exactly the same $\theta = 2.86$ rad for each T! But this is possible only if $\dot{R}$ and $\dot{\theta}$ both change in the same proportion for each T. So, plot the ratio $\dot{R}/\dot{\theta}$ just for the terminal values against T and find that the ratio is indeed constant (see fig. 19.54).

SAMPLE 19.18 Dynamics of a sliding-base pendulum. A cart of mass M slides down a frictionless inclined plane as shown in the figure. A simple pendulum of mass m and length ℓ hangs from the center-of-mass of the cart. Find the equation of motion of the pendulum.

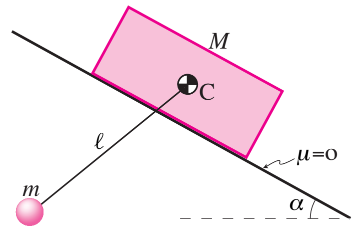

Figure 19.55

Solution Let us measure the angular displacement of the pendulum with respect to the cart with angle θ measured anticlockwise from the normal to the inclined plane. Let s be the position of the cart along the inclined plane from some reference point. Then, the acceleration of the cart can be written as $\vec{\boldsymbol{a}}_C = \ddot{s}\hat{\boldsymbol{\lambda}}$ and the acceleration of the pendulum mass as $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_C + \vec{\boldsymbol{a}}_{\text{rel}} = \ddot{s}\hat{\boldsymbol{\lambda}} + \ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R$.

The free-body diagram of the cart and the pendulum system is shown in fig. 19.56. Writing angular momentum balance of the system about point C, we get

$$\begin{aligned}
\vec{\boldsymbol{M}}_C &= \dot{\vec{\boldsymbol{H}}}_C \\
\ell\hat{\boldsymbol{e}}_R \times (-mg\hat{\boldsymbol{\jmath}}) &= \ell\hat{\boldsymbol{e}}_R \times m(\ddot{s}\hat{\boldsymbol{\lambda}} + \ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R) \\
-mgl\sin(\theta-\alpha)\hat{\boldsymbol{k}} &= m\ddot{s}\ell\cos\theta\hat{\boldsymbol{k}} + m\ell^2\ddot{\theta}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \ddot{\theta} &= -\frac{g}{\ell}\sin(\theta-\alpha) - \frac{\ddot{s}}{\ell}\cos\theta. \qquad (19.62)
\end{aligned}$$

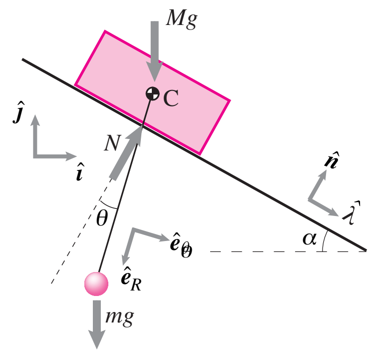

Figure 19.56

To find $\ddot{s}$, we write the linear momentum balance for the whole system in the $\hat{\boldsymbol{\lambda}}$ (so that we do not involve the unknown normal reaction N) direction.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot (-Mg\hat{\boldsymbol{\jmath}} - mg\hat{\boldsymbol{\jmath}}) &= \hat{\boldsymbol{\lambda}} \cdot \left[M\ddot{s}\hat{\boldsymbol{\lambda}} + m(\ddot{s}\hat{\boldsymbol{\lambda}} + \ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R)\right] \\
\Rightarrow \quad (M+m)g\sin\sin\alpha &= (M+m)\ddot{s} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta \\
\Rightarrow \quad \ddot{s} &= g\sin\alpha - \frac{m\ell}{M+m}(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta)
\end{aligned}$$

(19.63)

Substituting eqn. (19.63) in eqn. (19.62) and rearranging terms, we get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta\frac{(1+\frac{m}{M})\cos\alpha}{1+\frac{m}{M}\sin^2\theta} + \frac{\frac{m}{M}\dot{\theta}^2\sin\theta\cos\theta}{1+\frac{m}{M}\sin^2\theta}$$

$$\boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta\frac{(1+\frac{m}{M})\cos\alpha}{1+\frac{m}{M}\sin^2\theta} + \frac{\frac{m}{M}\dot{\theta}^2\sin\theta\cos\theta}{1+\frac{m}{M}\sin^2\theta}}$$

Note that if we set $\alpha = 0$ and let $M \to \infty$ so that the cart behaves like a fixed ground, then we recover the equation of simple pendulum, $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, from the equation of motion above. It is a good practice to carry out such simple checks wherever possible.

