19.2 Mechanics of one-degree-of-freedom 2-D mechanisms

A one-degree-of-freedom mechanism is a collection of parts linked together so that they can move in only one way①. The word "freedom" has to be taken lightly here because in practice even the one "freedom" is often controlled or restricted. Frankly, most machine designers don't trust the laws of mechanics to enforce motions that they want. Instead they choose kinematic restrictions that enforce the desired motions and then use a motor, a computer controlled actuator or big flywheel to keep that motion moving at a prescribed rate.

We consider here machines that can move in just one way, whether or not that one motion is free. So in this sense, "one"-degree-of-freedom machines include machines with no freedom at all, just so long as they move in only one way.

Some familiar examples of one-degree-of-freedom mechanisms are a 1-D spring and mass, a pendulum, a slider-crank, a grounded 4-bar linkage, and a gear train.

①No physical system can move in only one way. The idea that a machine has one degree of freedom is an idealization that takes the rigid-object description of the parts and the ideal nature of the connections literally. In fact, all parts deform at least a little, and no connections are so precise as to be exact geometric constraints. It may be reasonable to respect the standard rigid-object idealizations and consider a machine as one-degree-of-freedom for basic analysis, as we do in this section. When, for example, trying to figure out why a machine vibrates in an undesirable way it may also be reasonable to relax some of the assumptions we make here and consider more degrees of freedom.

Most ideal constraints are workless constraints

A fruitful equation for studying one-degree-of-freedom mechanisms is power balance, or for conservative systems, energy balance. The reason these equations are so useful is because most ideal connections are workless. That is:

> the net work of the interaction forces (and moments) of a pair of parts that are connected with the standard ideal connections is zero. This includes welds, frictionless hinges, frictionless sliding contact, rolling contact, or parts connected by a massless inextensible link.

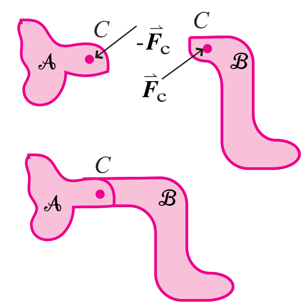

Figure 19.27: A frictionless hinge is a workless constraint. The *net* work of the interaction force on the two contacting bodies is zero.

Example: A frictionless hinge is a workless constraint

Body $\mathcal{A}$ is connected to body $\mathcal{B}$ by a frictionless hinge at C (see fig. 19.27). The force on body $\mathcal{B}$ at C from $\mathcal{A}$ is $\vec{F}_C$ and the force on $\mathcal{A}$ from body $\mathcal{B}$ at C is $-\vec{F}_C$. The power of the interaction force on body $\mathcal{B}$ is $P_{\mathcal{A}\text{on}\mathcal{B}} = \vec{F}_C \cdot \vec{v}_C$. This power contributes to the increase in the kinetic energy of $\mathcal{B}$. The power of the interaction force on $\mathcal{A}$ is $P_{\mathcal{B}\text{on}\mathcal{A}} = -\vec{F}_C \cdot \vec{v}_C = -P_{\mathcal{A}\text{on}\mathcal{B}}$. So the contribution to the increase in the kinetic energy of body $\mathcal{A}$ is minus the contribution to body

$\mathcal{B}$ and the net power on the system of two bodies is zero. Writing this out,

$$\begin{aligned}
\overbrace{P_{\text{total}}}^{\boxed{\text{The net power of the pair of interaction forces on the pair of bodies}}} &= P_{\mathcal{B}\text{on}\mathcal{A}} + P_{\mathcal{A}\text{on}\mathcal{B}} \\
&= \vec{\boldsymbol{F}}_C \cdot \vec{\boldsymbol{v}}_{\mathrm{C}} + (-\vec{\boldsymbol{F}}_C) \cdot \vec{\boldsymbol{v}}_{\mathrm{C}} \\
&= (\vec{\boldsymbol{F}}_C - \vec{\boldsymbol{F}}_C) \cdot \vec{\boldsymbol{v}}_{\mathrm{C}} \\
&= 0.
\end{aligned}$$

Basically the same situation holds for all the standard ideal connections as explained in the box on page 2.

