SAMPLE 19.11 A plate pendulum. A $2a \times 2b$ rectangular plate of mass m hangs from two parallel, massless links EA and FD of length ℓ each. The links are hinged at both ends so that when the plate swings, its edges AD and BC remain horizontal at all times. The only driving force present is gravity. Find the equation of motion of the plate.

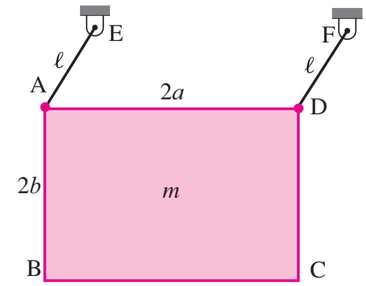

Figure 19.33

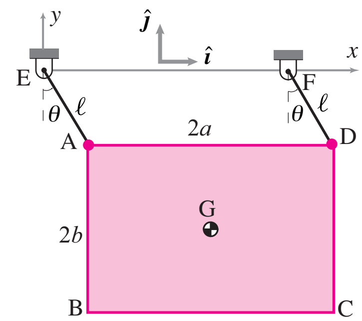

Figure 19.34

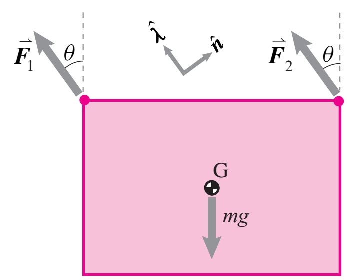

Figure 19.35

Solution The given system is a single DOF system. So, we need just one configuration variable and the equation of motion will be just one scalar equation in this variable. Let us take angle θ (fig. 19.34) as our configuration variable.

The free-body diagram of the plate is shown in fig. 19.35. Note that the link forces $\vec{F}_1$ and $\vec{F}_2$ act along the links because massless links are two-force bodies. Let (x, y) be the coordinates of the center-of-mass. Then the linear momentum balance for the plate gives

$$\vec{F}_1 + \vec{F}_2 - mg\hat{\jmath} = m\vec{a} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}).$$

Now, we can eliminate both the unknown link forces from this equation by dotting the equation with $\hat{n} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$, a unit vector normal to the links. Then, we have

$$\begin{aligned} -mg(\hat{\jmath}\cdot\hat{n}) &= m\left[\ddot{x}(\hat{\imath}\cdot\hat{n}) + \ddot{y}(\hat{\jmath}\cdot\hat{n})\right] \\ -g\sin\theta &= \ddot{x}\cos\theta + \ddot{y}\sin\theta. \end{aligned} \tag{19.34}$$

Now we need to find a relationship between x and θ, and y and θ, so that we can write $\ddot{x}$ and $\ddot{y}$ in terms of our configuration variable θ and its derivatives. From fig. 19.34, we have

$$\begin{aligned} \vec{r}_G &= \overbrace{(\ell\sin\theta + a)}^{x}\,\hat{\imath} + \overbrace{(-\ell\cos\theta - b)}^{y}\,\hat{\jmath} \\ \Rightarrow\quad \ddot{x} &= \ell(\cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2) \\ \ddot{y} &= \ell(\sin\theta\cdot\ddot{\theta} + \cos\theta\cdot\dot{\theta}^2) \end{aligned}$$

Substituting these expressions for $\ddot{x}$ and $\ddot{y}$ in eqn. (19.34) , we get

$$\begin{aligned} -g\sin\theta &= \ell\ddot{\theta} \\ \Rightarrow\quad \ddot{\theta} + \frac{g}{\ell}\sin\theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \tfrac{g}{\ell}\sin\theta = 0}$$

This is the equation of a simple pendulum! Well, the plate does behave just like a simple pendulum in the given mechanism. From the expressions for the x and y coordinates of the center-of-mass, we have

$$\begin{aligned} x - a &= \ell\sin\theta \\ y + b &= -\ell\cos\theta \\ \Rightarrow\quad (x-a)^2 &+ (y+b)^2 = \ell^2 \end{aligned}$$

that is, the center-of-mass follows a circle of radius ℓ centered at $(-a, b)$. Since the orientation of the plate never changes (AD and BC always remain horizontal), the plate has no angular velocity. Thus the motion of the plate is equivalent to the motion of a particle of mass m going in a circle centered at $(-a, b)$ and driven by gravity. That is the simple pendulum.