Remarks: We could write the equations of motion, eqn. (19.62) and eqn. (19.63) in the coupled form as

$$\begin{bmatrix} \ell & \cos\theta \\ m\ell\cos\theta & M+m \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} -\frac{g}{\ell}\sin(\theta-\alpha) \\ (M+m)g\sin\alpha - m\ell\dot{\theta}^2\sin\theta \end{pmatrix}$$

and leave it at that, since for most computational purposes, it is enough. It is not so hard to find expressions for $\ddot{\theta}$ and $\ddot{s}$ from here by solving the matrix equation, even by hand.

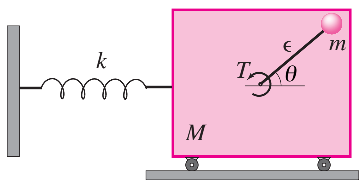

Figure 19.57

SAMPLE 19.19 **Resonant capture.** A slightly unbalanced motor mounted on an elastic machine part is modeled as a spring mass system with a simple pendulum of mass m and length ϵ driven by a constant torque T as shown in the figure. The spring has stiffness k and the motor has mass M. There is no friction between mass M and the horizontal surface.

1. Find the equation of motion of the system.
2. Take $M = m = 1\,\text{kg}$, $k = 1\,\text{N/m}$, $T = 15 \times 10^{-3}\,\text{N·m}$. Solve (numerically) the equations of motion with zero initial conditions and plot $x(t)$ and $\theta(t)$ for $t = 0$ to $100\,\text{s}$.

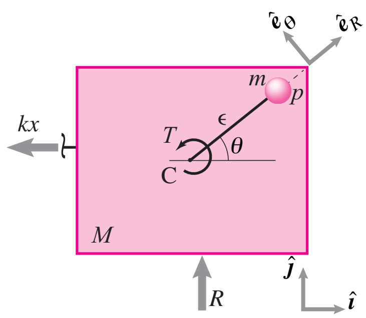

Figure 19.58

Solution

1. The free-body diagram of the system is shown in fig. 19.58. Let the angular displacement of the eccentric mass m at some instant t be θ. At the same instant, let the displacement of the motor be x from the relaxed state of the spring. Then we can write the acceleration of the motor as $\ddot{x}\hat{\imath}$ and that of the eccentric mass as $\vec{a}_P = \ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R$. Now, we can write the angular momentum balance for the system about point C (fixed in the stationary frame of reference but instantly coincident with the center-of-mass of motor M) as

$$
\begin{aligned}
T\hat{k} &= \epsilon\hat{e}_R \times m(\ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R) \\
&= m\epsilon\ddot{x}(\hat{e}_R \times \hat{\imath}) + m\epsilon^2\ddot{\theta}(\hat{e}_R \times \hat{e}_\theta) \\
&= m\epsilon(-\ddot{x}\sin\theta + \epsilon\ddot{\theta})\hat{k} \\
\Rightarrow \quad \epsilon\ddot{\theta} - \sin\theta\,\ddot{x} &= \frac{T}{m\epsilon}
\end{aligned}
\tag{19.64}
$$

This is just one scalar equation in $\ddot{\theta}$ and $\ddot{x}$. We need one more independent equation $\ddot{\theta}$ and $\ddot{x}$ without involving any other unknowns. So, we write the linear momentum balance for the system in the x-direction:

$$
\begin{aligned}
-kx &= M\ddot{x} + m(\ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R)\cdot\hat{\imath} \\
&= (M+m)\ddot{x} - m\epsilon\ddot{\theta}\sin\theta - m\epsilon\dot{\theta}^2\cos\theta \\
\Rightarrow \quad \epsilon\sin\theta\,\ddot{\theta} - \left(\frac{M+m}{m}\right)\ddot{x} &= \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta.
\end{aligned}
\tag{19.65}
$$

Thus we have the required equations of motion. We can write eqn. (19.64) and (19.65) compactly as

$$
\begin{bmatrix} \epsilon & -\sin\theta \\ \epsilon\sin\theta & -\frac{m+M}{M} \end{bmatrix}
\begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix}
=
\begin{pmatrix} \frac{T}{m\epsilon} \\ \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta. \end{pmatrix}
$$