If one of two interacting bodies is known to be stationary, like the ground, then the work of the constraint forces is zero on both of the bodies. Thus the work of the hinge force on a pendulum, and the ground reaction forces on a frictionlessly sliding body or the ground force on a perfectly rolling body is zero. But be careful with the words "workless constraint forces", however.

> The workless constraint connecting moving bodies $\mathcal{A}$ and $\mathcal{B}$ is likely to do positive work on one of the bodies and negative work on the other.

It is just the net work on the two bodies which is zero.

Energy method: single degree of freedom systems

Although linear and angular momentum balance apply to a single degree of freedom system and all of its parts, often one finds what one wants with a single scalar equation, namely energy or power balance.

19.2 Ideal constraints and workless constraints

All of the ideal constraints we consider are interactions between two bodies $\mathcal{A}$ and $\mathcal{B}$. One of these could be the ground. Let's take the interaction force $\vec{\boldsymbol{F}}$ and moment $\vec{\boldsymbol{M}}$ to be the force and moment of $\mathcal{A}$ on $\mathcal{B}$. The point of interaction is A on $\mathcal{A}$ and B on $\mathcal{B}$. By the principle of action and reaction, the net power of the interaction force on the two bodies is

$$\begin{aligned}
P &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_{\mathrm{B}} + \vec{\boldsymbol{M}} \cdot \vec{\boldsymbol{\omega}}_{\mathcal{B}} + (-\vec{\boldsymbol{F}}) \cdot \vec{\boldsymbol{v}}_{\mathrm{A}} + (-\vec{\boldsymbol{M}}) \cdot \vec{\boldsymbol{\omega}}_{\mathcal{A}} \\
&= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_{\mathrm{B/A}} + \vec{\boldsymbol{M}} \cdot \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}}.
\end{aligned}$$

All of our ideal constraints are designed to exactly make these dot products zero. The ideal hinge is considered in the text. Another example is perfect rolling. In that case the interaction moment is assumed to be zero. The no-slip condition means that $\vec{\boldsymbol{v}}_{\mathrm{B/A}} = \boldsymbol{0}$. On the other hand for frictionless sliding there $\vec{\boldsymbol{v}}_{\mathrm{B/A}}$ can have a component tangent to the surfaces. But that is exactly the direction where the friction force is assumed to be zero.

And so it is for all of the ideal "workless" constraints.

Examples of non-workless constraints, that is, interactions that contribute to the energy equations are: sliding with non-zero friction, joints with non-zero friction torques, joints with motors, or interactions mediated by springs, dampers or actuators.

Imagine a complex machine that only has one degree of freedom, meaning the position of the whole machine is determined by a single configuration variable, call it q. Further assume that the machine has no motion when $\dot{q} = 0$. The variable q could be, for example, the angle of one of the linked-together machine parts. Also, assume that the machine has no dissipative parts: no friction, no collisions, no inelastic deformation. Because q characterizes the position of all of the parts of the system we can, in principle, calculate the potential energy of the system as a function of q,

$$E_\mathrm{P} = E_\mathrm{P}(q).$$

We find this function by adding up the potential energies of all the springs in the machine and the gravitational potential energies of the parts. Similarly we can write the system's kinetic energy in terms of q and its rate of change $\dot{q}$. Because at any configuration the velocity of every point in the system is proportional to $\dot{q}$ we can write the kinetic energy as:

$$E_\mathrm{K} = M(q)\dot{q}^2/2$$

where $M(q)$ is a function that one can determine by calculating the machine's total kinetic energy in terms of q and $\dot{q}$ and then factoring $\dot{q}^2$ out of the resulting expression.

Now, if we accept the equation of mechanical energy conservation we have

$$\begin{aligned}
\text{constant} &= E_\mathrm{T} \quad \text{conservation of energy,}\\
\Rightarrow \quad 0 &= \frac{d}{dt}E_\mathrm{T} \quad \text{taking one time derivative,}\\
&= \frac{d}{dt}[E_\mathrm{P} + E_\mathrm{K}] \quad \text{total energy is potential plus kinetic}\\
&= \frac{d}{dt}[E_\mathrm{P}(q) + \frac{1}{2}M(q)\dot{q}^2] \quad \text{substituting from paragraphs above}\\
\Rightarrow \quad 0 &= \frac{d}{dq}[E_\mathrm{P}(q)]\,\dot{q} + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}\dot{q}^2 + M(q)\dot{q}\ddot{q}\\
0 &= \frac{d}{dq}[E_\mathrm{P}(q)] + \left(\frac{1}{2}\frac{d}{dq}[M(q)]\right)\dot{q}^2 + M(q)\,\ddot{q} \quad \text{cancelling } \dot{q}\\
0 &= f_1(q) + f_2(q)\,\dot{q}^2 + f_3(q)\,\ddot{q} \qquad (19.20)
\end{aligned}$$