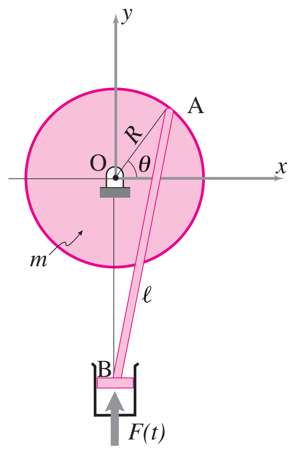

Figure 19.36

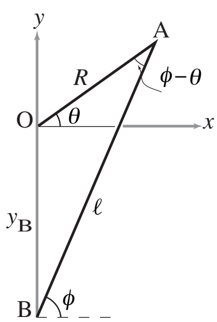

Figure 19.37

SAMPLE 19.12 Equation of motion from power balance. A slider crank mechanism is shown in fig. 19.36 where the crank is a uniform wheel of mass m and radius R anchored at the center and the connecting rod AB is a massless rod of length ℓ. The rod is driven by a piston at B with a known force $F(t) = F_0 \cos \Omega t$. There is no gravity. Find the equation of motion of the wheel.

Solution The given mechanism is a one DOF system. So, let us choose a single configuration variable, θ for specifying the configuration of the system and derive an equation that determines θ. Since the applied force is given and the point of application of the force has a simple motion (vertical), it will be easy to calculate power of this force. Also, the connecting rod is massless, so it does not enter into dynamic calculations. The wheel rotates about its center and, therefore, it is easy to calculate its kinetic energy. So, we use the power balance, $\dot{E}_K = P$, here to find the equation of motion of the wheel. Since $E_K = \frac{1}{2} I_{zz}^{cm} \dot{\theta}^2$ and $P = \vec{F} \cdot \vec{v}_B$, we have

$$I_{zz}^{cm} \ddot{\theta}\, \dot{\theta} = F(t)\hat{\jmath} \cdot v_B \hat{\jmath} = F(t) v_B \tag{19.35}$$

Now, we need to find v_B and express it using the configuration variable θ and its derivatives. There are several ways we could find v_B. Vectorially, we could write, $\vec{v}_B \equiv v_B \hat{\jmath} = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$ where $\vec{\omega}_{AB} = \dot{\phi}\hat{k}$. Dotting both sides of this equation with $\hat{\imath}$ and $\hat{\jmath}$ we can find $\dot{\phi}$ in terms of θ and v_B in terms of θ and $\dot{\theta}$. But, for a change, let us use geometry here.

See fig. 19.37. From triangle ABO, we have

$$\begin{aligned} \frac{R}{\sin(90^o - \phi)} &= \frac{\ell}{\sin(90^o + \theta)} \\ \Rightarrow \quad \ell \cos\phi &= R\cos\theta \\ \Rightarrow \quad -\ell \sin\phi \cdot \dot{\phi} &= -R\sin\theta \cdot \dot{\theta} \\ \dot{\phi} &= \frac{R\sin\theta}{\ell \sin\phi}\dot{\theta} \end{aligned}$$

Now,

$$\begin{aligned} y_B &= R\sin\theta - \ell\sin\phi \\ \Rightarrow \quad v_B &\equiv \dot{y}_B = R\cos\theta \cdot \dot{\theta} - \ell\cos\phi \cdot \dot{\phi} \\ &= R\dot{\theta}\cos\theta - R\cos\theta \cdot \frac{R\sin\theta}{\ell\sin\phi}\dot{\theta} \\ &= R\dot{\theta}\left(\cos\theta - \frac{R\sin\theta\cos\theta}{\sqrt{\ell^2 - R^2\cos^2\theta}}\right) \end{aligned}$$

Substituting this expression for v_B in power balance eqn. (19.35), we get

$$\begin{aligned} I_{zz}^{cm}\ddot{\theta}\dot{\phi} &= F(t) \cdot R\dot{\phi}\left(\cos\theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2\theta}}\right) \\ \ddot{\theta} &= \frac{RF_0\cos\Omega t \cdot}{\frac{1}{2}mR^2}\left(\cos\theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2\theta}}\right). \end{aligned}$$

This is the required equation of motion. As is evident, it is a nonlinear ODE which requires numerical solution on a computer if we would like to plot $\theta(t)$.

$$\ddot{\theta} = \frac{2F_0\cos\Omega t \cdot}{mR}\left(\cos\theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2\theta}}\right)$$

SAMPLE 19.13 **Instantaneous dynamics of slider crank.** A uniform rigid rod AB of mass m and length $\ell = 4R$ has one of its ends pinned to the rim of a disk of radius R. The other end of the bar is free to slide on a frictionless horizontal surface. A motor, connected to the center of the disk at O, keeps the disk rotating at a constant angular speed ω_D. At the instant shown, end B of the rod is directly above the center of the disk making θ to be 30^o.