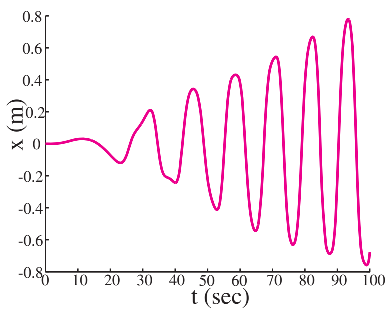

Figure 19.59

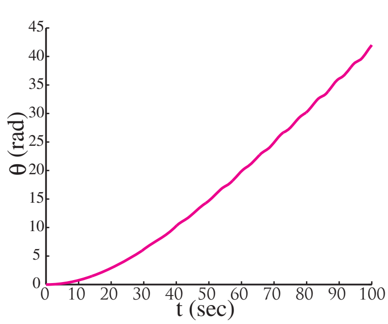

Figure 19.60

2. We use the following pseudocode to solve the equations of motion. Note that we first convert the two second order ODEs into four first order ODEs by introducing new variables $\omega = \dot{\theta}$ and $u = \dot{x}$.

```
Set T = 0.015, m = 1, M = 1, k = 1, e = 1
A=[e -sin(theta); e*sin(theta) -(1+M/m)];
b = [T/(m*e); k/m*x-e*omega^2*cos(theta)];
solve A*acln = b for acln  % acln = accelerations
ODEs = {omega = thetadot, u  = xdot,
        omegadot = acln(1), udot = acln(2)}
IC   = {theta = 0, x = 0, omega = 0, u = 0}
Solve ODEs with IC for t=0 to t=100
```

The plots of $x(t)$ and $\theta(t)$ obtained from the numerical solution are shown in fig. 19.59 and fig. 19.60, respectively. Note the resonance of M for the given values of the system.

SAMPLE 19.20 Dynamics using a rotating and translating coordinate system. Consider the rotating wheel of Sample 18.11 which is shown here again in Figure 19.61. At the instant shown in the figure find

1. the linear momentum of the mass P and
2. the net force on the mass P.

For calculations, use a frame $\mathcal{B}$ attached to the rod and a coordinate system in $\mathcal{B}$ with origin at point A of the rod OA.

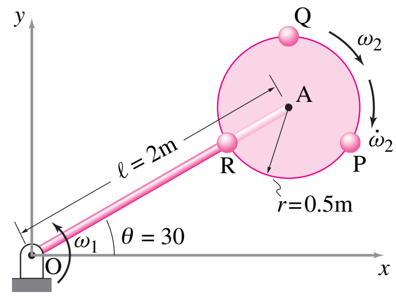

Figure 19.61

Solution We attach a frame $\mathcal{B}$ to the rod. We choose a coordinate system $x'y'z'$ in this frame with its origin O′ at point A. We also choose the orientation of the primed coordinate system to be parallel to the fixed coordinate system xyz (see Fig. 19.62), *i.e.*, $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}$.

1. **Linear momentum of P:** The linear momentum of the mass P is given by

$$\vec{\boldsymbol{L}} = m\vec{\boldsymbol{v}}_P.$$

Clearly, we need to calculate the velocity of point P to find $\vec{\boldsymbol{L}}$. Now,

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}} = \underbrace{\vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{v}}_{P'/O'}}_{\vec{\boldsymbol{v}}_{P'}} + \vec{\boldsymbol{v}}_{\text{rel}}.$$

Note that O′ and P′ are two points on the same (imaginary) rigid body OAP′. Therefore, we can find $\vec{\boldsymbol{v}}_{P'}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{P'} &= \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{O'/O}}^{\vec{\boldsymbol{v}}_{O'}} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'}}^{\vec{\boldsymbol{v}}_{P'/O'}} \\
&= \omega_1 \hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + \omega_1 \hat{\boldsymbol{k}} \times r(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= \omega_1[(\ell + r)\cos\theta\hat{\boldsymbol{\jmath}} - (\ell - r)\sin\theta\hat{\boldsymbol{\imath}}] \\
&= 3\,\text{rad/s} \cdot [2.5\,\text{m} \cdot \cos 30^\circ \hat{\boldsymbol{\jmath}} - 1.5\,\text{m} \cdot \sin 30^\circ \hat{\boldsymbol{\imath}}] \\
&= (6.50\hat{\boldsymbol{\jmath}} - 2.25\hat{\boldsymbol{\imath}})\,\text{m/s} \quad \text{(same as in Sample 18.11.),}
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\text{rel}} &= \vec{\boldsymbol{v}}_{P/\mathcal{B}} \\
&= -\omega_2 \hat{\boldsymbol{k}}' \times r(\cos\theta\hat{\boldsymbol{\imath}}' - \sin\theta\hat{\boldsymbol{\jmath}}') \\
&= -\omega_2 r(\cos\theta\hat{\boldsymbol{\jmath}}' + \sin\theta\hat{\boldsymbol{\imath}}') \\
&= -(2.16\hat{\boldsymbol{\jmath}}' + 1.25\hat{\boldsymbol{\imath}}')\,\text{m/s} \\
&= -(2.16\hat{\boldsymbol{\jmath}} + 1.25\hat{\boldsymbol{\imath}})\,\text{m/s.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{v}}_P &= \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}} \\
&= (4.34\hat{\boldsymbol{\jmath}} - 3.50\hat{\boldsymbol{\imath}})\,\text{m/s} \qquad \text{and} \\
\vec{\boldsymbol{L}} &= m\vec{\boldsymbol{v}}_P \\
&= 0.5\,\text{kg} \cdot (4.34\hat{\boldsymbol{\jmath}} - 3.50\hat{\boldsymbol{\imath}})\,\text{m/s} \\
&= (-1.75\hat{\boldsymbol{\imath}} + 2.17\hat{\boldsymbol{\jmath}})\,\text{kg·m/s.}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{L}} = (-1.75\hat{\boldsymbol{\imath}} + 2.17\hat{\boldsymbol{\jmath}})\,\text{kg·m/s}}$$