$$\begin{aligned}
\text{with } f_1(q) &\equiv \frac{d}{dq}[E_\mathrm{P}(q)],\\
f_2(q) &\equiv \frac{1}{2}\frac{d}{dq}[M(q)], \quad \text{and}\\
f_3(q) &\equiv M(q).
\end{aligned}$$

The cancellation of $\dot{q}$ above lacks mathematical rigor, but doesn't cause problems②.The equation of motion is complicated because when we take the time derivative of a function of $M(q)$ and $E_\mathrm{P}(q)$ we have to use the chain rule. Also, because we have products of terms, we had to use the

②The cancellation of the factor $\dot{q}$ from equation 19.20 depends on $\dot{q}$ being other than zero. While moving, $\dot{q}$ is not zero. Strictly we cannot cancel the $\dot{q}$ term from the equation at the instants when $\dot{q} = 0$. However, to say that a differential equation is true except for certain instants in time is, in practice to say that it is always true, at least if we make reasonable assumptions about the smoothness of the motions.

product rule. Eqn. 19.20 is the general equation of motion of a conservative one-degree-of-freedom system. It is really just a special case of the equation of motion for one-degree-of-freedom systems found from power balance. Rather than memorizing eqn. (19.20) it is probably best to look at its derivation as an algorithm to be reproduced on a problem by problem basis.

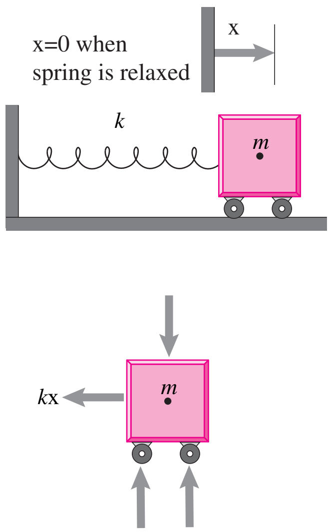

Figure 19.28: The familiar one degree of freedom spring and mass system.

Example: Spring and mass

Although the motion of a spring and mass system can be found easily enough from linear momentum balance, it is also a good example for energy balance (see fig. 19.28). Using conservation of energy for the spring and mass system:

$$\begin{aligned}
E_{\mathrm{T}} &= constant \\
0 &= \frac{d}{dt}E_{\mathrm{T}} \\
&= \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} \\
&= \frac{d}{dt}(mv^2/2) + \frac{d}{dt}(kx^2/2) \\
&= mv\dot{v} + kx\dot{x} \\
v = \dot{x} \quad \Rightarrow \quad 0 &= m\ddot{x} + kx.
\end{aligned}$$

Similarly power balance could have been used to get the same result, looking at just the mass

$$\begin{aligned}
P &= \frac{d}{dt}E_{\mathrm{K}} \\
(-kx)(\dot{x}) &= \frac{d}{dt}(mv^2/2) = mv\dot{v} \\
\Rightarrow \quad 0 &= kx + m\ddot{x}
\end{aligned}$$

as before.

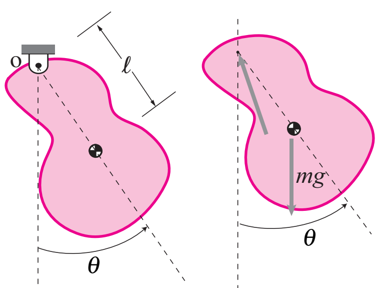

Figure 19.29: A rigid body suspended from a frictionless hinge is an energy conserving one-degree-of-freedom mechanism.