1. Find all the forces acting on the rod.
2. Is there a value of ω_D which makes end A of the rod lift off the horizontal surface when $\theta = 30^o$?

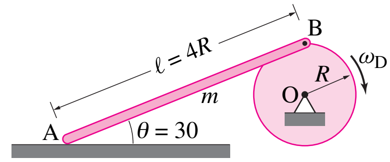

Figure 19.38: End A of bar AB is free to slide on the frictionless horizontal surface while end B is going in circles with a disk rotating at a constant rate.

Solution The disk is rotating at constant speed. Since end B of the rod is pinned to the disk, end B is going in circles at constant rate. The motion of end B of the rod is completely prescribed. Since end A can only move horizontally (assuming it has not lifted off yet), the orientation (and hence the position of each point) of the rod is completely determined at any instant during the motion. Therefore, the rod represents a zero degree of freedom system.

1. **Forces on the rod:** The free-body diagram of the rod is shown in Fig. 19.39. The pin at B exerts two forces B_x and B_y while the surface in contact at A exerts only a normal force N because there is no friction. Now, we can write the momentum balance equations for the rod. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the rod gives

$$B_x\hat{\imath} + (B_y + N - mg)\hat{\jmath} = m\vec{a}_G. \tag{19.36}$$

The angular momentum balance about the center-of-mass G ($\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$) of the rod gives

$$\vec{r}_{A/G} \times N\hat{\jmath} + \vec{r}_{B/G} \times (B_x\hat{\imath} + B_y\hat{\jmath}) = I_{zz/G}\,\alpha_{rod}\hat{k}. \tag{19.37}$$

From these two vector equations we can get three scalar equations (the Angular Momentum Balance gives only one scalar equation in 2-D since the quantities on both sides of the equation are only in the $\hat{k}$ direction), but we have six unknowns — B_x, B_y, N, $\vec{a}_G$ (counts as two unknowns), and α_{rod}. Therefore, we need more equations. We have already used the momentum balance equations, hence, the extra equations have to come from kinematics.

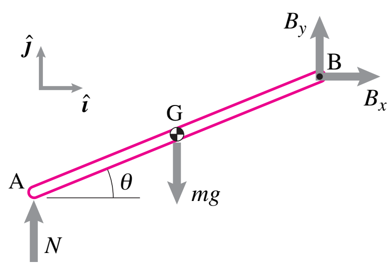

Figure 19.39: Free-body diagram of the bar.

$$\begin{aligned}
\vec{v}_A &= \vec{v}_B + \overbrace{\vec{\omega}_{rod} \times \vec{r}_{A/B}}^{\vec{v}_{A/B}} \\
\text{or} \qquad v_A\hat{\imath} &= \omega_D R\hat{\imath} + \omega_{rod}\hat{k} \times \ell(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= (\omega_D R + \omega_{rod}\ell\sin\theta)\hat{\imath} - \omega_{rod}\ell\cos\theta\hat{\jmath}
\end{aligned}$$

Dotting both sides of the equation with $\hat{\jmath}$ we get

$$0 = \omega_{rod}\ell\cos\theta \qquad \Rightarrow \qquad \omega_{rod} = 0.$$

Also,

$$\begin{aligned}
\vec{a}_A &= \vec{a}_B + \overbrace{\dot{\vec{\omega}} \times \vec{r}_{A/B} + \underbrace{\vec{\omega}_{rod}}_{\vec{0}} \times (\vec{\omega}_{rod} \times \vec{r}_{A/B})}^{\vec{a}_{A/B}} \\
\text{or} \qquad a_A\hat{\imath} &= -\omega_D^2 R\hat{\jmath} + \dot{\omega}_{rod}\hat{k} \times \ell(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -(\omega_D^2 R + \dot{\omega}_{rod}\ell\cos\theta)\hat{\jmath} + \dot{\omega}_{rod}\ell\sin\theta\hat{\imath}.
\end{aligned}$$