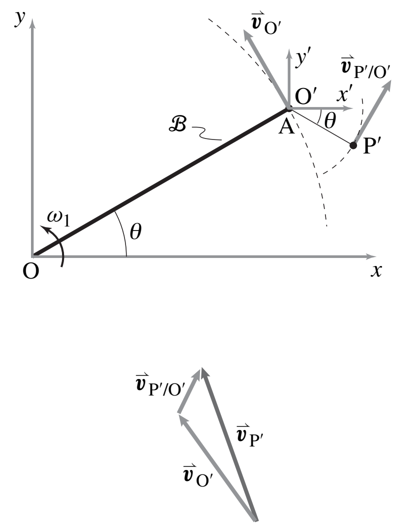

Figure 19.62: The velocity of point P' is the sum of two terms: the velocity of O' and the velocity of P' relative to O'.

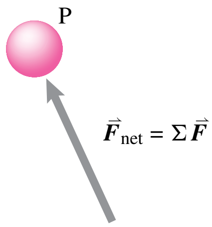

Figure 19.63

2. **Net force on P:** From the

$$\sum \vec{F} = m\vec{a}$$

for the mass P we get $\sum \vec{F} = m\vec{a}_P$. Thus to find the net force $\sum \vec{F}$ we need to find $\vec{a}_P$. The calculation of $\vec{a}_P$ is the same as in Sample 18.11 except that $\vec{a}_{P'}$ is now calculated from

$$\vec{a}_{P'} = \vec{a}_{O'} + \vec{a}_{P'/O'}$$

where

$$\begin{aligned}
\vec{a}_{O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}) \\
&= -\omega_1^2 \vec{r}_{O'/O} \\
&= -\omega_1^2 \ell(\cos\theta \hat{\imath} + \sin\theta \hat{\jmath}) \\
&= -(3\,\mathrm{rad/s})^2 \cdot 2\,\mathrm{m} \cdot (\cos 30^\circ \hat{\imath} + \sin 30^\circ \hat{\jmath}) \\
&= -(15.59\hat{\imath} + 9.00\hat{\jmath})\,\mathrm{m/s^2},
\end{aligned}$$

$$\begin{aligned}
\vec{a}_{P'/O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\
&= -\omega_1^2 \vec{r}_{P'/O'} \\
&= -\omega_1^2 r(\cos\theta \hat{\imath} - \sin\theta \hat{\jmath}) \\
&= -(3\,\mathrm{rad/s})^2 \cdot 0.5\,\mathrm{m} \cdot (\cos 30^\circ \hat{\imath} - \sin 30^\circ \hat{\jmath}) \\
&= -(3.90\hat{\imath} - 2.25\hat{\jmath})\,\mathrm{m/s^2}.
\end{aligned}$$

Thus,

$$\vec{a}_{P'} = -(19.49\hat{\imath} + 6.75\hat{\jmath})\,\mathrm{m/s^2}$$

which, of course, is the same as calculated in Sample 18.11. The other two terms, $\vec{a}_{\rm cor}$ and $\vec{a}_{\rm rel}$, are exactly the same as in Sample 18.11. Therefore, we get the same value for $\vec{a}_P$ by adding the three terms:

$$\vec{a}_P = -(17.83\hat{\imath} + 3.63\hat{\jmath})\,\mathrm{m/s^2}.$$

The net force on P is

$$\begin{aligned}
\sum \vec{F} &= m\vec{a}_P \\
&= 0.5\,\mathrm{kg} \cdot (-17.83\hat{\imath} - 3.63\hat{\jmath})\,\mathrm{m/s^2} \\
&= -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\mathrm{N}.
\end{aligned}$$

$$\boxed{\sum \vec{F} = -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\mathrm{N}}$$

SAMPLE 19.21 Inverse dynamics of a four bar mechanism. A four bar mechanism ABCD consists of three uniform bars AB, BC, and CD of length ℓ_1, ℓ_2, ℓ_3, and mass m_1, m_2, m_3, respectively. The mechanism is driven by a torque T applied at A such that bar AB rotates at constant angular speed. Write equations to find the torque T at some instant t.