Example: Pendulum

Consider a rigid body with mass m and moment of inertia I^o about a hinge which is a distance ℓ from the center-of-mass (see fig. 19.29). The familiar simple pendulum is another single degree of freedom system for which the equation of motion can be found from conservation of energy.

$$\begin{aligned}
E_{\mathrm{T}} &= constant \\
\rightarrow 0 &= \frac{d}{dt}E_{\mathrm{T}} = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} \\
&= \frac{d}{dt}(I^o\omega^2/2) + \frac{d}{dt}(-mg\ell\cos\theta) \\
&= I^o\omega\dot{\omega} + -mg\ell(-\sin\theta)\dot{\theta} \\
\omega = \dot{\theta} \quad \Rightarrow \quad 0 &= \ddot{\theta} + \frac{mg\ell}{I^o}\sin\theta.
\end{aligned}$$

the pendulum equation that we have derived before by this and other means (angular momentum balance about point o).

The above examples are old friends which are handled easily with other techniques. Here is a problem which is much more difficult without the energy method.

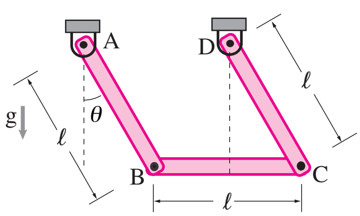

Figure 19.30: Three identical uniform bars are pinned with frictionless hinges and swing, obviously, something like a simple pendulum.

Example: Three bars act like a simple pendulum

Assume all three bars in the structure shown in fig. 19.30are of equal length ℓ

and have mass m uniformly distributed along their length. It is intuitively obvious that this device swings back and forth something like a simple pendulum. But how can we get the laws of mechanics to tell us this? One approach, which will work in the end, is to draw free-body diagrams of all the parts, write linear and angular momentum balance for each, and then add and subtract equations to eliminate the unknown constraint forces at the various hinges.

The more direct approach is to write the energy equation, adding up the potential and kinetic energies of the parts, all evaluated in terms of the single configuration variable θ. Taking the potential energy to be zero at $\theta = \pi/2$ (when all centers of mass are at hinge height) we have

$$
\begin{aligned}
E_\mathrm{T} &= \mathit{constant}\\
0 &= \frac{d}{dt}E_\mathrm{T} = \dot{E}_\mathrm{P} + \dot{E}_\mathrm{K}\\
&= \frac{d}{dt}\left((I^o\omega^2/2) + (I^o\omega^2/2) + (m(\ell\omega)^2/2) \right)\\
&\quad + \frac{d}{dt}\left((-gm(\ell/2)\cos\theta \right.\\
&\quad \left. -gm(\ell/2)\cos\theta) - gm\ell\cos\theta \right)\\
I^o = m\ell^2/3 \quad \Rightarrow \quad 0 &= \frac{d}{dt}\left(5m\ell^2\omega^2/6 \right) + \frac{d}{dt}\left(-2gm\ell\cos\theta \right)\\
&= (5m\ell^2\omega\dot{\omega}/3 + 2gm\ell\omega\sin\theta)\\
\Rightarrow \quad 0 &= 5\ell\dot{\omega}/3 + 2g\sin\theta\\
\Rightarrow \quad 0 &= \ddot{\theta} + \frac{6g}{5\ell}\sin\theta
\end{aligned}
$$

which is the same governing equation as for a point-mass pendulum with length $5\ell/6$. This is just half way between the following two cases. If the side links had no mass the equation would have been the same as for a point mass pendulum with length ℓ

$$0 = \ddot{\theta} + \frac{g}{\ell}\sin\theta$$

and if the bottom link had no mass the equation would be the same as a stick hanging from one end which goes back and forth like a point mass pendulum with length $2\ell/3$ according to

$$0 = \ddot{\theta} + \frac{3g}{2\ell}\sin\theta.$$

Example: One on the rim is like two on the frame.

A bicycle transmission is such that the speed of the bike relative to the ground is n times the speed of the pedal relative to the frame :

$$v_\mathrm{bike} = n v_\mathrm{pedal/bike}$$

Assume the kinetic energy of the relative motion of a rider's legs can be neglected, as can be the weight of the rider's leg. At the moment in question the velocity of the pedal is parallel to the direction from the seat to the leg. Thus the free-body diagram of the bike/person system, leaving out the pedaling leg is as shown in fig. 19.31. Let's assume the bike and rider have mass M and that the wheels have mass m_r and m_f concentrated on the rim (the hubs are considered part of the frame and the spokes are neglected). Neglecting air resistance *etc.*the power balance equation is:

$$P = \dot{E}_\mathrm{K} \tag{19.21}$$

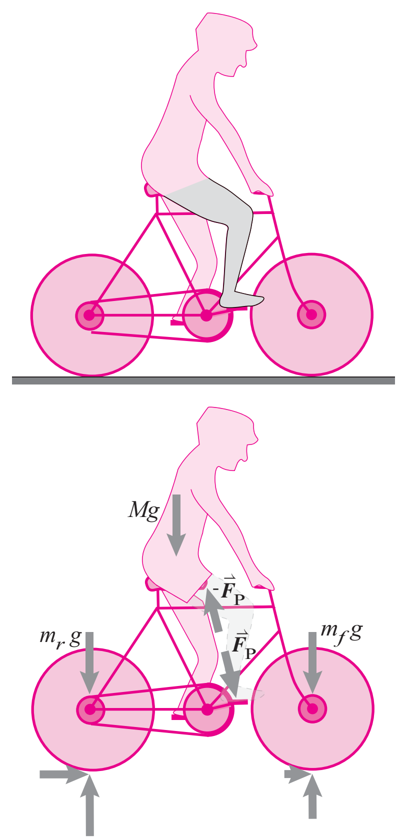

Figure 19.31: A person rides a bike. The pedaling leg is idealized as a pair of equal and opposite forces acting on the seat and pedal.

Let's do some side calculations for evaluating the terms in eqn. (19.21). First, the only forces that do work on the system as drawn are the force on the pedal and the force on the seat.

$$
\begin{aligned}
P &= -\vec{F}_{\mathrm{P}} \cdot \vec{v}_{\mathrm{seat}} + \vec{F}_{\mathrm{P}} \cdot \vec{v}_{\mathrm{pedal}} \\
&= -\vec{F}_{\mathrm{P}} \cdot \vec{v}_{\mathrm{seat}} + \vec{F}_{\mathrm{P}} \cdot \left(\vec{v}_{\mathrm{bike}} + \vec{v}_{\mathrm{pedal/bike}} \right) \\
&= \underbrace{-\vec{F}_{\mathrm{P}} \cdot \vec{v}_{\mathrm{seat}} + \vec{F}_{\mathrm{P}} \cdot \vec{v}_{\mathrm{bike}}}_{0} + \underbrace{\vec{F}_{\mathrm{P}} \cdot \vec{v}_{\mathrm{pedal/bike}}}_{F_P v_{\mathrm{pedal/bike}}} \\
&= F_P v_{\mathrm{bike}}/n
\end{aligned}
$$

The net power of the leg is expressed by the compression it carries times its extension rate. The kinetic energy of the wheel comes from both its rotation and its translation. The moment of inertia of a hoop about its center is $I = mr^2$. For rolling contact $|\omega R| = v$ so, for *one* wheel:

$$
\begin{aligned}
E_{\mathrm{K wheel}} &= m v_{\mathrm{bike}}^2/2 + I\omega^2/2 \\
&= m v_{\mathrm{bike}}^2/2 + (mR^2)(v/R)^2/2 \\
&= m v_{\mathrm{bike}}^2.
\end{aligned}
$$

The kinetic energy of a rolling hoop is twice that of a point mass moving at the same speed. Putting these results back in to eqn. (19.21) we have

$$
\begin{aligned}
P &= \dot{E}_{\mathrm{K}} && (19.22) \\
F_P v_{\mathrm{bike}}/n &= \frac{d}{dt}\left(M v_{\mathrm{bike}}^2/2 + (m_r + m_f) v_{\mathrm{bike}}^2 \right) && (19.23) \\
&= \left(M \dot{v}_{\mathrm{bike}} + 2(m_r + m_f)\dot{v}_{\mathrm{bike}} \right) v_{\mathrm{bike}} && (19.24) \\
\Rightarrow \quad F_P/n &= (M + 2(m_r + m_f))\dot{v}_{\mathrm{bike}} && (19.25) \\
\Rightarrow \quad \dot{v}_{\mathrm{bike}} &= \frac{F_P}{n(M + 2(m_r + m_f))}. && (19.26)
\end{aligned}
$$

The bigger the pedal force, the bigger the acceleration, obviously. The higher the gear ratio, the less the acceleration; the faster gears let you pedal slower for a given bike speed, but demand more pedal force for a given acceleration. A heavier bike accelerates less. But the contribution to slowing a bike is twice as much for mass added to the rim as for mass added to the frame or body.