Dotting both sides of this equation by $\hat{\boldsymbol{\jmath}}$ we get

$$\dot{\omega}_{rod} = -\frac{\omega_D^2 R}{\ell \cos\theta}. \tag{19.38}$$

Now, we can find the acceleration of the center-of-mass:

$$\begin{aligned}
\vec{\boldsymbol{a}}_G &= \vec{\boldsymbol{a}}_B + \overbrace{\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{G/B} + \underbrace{\vec{\boldsymbol{\omega}}_{rod}}_{\vec{\mathbf{0}}} \times (\vec{\boldsymbol{\omega}}_{rod} \times \vec{\boldsymbol{r}}_{G/B})}^{\vec{\boldsymbol{a}}_{G/B}} \\
&= -\omega_D^2 R \hat{\boldsymbol{\jmath}} + \dot{\omega}_{rod} \hat{\boldsymbol{k}} \times \frac{1}{2}\ell(-\cos\theta \hat{\boldsymbol{\imath}} - \sin\theta \hat{\boldsymbol{\jmath}}) \\
&= -(\omega_D^2 R + \frac{1}{2}\dot{\omega}_{rod} \ell \cos\theta) \hat{\boldsymbol{\jmath}} + \frac{1}{2}\dot{\omega}_{rod} \ell \sin\theta \hat{\boldsymbol{\imath}}.
\end{aligned}$$

Substituting for $\dot{\omega}_{rod}$ from eqn. (19.38) and 30^o for θ above, we obtain

$$\vec{\boldsymbol{a}}_G = -\frac{1}{2}\omega_D^2 R(\frac{1}{\sqrt{3}}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).$$

Substituting this expression for $\vec{\boldsymbol{a}}_G$ in eqn. (19.36) and dotting both sides by $\hat{\boldsymbol{\imath}}$ and then by $\hat{\boldsymbol{\jmath}}$ we get

$$\begin{aligned}
B_x &= -\frac{1}{2\sqrt{3}} m\omega_D^2 R, \\
B_y + N &= -\frac{1}{2} m\omega_D^2 R + mg
\end{aligned} \tag{19.39}$$

From eqn. (19.37)

$$\begin{aligned}
\frac{1}{2}\ell[(B_y - N)\cos\theta - B_x \sin\theta]\hat{\boldsymbol{k}} &= \frac{1}{12} m\ell^2 (-\frac{\omega_D^2 R}{\ell \cos\theta})\hat{\boldsymbol{k}} \\
\text{or} \qquad B_y - N &= -\frac{1}{6}\frac{m\omega_D^2 R}{\cos^2\theta} + B_x \tan\theta \\
&= -\frac{2}{9} m\omega_D^2 R - \frac{1}{6} m\omega_D^2 R \\
&= -\frac{7}{18} m\omega_D^2 R
\end{aligned} \tag{19.40}$$

From eqns. (19.39) and (19.40)

$$B_y = \frac{1}{2}\left(mg - \frac{8}{9} m\omega_D^2 R\right)$$

and

$$N = \frac{1}{2}\left(mg - \frac{1}{9} m\omega_D^2 R\right).$$

2. **Lift off of end A:** End A of the rod loses contact with the ground when normal force N becomes zero. From the expression for N from above, this condition is satisfied when

$$\begin{aligned}
\frac{2}{9} m\omega_D^2 R &= mg \\
\Rightarrow \qquad \omega_D &= 3\sqrt{\frac{g}{R}}.
\end{aligned}$$

SAMPLE 19.14 A bar sliding on a sliding wedge. A bar AB of mass m and length ℓ is hinged at end A and rests on a wedge of mass M at the other end B. The contact at B is frictionless. The wedge is free to slide horizontally without any friction. The motion of the system is driven only by gravity. Find the equation of motion of the bar using

1. momentum balance
2. energy (or power) balance.

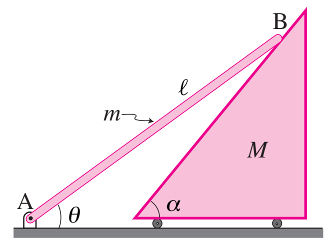

Figure 19.40

Solution (a) **Momentum Balance:** The bar and the wedge make up a single DOF system. To derive the equations of motion of the bar, let us choose θ as the configuration variable. The free-body diagram of the bar and the wedge are shown in fig. 19.41. Note that the normal reaction at B is normal to the wedge surface, *i.e.,* $\vec{\boldsymbol{N}} = N\hat{\boldsymbol{n}}$. Now, the angular momentum balance for the bar about point A gives,