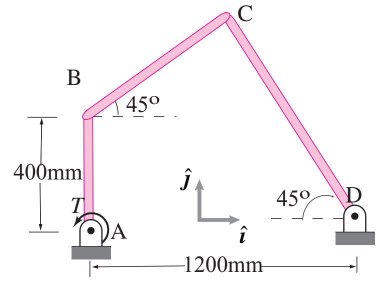

Figure 19.64

Solution This is an inverse dynamics problem, that is, we are given the motion and we are supposed to find the forces (torque T in this case) that cause that motion. We are given that rod AB rotates at constant angular speed, say $\dot{\theta}$. From kinematics, we can find out angular velocities and angular accelerations of the other two bars as well as the accelerations of center-of-mass of each rod. Then we can write the momentum balance equations and compute the forces and moments required to generate this motion. So, in contrast to what we usually do, let us do the kinematics first. Please see Sample 18.5 on page 990. We found the angular velocities, $\dot{\beta}$ (eqn. (18.71)) and $\dot{\phi}$ (eqn. (18.70)), of rods BC and CD, respectively, in terms of $\dot{\theta}$. We can rewrite those equations as

$$\begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix}\dot{\beta}\\ \dot{\phi}\end{pmatrix} = \dot{\theta}\begin{pmatrix}\ell_1\sin\theta\\ \ell_1\cos\theta\end{pmatrix} \tag{19.66}$$

We wrote this equation in matrix form to make it easier for us to find the angular accelerations which we do by simply differentiating this equation once:

$$\begin{aligned}\begin{bmatrix} -\ell_2\cos\beta\,\dot{\beta} & \ell_3\cos\phi\,\dot{\phi} \\ \ell_2\sin\beta\,\dot{\beta} & -\ell_3\sin\phi\,\dot{\phi} \end{bmatrix}\begin{pmatrix}\dot{\beta}\\ \dot{\phi}\end{pmatrix} &+ \begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix}\ddot{\beta}\\ \ddot{\phi}\end{pmatrix} \\ &= \ell_1\begin{pmatrix}\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\\ \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\end{pmatrix}\end{aligned}$$

Rearranging terms we get

$$\begin{aligned}\begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix}\ddot{\beta}\\ \ddot{\phi}\end{pmatrix} &= -\begin{bmatrix} -\ell_2\cos\beta & \ell_3\cos\phi \\ \ell_2\sin\beta & \ell_3\sin\phi \end{bmatrix}\begin{pmatrix}\dot{\beta}^2\\ \dot{\phi}^2\end{pmatrix} \\ &\quad +\ell_1\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{pmatrix}\ddot{\theta}\\ \dot{\theta}^2\end{pmatrix}\end{aligned} \tag{19.67}$$

Thus, we can find the angular accelerations of BC and CD, $\ddot{\beta}$ and $\ddot{\phi}$, because the quantities on the right hand side are known ($\ddot{\theta}(=0)$ and $\dot{\theta}$ are given, and $\dot{\beta}$ and $\dot{\phi}$ are determined by eqn. (19.66)). Now, we can find the accelerations of center-of-mass of each rod as follows.

$$\vec{\boldsymbol{a}}_{G_1} = -\frac{\ell_1}{2}\dot{\theta}^2\hat{\boldsymbol{\lambda}}_1 \tag{19.68}$$

$$\vec{\boldsymbol{a}}_{G_2} = \vec{\boldsymbol{a}}_B + \vec{\boldsymbol{a}}_{G_2/B} = -\ell_1\dot{\theta}^2\hat{\boldsymbol{\lambda}}_1 - \frac{\ell_2}{2}\dot{\beta}^2\hat{\boldsymbol{\lambda}}_2 + \ell_2\ddot{\beta}\hat{\boldsymbol{n}}_2 \tag{19.69}$$

$$\vec{\boldsymbol{a}}_{G_3} = -\frac{\ell_3}{2}\dot{\phi}^2\hat{\boldsymbol{\lambda}}_3 \tag{19.70}$$

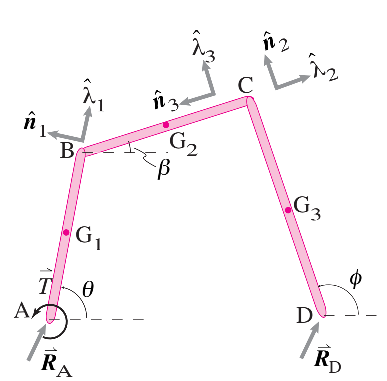

Figure 19.65

We are now ready to write momentum balance equations. Since we are only interested in finding the torque T, we should try to write equations involving minimum number of unknown forces. So, we draw free-body diagrams of the whole mechanism, of part BCD, and of bar CD alone; and write angular momentum balance equations about appropriate points so that we involve only the unknown torque T and the unknown reaction $\vec{\boldsymbol{R}}_D$ at D. Thus, we will have only three scalar unknowns T, R_{D_x} and R_{D_y} (since $\vec{\boldsymbol{R}}_D = R_{D_x}\hat{\boldsymbol{\imath}} + R_{D_y}\hat{\boldsymbol{\jmath}}$). So, we will need only three independent equations.