Some comments. n typically ranges from about 1.7 to 8 for a new 21 speed bike and is about 5 for an adult European, Indian or Chinese 1-speed. For a given speed of bicycle riding your feet go n times slower relative to your body than for walking or running at that speed. This calculation is for accelerating a bike on level ground with no wind and rolling resistance. The net speed of a bike in a bike race is not so dependent on weight, because the main enemy is wind resistance. To the extent that weight is a problem it is for steady uphill travel. In this case the mass on the rim makes the same contribution as mass on the frame.

Vibrations

The preponderance of systems where vibrations occur is not due to the fact that so many systems look like a spring connected to a mass, a simple pendulum, or a torsional oscillator. Instead there is a general class of systems which can be expected to vibrate sinusoidally near some equilibrium position. These systems are one-degree -of-freedom (one DOF) near an energy minimum.

In detail why this works out is explained in Box 19.3.

Examples of 1 DOF harmonic oscillators

In the previous section, we have shown that any non-dissipative one-degree-of-freedom system that is near a potential energy minimum can be expected to have simple harmonic motion. Besides the three examples we have given so far, namely,

- a spring and mass,
- a simple pendulum, and
- a rigid body and a torsional spring,

there are examples that are somewhat more complex, such as

- a cylinder rolling near the bottom of a valley,
- a cart rolling near the bottom of a valley, and a
- a four bar linkage swinging freely near its energy minimum.

The restriction of this theory to systems with only one-degree-of-freedom is not so bad as it seems at first sight. First of all, it turns out that sim-

19.3 One degree of freedom systems near a potential energy minimum are harmonic oscillators

In order to specialize to the case of oscillations, we want to look at a one degree of freedom system near a stable equilibrium point, a potential energy minimum.

At a potential energy minimum we have, as you will recall from 'max-min' problems in calculus, that $dE_{\rm P}(q)/dq = 0$. To keep our notation simple, let's assume that we have defined q so that $q = 0$ at this minimum. Physically this means that q measures how far the system is from its equilibrium position. That means that if we take a Taylor series approximation of the potential energy the expression for potential energy can be expressed as follows:

$$\begin{aligned} E_{\rm P} &\approx \text{const} + \underbrace{\frac{dE_{\rm P}}{dq}}_{0} \cdot q + \frac{1}{2} \underbrace{\frac{d^2E_{\rm P}}{dq^2}}_{K_{\rm equiv}} \cdot q^2 + \ldots \\ \Rightarrow \quad \frac{dE_{\rm P}}{dq} &\approx K_{\rm equiv} \cdot q \end{aligned} \tag{19.27}$$

Applying this result to equation(19.20) we get:

$$0 = K_{\rm equiv} q + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}^2 + M(q)\ddot{q}. \tag{19.28}$$

We now write $M(q)$ in terms of its Taylor series. We have

$$M(q) = M(0) + dM/dq|_0 \cdot q + \ldots \tag{19.29}$$

and substitute this result into equation 19.28. We have not finished using our assumption that we are only going to look at motions that are close to the equilibrium position $q = 0$ where q is small. The nature of motion close to an equilibrium is that when the deflections are small, the rates and accelerations are also small. Thus, to be consistent in our approximation we should neglect any terms that involve products of $q, \dot{q}$, or $\ddot{q}$. Thus the middle term involving $\dot{q}^2$ is negligibly smaller than other terms. Similarly, using the Taylor series for $M(q)$, the last term is well approximated by $M(0)\ddot{q}$, where $M(0)$ is a constant which we will call $M_{\rm equiv}$. Now we have for the equation of motion:

$$0 = \frac{d}{dt}E_{\rm T} \qquad \Rightarrow \qquad 0 = K_{\rm equiv} q + M_{\rm equiv}\ddot{q}, \tag{19.30}$$

which you should recognize as the harmonic oscillator equation. So we have found that for any energy conserving one degree of freedom system near a position of stable equilibrium, the equation governing small motions is the harmonic oscillator equation. The effective stiffness is found from the potential energy by $K_{\rm equiv} = d^2E_{\rm P}/dq^2$ and the effective mass is the coefficient of $\dot{q}^2/2$ in the expansion for the kinetic energy $E_{\rm K}$. The displacement of any part of the system from equilibrium will thus be given by

$$A\sin(\lambda t) + B\cos(\lambda t) \tag{19.31}$$

with $\lambda^2 = K_{\rm equiv}/M_{\rm equiv}$, and A and B determined by the initial conditions. So we have found that *all stable non-dissipative one-degree-of-freedom systems oscillate when disturbed slightly from equilibrium* and we have found how to calculate the frequency of vibration.