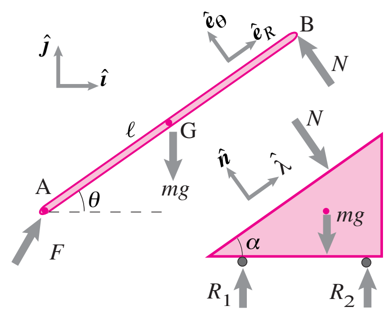

Figure 19.41

$$
\begin{aligned}
\dot{\boldsymbol{H}}_{\mathrm{A}} &= \sum \vec{\boldsymbol{M}}_{\mathrm{A}} \\
I_{zz}^{A}\ddot{\theta}\hat{\boldsymbol{k}} &= \frac{\ell}{2}\hat{\boldsymbol{e}}_R \times (-mg\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{e}}_R \times N\hat{\boldsymbol{n}} \\
&= -\frac{1}{2}mg\ell\cos\theta\hat{\boldsymbol{k}} + N\ell\cos(\alpha-\theta)\hat{\boldsymbol{k}}
\end{aligned}
\qquad (19.41)
$$

where the last line follows from the fact that $\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$, the unit normal $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\boldsymbol{\imath}} + \cos\alpha\hat{\boldsymbol{\jmath}}$, and so, $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{\jmath}} = \cos\theta\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{n}} = \cos(\alpha-\theta)\hat{\boldsymbol{k}}$. We now need to eliminate the unknown normal reaction N from the above equation. Since the wedge is constrained to move only horizontally, we can write the linear momentum balance for the wedge as

$$
N\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\imath}} = M\ddot{x} \quad\Rightarrow\quad N = \frac{M\ddot{x}}{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\imath}}} = \frac{M\ddot{x}}{\sin\alpha} \qquad (19.42)
$$

Thus, we have found N in terms of $\ddot{x}$ that we can use in eqn. (19.41) to get rid of N. But, we now need to express $\ddot{x}$ in terms of our configuration variable θ and its derivatives. Consider the triangle ABC formed by the bar and the slanted edge of the wedge. Let $x =$ AC denote the horizontal position of the wedge. Then, from the law of sines, we have $\frac{x}{\sin(\alpha-\theta)} = \frac{\ell}{\sin\alpha}$, so that

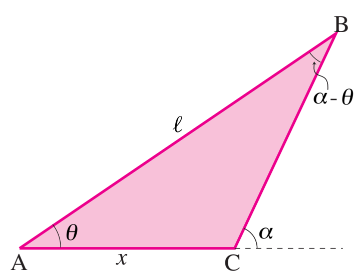

Figure 19.42

$$
\begin{aligned}
x &= \frac{\ell}{\sin\alpha}\sin(\alpha-\theta) \\
\Rightarrow\quad \dot{x} &= \frac{\ell}{\sin\alpha}\cos(\alpha-\theta)\cdot(-\dot{\theta}) && (19.43)\\
\Rightarrow\quad \ddot{x} &= \frac{\ell}{\sin\alpha}\left[\ddot{\theta}\cos(\alpha-\theta) + \dot{\theta}^2\sin(\alpha-\theta)\right]. && (19.44)
\end{aligned}
$$

Now, substituting for N in eqn. (19.41) from eqn. (19.42), using the expression for $\ddot{x}$ from above, and dotting the resulting equation with $\hat{\boldsymbol{k}}$, we get

$$
\begin{aligned}
\frac{1}{3}m\ell^2\ddot{\theta} &= -\frac{mg\ell}{2}\cos\theta + \frac{M\ell^2\cos(\alpha-\theta)}{\sin^2\alpha}\left[\ddot{\theta}\cos(\alpha-\theta) + \dot{\theta}^2\sin(\alpha-\theta)\right] \\
\Rightarrow\quad \ddot{\theta} &= -\frac{3mg\sin^2\alpha\cos\theta + 3M\ell\dot{\theta}^2\sin 2(\alpha-\theta)}{2m\ell\sin^2\alpha + 6M\ell\cos^2(\alpha-\theta)} \\
&= -\frac{3\left[(g/\ell)\sin^2\alpha\cos\theta + (M/m)\dot{\theta}^2\sin 2(\alpha-\theta)\right]}{2\left[\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)\right]} && (19.45)
\end{aligned}
$$

$$
\boxed{\ddot{\theta} = -\frac{3\left[(g/\ell)\sin^2\alpha\cos\theta + (M/m)\dot{\theta}^2\sin 2(\alpha-\theta)\right]}{2\left[\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)\right]}}
$$