Consider the free-body diagram of the whole mechanism. We can write angular momentum balance about point A for the whole mechanism as

$$T\hat{\boldsymbol{k}} + \vec{\boldsymbol{r}}_{D/A}\times\vec{\boldsymbol{R}}_D = \dot{\vec{\boldsymbol{H}}}_A = \dot{\vec{\boldsymbol{H}}}_{1/A} + \dot{\vec{\boldsymbol{H}}}_{2/A} + \dot{\vec{\boldsymbol{H}}}_{3/A} \tag{19.71}$$

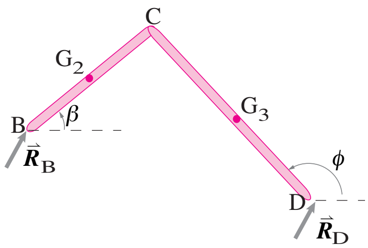

Figure 19.66

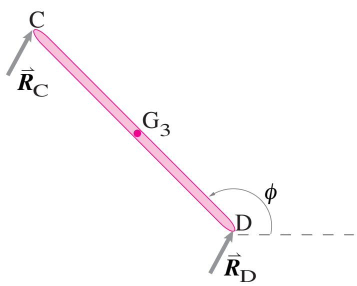

Figure 19.67

where

$$
\begin{aligned}
\dot{\vec{H}}_{1/\mathrm{A}} &= I_1\ddot{\theta}\hat{k} + \vec{r}_{\mathrm{G}_1} \times m_1\vec{a}_{\mathrm{G}_1} \\
\dot{\vec{H}}_{2/\mathrm{A}} &= I_2\ddot{\beta}\hat{k} + \vec{r}_{\mathrm{G}_2} \times m_2\vec{a}_{\mathrm{G}_2} = I_2\ddot{\beta}\hat{k} + (\vec{r}_{\mathrm{B}} + \vec{r}_{\mathrm{G}_2/\mathrm{B}}) \times m_2\vec{a}_{\mathrm{G}_2} \\
\dot{\vec{H}}_{3/\mathrm{A}} &= I_3\ddot{\phi}\hat{k} + \vec{r}_{\mathrm{G}_3} \times m_3\vec{a}_{\mathrm{G}_3} = I_3\ddot{\phi}\hat{k} + (\vec{r}_{\mathrm{D}} + \vec{r}_{\mathrm{G}_3/\mathrm{D}}) \times m_3\vec{a}_{\mathrm{G}_3}.
\end{aligned}
$$

Similarly, the angular momentum balance about point B for BCD gives

$$
\vec{r}_{\mathrm{D}/\mathrm{B}} \times \vec{R}_{\mathrm{D}} = \dot{\vec{H}}_{2/\mathrm{B}} + \dot{\vec{H}}_{3/\mathrm{B}} \tag{19.72}
$$

where

$$
\begin{aligned}
\dot{\vec{H}}_{2/\mathrm{B}} &= I_2\ddot{\beta}\hat{k} + \vec{r}_{\mathrm{G}_2/\mathrm{B}} \times m_2\vec{a}_{\mathrm{G}_2} \\
\dot{\vec{H}}_{3/\mathrm{B}} &= I_3\ddot{\phi}\hat{k} + \vec{r}_{\mathrm{G}_3/\mathrm{B}} \times m_3\vec{a}_{\mathrm{G}_3}
\end{aligned}
$$

and angular momentum balance of bar CD about point C gives

$$
\vec{r}_{\mathrm{D}/\mathrm{C}} \times \vec{R}_{\mathrm{D}} = \dot{\vec{H}}_{3/\mathrm{C}} = I_3\ddot{\phi}\hat{k} + \vec{r}_{\mathrm{G}_3/\mathrm{C}} \times m_3\vec{a}_{\mathrm{G}_3}. \tag{19.73}
$$

Note that we can easily write the position vectors in terms of ℓ_1, ℓ_2, ℓ_3 and the unit vectors $(\hat{\lambda}_1, \hat{n}_1)$, $(\hat{\lambda}_2, \hat{n}_2)$ and $(\hat{\lambda}_3, \hat{n}_3)$ where

$$
\begin{aligned}
\hat{\lambda}_1 &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}, & \hat{n}_1 &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \\
\hat{\lambda}_2 &= \cos\beta\hat{\imath} + \sin\beta\hat{\jmath}, & \hat{n}_2 &= -\sin\beta\hat{\imath} + \cos\beta\hat{\jmath} \\
\hat{\lambda}_1 &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}, & \hat{n}_1 &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}.
\end{aligned}
$$