ple harmonic motion is important for systems with multiple-degrees-of-freedom. We will discuss this generalization in more detail later with regard to normal modes. Secondly, one can also get a good understanding of a vibrating system with multiple-degrees-of-freedom by modeling it as if it has only one-degree-of-freedom.

Example: Cylinder rolling in a valley
Consider the uniform cylinder with radius r rolling without slip in an cylindrical 'ideal' valley of radius R.

For this problem we can calculate E_K and E_P in terms of θ. Briefly,

$$\begin{aligned} E_P &= -mg(R-r)\cos\theta \\ E_K &= \frac{1}{2}\left(\frac{3}{2}mr^2\right)\left(\frac{\dot\theta(R-r)}{r}\right)^2 \\ &= \frac{3}{4}m(R-r)^2\dot\theta^2 \end{aligned}$$

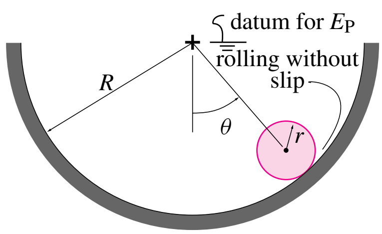

Figure 19.32: Cylinder rolling without slip in a cylinder. This gives the pendulum equation, but the effective pendulum length does not tend to R as $r \to 0$ (see text).

So we can derive the equation of motion using the fact of constant total energy.

$$\begin{aligned} 0 &= \frac{d}{dt}(E_T) = \frac{d}{dt}(E_K + E_P) \\ &= \frac{d}{dt}\left(\underbrace{-mg(R-r)\cos\theta}_{E_P} + \underbrace{\frac{3}{4}m(R-r)^2\dot\theta^2}_{E_K}\right) \\ &= (mg(R-r)\sin\theta)\dot\theta + \frac{3}{2}(R-r)^2\dot\theta\ddot\theta \\ \Rightarrow\quad 0 &= mg(R-r)\sin\theta + \frac{3}{2}(R-r)^2 m\ddot\theta \end{aligned}$$

Now, assuming small angles, so $\theta \approx \sin\theta$, we get

$$g(R-r)\theta + \frac{3}{2}(R-r)^2\ddot\theta = 0 \qquad (19.32)$$

$$\ddot\theta + \underbrace{\left(\frac{2}{3}\frac{g}{(R-r)}\right)}_{\lambda^2}\theta = 0 \qquad (19.33)$$

This is, naturally, our old friend the harmonic oscillator equation. The period is a funny combination of terms. If $r \ll R$ it looks like a point mass pendulum with length $3R/2$, more than R. That is, the rolling effect doesn't go away and make the roller act like a point mass even when the radius goes to zero. See page 889 for the angular momentum approach to this problem.

Although, in some abstract way the energy approach always works, practically speaking it has limitations for systems where the configuration is not easily found from one configuration variable.

Example: A four bar linkage analyzed with energy methods
Although probably not usually the best approach, energy methods can be used to find the motions of a 4-bar linkage. Take a four-bar linkage with one bar grounded. Assume the bars all have different lengths. This is a one-DOF system which can use the angle θ of one of the links as a configuration variable. But finding the potential energy as a single formula in terms of all of the links in terms of θ is more trigonometry than most of us like. And then finding the kinetic energy in terms of θ and $\dot\theta$ is close enough to impossible that people don't do it.

So, though it is true that there are functions $E_{\mathrm{P}}(\theta)$ and $E_{\mathrm{K}}(\theta, \dot{\theta})$ and that the equations of motion could be written in terms of them, it is really not practical to do so.

How do you find the motions of a 4-bar linkage in practice? It's more tricky. One approach is to solve the kinematics by integrating kinematic differential equations, as in Sample 18.5 on page 990. Then you set up and solve the balance equations of the separate parts as in Sample 19.21 on page 1051