(b) **Power balance:** Now we derive the equation of motion for the bar using power balance $\dot{E}_{\mathrm{K}} = P$. For power balance, we have to consider both the bar and the wedge. The bar rotates about the fixed point A, therefore, its kinetic energy is $(1/2)I_z^A z\dot{\theta}^2$. The wedge moves with horizontal speed $\dot{x}$, therefore, its kinetic energy is $(1/2)M\dot{x}^2$. Thus, $E_{\mathrm{K}} = 1/2)I_z^A z\dot{\theta}^2 + (1/2)M\dot{x}^2$. The only force that contributes to power is the force of gravity on the rod because $\vec{\boldsymbol{v}}_{\mathrm{G}} \cdot (-mg\hat{\boldsymbol{\jmath}})$ is non-zero. The sliding contact at B is frictionless and hence the net power due to the contact force there is zero. Now, $\vec{\boldsymbol{v}}_{\mathrm{G}} = \frac{1}{2}\ell\dot{\theta}\hat{\boldsymbol{e}}_\theta$. So, $P = -\frac{1}{2}mg\ell\dot{\theta}(\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{\jmath}}) = -\frac{1}{2}mg\ell\dot{\theta}\cos\theta$. Thus the power balance for the system gives

$$I_{zz}^A\ddot{\theta}\dot{\theta} + M\ddot{x}\dot{x} = -\frac{1}{2}mg\ell\dot{\theta}\cos\theta. \tag{19.46}$$

We can simplify this equation further. Note that, $\ddot{\theta}\dot{\theta} = \frac{d}{dt}(\frac{1}{2}\dot{\theta}^2)$ and $\ddot{x}\dot{x} = \frac{d}{dt}(\frac{1}{2}\dot{x}^2)$. So that we can write the above equation as

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}I_{zz}^A\dot{\theta}^2\right) + \frac{d}{dt}\left(\frac{1}{2}M\dot{x}^2\right) &= -\frac{1}{2}mg\ell\dot{\theta}\cos\theta \\ \text{or} \qquad \int d\big(I_{zz}^A\dot{\theta}^2 + M\dot{x}^2\big) &= -mg\ell\int\cos\theta\, d\theta \\ \Rightarrow \quad I_{zz}^A\dot{\theta}^2 + M\dot{x}^2 &= C - mg\ell\sin\theta \end{aligned}$$

where C is a constant of integration to be determined from initial conditions. For example, when the bar begins to slide from rest, we have, at $t = 0$, $\dot{x} = 0$ and $\dot{\theta} = 0$. At that instant, if $\theta(0) = \theta_0$, then $C = mg\ell\sin\theta_0$. So, we can write

$$I_{zz}^A\dot{\theta}^2 + M\dot{x}^2 = mg\ell(\sin\theta_0 - \sin\theta).$$

Now, replacing $\dot{x}$ in this equation with the expression we obtained in eqn. (19.43), we have

$$\begin{aligned} I_{zz}^A\dot{\theta}^2 + M\ell^2\dot{\theta}^2\frac{\cos^2(\alpha-\theta)}{\sin^2\alpha} &= mg\ell(\sin\theta_0 - \sin\theta) \\ \Rightarrow \quad \dot{\theta}^2 &= \frac{mg\ell(\sin\theta_0 - \sin\theta)}{I_{zz}^A + M\ell^2\frac{\cos^2(\alpha-\theta)}{\sin^2\alpha}} \\ &= \frac{mg\ell\sin^2\alpha(\sin\theta_0 - \sin\theta)}{(1/3)m\ell^2\sin^2\alpha + M\ell^2\cos^2(\alpha-\theta)} \\ \Rightarrow \quad \dot{\theta} &= \sqrt{\frac{3g}{\ell}}\sin\alpha\sqrt{\frac{\sin\theta_0 - \sin\theta}{\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)}}. \end{aligned}$$

This is a first order, nonlinear, ODE compared to the second order equation we got in eqn. (19.45). However, again, we need to resort to numerical solution if we wish to solve for $\theta(t)$, in which case, this reduction to the first order equation does not save much work.

$$\boxed{\dot{\theta} = \sqrt{\frac{3g}{\ell}}\sin\alpha\sqrt{\frac{\sin\theta_0 - \sin\theta}{\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)}}}$$