We can put all the three angular momentum balance equations, (19.71), (19.72), and (19.73), in one matrix equation by dotting both sides of the equations with $\hat{k}$ and assembling them as follows.

$$
\begin{bmatrix}
1 & 0 & \ell_4 \\
0 & \hat{k}\cdot(\ell_2\hat{\lambda}_2 - \ell_3\hat{\lambda}_3)\times\hat{\imath} & \hat{k}\cdot(\ell_2\hat{\lambda}_2 - \ell_3\hat{\lambda}_3)\times\hat{\jmath} \\
0 & \hat{k}\cdot(-\ell_3\hat{\lambda}_3\times\hat{\imath}) & \hat{k}\cdot(-\ell_3\hat{\lambda}_3\times\hat{\jmath})
\end{bmatrix}
\begin{pmatrix} T \\ R_{D_x} \\ R_{D_y} \end{pmatrix}
=
\begin{pmatrix} \dot{H}_{123/A} \\ \dot{H}_{23/B} \\ \dot{H}_{3/C} \end{pmatrix}
\tag{19.74}
$$

where $\dot{H}_{123/A} = \hat{k}\cdot(\dot{\vec{H}}_{1/\mathrm{A}} + \dot{\vec{H}}_{2/\mathrm{A}} + \dot{\vec{H}}_{3/\mathrm{A}})$, $\dot{H}_{23/B} = \hat{k}\cdot(\dot{\vec{H}}_{2/\mathrm{B}} + \dot{\vec{H}}_{3/\mathrm{B}})$, and $\dot{H}_{3/C} = \hat{k}\cdot\dot{\vec{H}}_{3/\mathrm{C}}$.

Note that we know the $\dot{\vec{H}}$'s on the right hand side and the matrix on the left side can be evaluated for any given (θ, β, ϕ). Thus we can solve for T, R_{D_x}, and R_{D_y}.

SAMPLE 19.22 Numerical solution of the inverse dynamics problem. Consider Sample 19.21 again. Using numerical solutions on a computer, find and plot torque T against θ for one complete cycle of the drive arm AB. Take $m_1 = m_2 = m_3 = 1\,\text{kg}$ and $\ell_1 = 400\,\text{mm}$, $\ell_2 = 400\sqrt{2}\,\text{mm}$, $\ell_3 = 800\sqrt{2}\,\text{mm}$, and $\ell_4 = 1200\,\text{mm}$.

Solution

Since we have to plot T against θ for one complete revolution, we need to find angular velocities, angular accelerations and center-of-mass accelerations for several values of θ and then solve for T for each of those θ's. We can do this several ways. One way would be to first solve kinematic equations to find $\theta(t), \beta(t)$, and $\phi(t)$ at discrete times over one complete cycle and then compute all other quantities at each $(\theta(t_i), \beta(t_i), \phi(t_i))$ where t_i represents a discrete time. So, let us follow this method step by step with pseudocodes. Here, we assume that we have vector functions called `dot` and `cross` that compute the dot product and the cross product of two vectors that are given as input arguments.

Step-1: solve for angular positions. Specify the given geometry

```
L1=0.4, L2=0.4*sqrt(2), L3=0.8*sqrt(2), L4=1.2
```

and use the pseudocode of Sample 18.5 to find $\theta(t_i), \beta(t_i), \phi(t_i)$ for, say, 100 values of t_i between 0 and 1 sec. Now, for each triad of $(\theta(t_i), \beta(t_i), \phi(t_i))$, follow all the steps below.

Step-2: solve for angular velocities. Since $\dot{\theta} = 2\pi\,\text{rad/s}$ is given, we only need to solve for $\dot{\beta}$ and $\dot{\phi}$. We use eqn. (18.71) and eqn. (18.70) to compute $\dot{\beta}$ and $\dot{\phi}$ as follows (or modify the pseudocode of Sample 18.5 to save $\dot{\beta}$ and $\dot{\phi}$ along with the values for β and ϕ.

```
define thdot=thetadot, bdot=betadot, pdot=phidot
thdot = 2*pi   % this is given
set th = theta(ti), b = beta(ti), p = phi(ti)
set unit vectors
  l1=[cos(th) sin(th) 0]', n1=[-sin(th) cos(th) 0]'
  l2=[cos(b) sin(b) 0]', n2=[-sin(b) cos(b) 0]'
  l3=[cos(p) sin(p) 0]', n3=[-sin(p) cos(p) 0]'
bdot = -(L1/L2)*(cross(n1,l3)/cross(n2,l3))*thdot
pdot = (L1/L3)*(cross(n1,l2)/cross(n3,l2))*thdot
```

Step-3: solve for angular accelerations. Now that we have (θ, β, ϕ) and the corresponding values of $(\dot{\theta}, \dot{\beta}, \dot{\phi})$, we can use eqn. (19.67) to calculate $\ddot{\beta}$ and $\ddot{\phi}$ (we are given $\ddot{\theta} = 0$).

```
define thddot=thetaddot, bddot=betaddot,
       pddot=phiddot
thddot = 0   % this is given
B = [-L2*sin(b) L3*sin(p); -L2*cos(b) L3*cos(p)]
C = L1*[sin(th) cos(th); cos(th) -sin(th)]
D = [-cos(b) cos(p); sin(b) -sin(p)]
c = [thddot thdot^2]',  d = [L2*bdot^2 L3*pdot^2]'
assume w = [bddot pddot]'
solve B*w = C*c + D*d for w
```

So, now we know $\ddot{\theta}, \ddot{\beta}, \ddot{\phi}$ also. We are now ready to compute $\dot{\boldsymbol{H}}$'s required for dynamic calculations.

Step-4: set up equations and solve for unknown forces. We need to set up and solve eqn. (19.74). Note that we need to compute several quantities for this equation but the vector computations are more or less straightforward.

```
% set mass and inertia properties
m1 = 1, m2 = 1, m3 = 1

I1 = m1*L1^2/12, I2 = m2*L2^2/12, I3 = m3*L3^2/12
% set fixed unit vectors
i = [1 0 0]', j = [0 1 0]', k = [0 0 1]'
% compute position vectors
rA = [0;0;0],  rB = rA+L1*l1
rC = rB+L2*l2, rD = L4*l4
rG1 = L1/2*l1
rG2 = rB+L2/2*l2
rG3 = rD+L3/2*l3
% compute center-of-mass accelerations
aG1 = 0.5*L1*(tddot*n1-tdot^2*l1)
aG2 = 2*aG1+0.5*L2*(bddot*n2-bdot^2*l2)    % aB = 2*aG1
aG3 = 0.5*L3*(pddot*n3-pdot^2*l3)

% compute Hdot_cms

Hdot_cm1 = I1*tddot*uk
Hdot_cm2 = I2*bddot*uk
Hdot_cm3 = I3*pddot*uk
% compute Hdots
Hdot_123_A = Hdot_cm1 + cross(rG1, m1*aG1)
              + Hdot_cm2 + cross(rG2, m2*aG2)
                + Hdot_cm3 + cross(rG3, m3*aG3)
Hdot_23_B = Hdot_cm2 + cross(rG2-rB, m2*aG2)
              + Hdot_cm3 + cross(rG3-rB, m3*aG3)
Hdot_3_C = Hdot_cm3 + cross(rG3-rC, m3*aG3)
% set up the linear eqns for torque and RD
b = [dot(Hdot_123_A,k) dot(Hdot_23_B,k) dot(Hdot_3_C,k)]
A = [1  dot(k,cross(rD,i))     dot(k,cross(rD,j))
     0  dot(k,cross(rD-rB,i))  dot(k,cross(rD-rB,j))
      0  dot(k,cross(rD-rC,i))  dot(k,cross(rD-rC,j))]
% let forces = [T  RDx  RDy]'
solve A*forces = b for forces
```

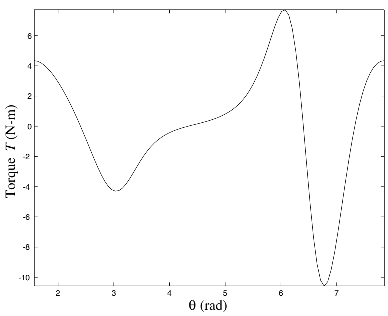

Figure 19.68: Torque T as a function of θ over one complete cycle of motion ($\theta(0) = \pi/2, \theta(1) = 5\pi/2$).

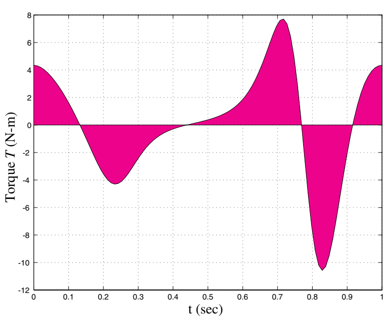

Figure 19.69: Torque T as a function of time over one complete cycle of motion.

Step-5, repeat calculations. Now repeat Step-2 – Step-4 for each triad (θ, β, ϕ) obtained in Step-1 and save the corresponding values of T in a vector. Finally,

```
plot T vs theta
```

The plot thus obtained is shown in fig. 19.68. We can also plot T vs time (as shown in fig. 19.69), and, of course, expect to see the same graph of T since θ is just a linear function of t. Note that the area under the graph of T over one complete cycle must equal zero since the net impulse must be zero over one cycle.