

19.2 Mechanics of one-degree-of-freedom 2-D mechanisms

A one-degree-of-freedom mechanism is a collection of parts linked together so that they can move in only one way^①. The word “freedom” has to be taken lightly here because in practice even the one “freedom” is often controlled or restricted. Frankly, most machine designers don’t trust the laws of mechanics to enforce motions that they want. Instead they choose kinematic restrictions that enforce the desired motions and then use a motor, a computer controlled actuator or big flywheel to keep that motion moving at a prescribed rate.

We consider here machines that can move in just one way, whether or not that one motion is free. So in this sense, “one”-degree-of-freedom machines include machines with no freedom at all, just so long as they move in only one way.

Some familiar examples of one-degree-of-freedom mechanisms are a 1-D spring and mass, a pendulum, a slider-crank, a grounded 4-bar linkage, and a gear train.

Most ideal constraints are workless constraints

A fruitful equation for studying one-degree-of-freedom mechanisms is power balance, or for conservative systems, energy balance. The reason these equations are so useful is because most ideal connections are workless. That is:

the net work of the interaction forces (and moments) of a pair of parts that are connected with the standard ideal connections is zero. This includes welds, frictionless hinges, frictionless sliding contact, rolling contact, or parts connected by a massless inextensible link.

Example: A frictionless hinge is a workless constraint

Body $\mathcal{A}$ is connected to body $\mathcal{B}$ by a frictionless hinge at C (see fig. 19.27). The force on body $\mathcal{B}$ at C from $\mathcal{A}$ is $\vec{F}_C$ and the force on $\mathcal{A}$ from body $\mathcal{B}$ at C is $-\vec{F}_C$. The power of the interaction force on body $\mathcal{B}$ is $P_{\mathcal{A}on\mathcal{B}} = \vec{F}_C \cdot \vec{v}_C$. This power contributes to the increase in the kinetic energy of $\mathcal{B}$. The power of the interaction force on $\mathcal{A}$ is $P_{\mathcal{B}on\mathcal{A}} = -\vec{F}_C \cdot \vec{v}_C = -P_{\mathcal{A}on\mathcal{B}}$. So the contribution to the increase in the kinetic energy of body $\mathcal{A}$ is minus the contribution to body

①No physical system can move in only one way. The idea that a machine has one degree of freedom is an idealization that takes the rigid-object description of the parts and the ideal nature of the connections literally. In fact, all parts deform at least a little, and no connections are so precise as to be exact geometric constraints. It may be reasonable to respect the standard rigid-object idealizations and consider a machine as one-degree-of-freedom for basic analysis, as we do in this section. When, for example, trying to figure out why a machine vibrates in an undesirable way it may also be reasonable to relax some of the assumptions we make here and consider more degrees of freedom.

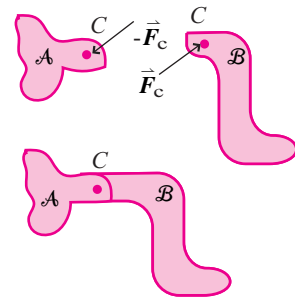


Figure 19.27: A frictionless hinge is a workless constraint. The *net* work of the interaction force on the two contacting bodies is zero.

$\mathcal{B}$ and the net power on the system of two bodies is zero. Writing this out,

The net power of the pair of interaction forces on the pair of bodies

$$\begin{aligned} \overbrace{P_{\text{total}}} &= P_{\mathcal{B} \text{ on } \mathcal{A}} + P_{\mathcal{A} \text{ on } \mathcal{B}} \\ &= \vec{\mathbf{F}}_C \cdot \vec{\mathbf{v}}_C + (-\vec{\mathbf{F}}_C) \cdot \vec{\mathbf{v}}_C \\ &= (\vec{\mathbf{F}}_C - \vec{\mathbf{F}}_C) \cdot \vec{\mathbf{v}}_C \\ &= 0. \end{aligned}$$

Basically the same situation holds for all the standard ideal connections as explained in the box on page 2.

If one of two interacting bodies is known to be stationary, like the ground, then the work of the constraint forces is zero on both of the bodies. Thus the work of the hinge force on a pendulum, and the ground reaction forces on a frictionlessly sliding body or the ground force on a perfectly rolling body is zero. But be careful with the words “workless constraint forces”, however.

The workless constraint connecting moving bodies $\mathcal{A}$ and $\mathcal{B}$ is likely to do positive work on one of the bodies and negative work on the other.

It is just the net work on the two bodies which is zero.

Energy method: single degree of freedom systems

Although linear and angular momentum balance apply to a single degree of freedom system and all of its parts, often one finds what one wants with a single scalar equation, namely energy or power balance.

19.2 Ideal constraints and workless constraints

All of the ideal constraints we consider are interactions between two bodies $\mathcal{A}$ and $\mathcal{B}$. One of these could be the ground. Let's take the interaction force $\vec{\mathbf{F}}$ and moment $\vec{\mathbf{M}}$ to be the force and moment of $\mathcal{A}$ on $\mathcal{B}$. The point of interaction is A on $\mathcal{A}$ and B on $\mathcal{B}$. By the principle of action and reaction, the net power of the interaction force on the two bodies is

$$\begin{aligned} P &= \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}_B + \vec{\mathbf{M}} \cdot \vec{\boldsymbol{\omega}}_B + (-\vec{\mathbf{F}}) \cdot \vec{\mathbf{v}}_A + (-\vec{\mathbf{M}}) \cdot \vec{\boldsymbol{\omega}}_A \\ &= \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}_{B/A} + \vec{\mathbf{M}} \cdot \vec{\boldsymbol{\omega}}_{B/A}. \end{aligned}$$

All of our ideal constraints are designed to exactly make these dot products zero. The ideal hinge is considered in the text. Another

example is perfect rolling. In that case the interaction moment is assumed to be zero. The no-slip condition means that $\vec{\mathbf{v}}_{B/A} = \mathbf{0}$. On the other hand for frictionless sliding there $\vec{\mathbf{v}}_{B/A}$ can have a component tangent to the surfaces. But that is exactly the direction where the friction force is assumed to be zero.

And so it is for all of the ideal “workless” constraints.

Examples of non-workless constraints, that is, interactions that contribute to the energy equations are: sliding with non-zero friction, joints with non-zero friction torques, joints with motors, or interactions mediated by springs, dampers or actuators.

Imagine a complex machine that only has one degree of freedom, meaning the position of the whole machine is determined by a single configuration variable, call it q . Further assume that the machine has no motion when $\dot{q} = 0$. The variable q could be, for example, the angle of one of the linked-together machine parts. Also, assume that the machine has no dissipative parts: no friction, no collisions, no inelastic deformation. Because q characterizes the position of all of the parts of the system we can, in principle, calculate the potential energy of the system as a function of q ,

$$E_p = E_p(q).$$

We find this function by adding up the potential energies of all the springs in the machine and the gravitational potential energies of the parts. Similarly we can write the system's kinetic energy in terms of q and its rate of change $\dot{q}$. Because at any configuration the velocity of every point in the system is proportional to $\dot{q}$ we can write the kinetic energy as:

$$E_K = M(q)\dot{q}^2/2$$

where $M(q)$ is a function that one can determine by calculating the machine's total kinetic energy in terms of q and $\dot{q}$ and then factoring $\dot{q}^2$ out of the resulting expression.

Now, if we accept the equation of mechanical energy conservation we have

constant = E_T conservation of energy,

$$\begin{aligned} \Rightarrow 0 &= \frac{d}{dt}E_T \quad \text{taking one time derivative,} \\ &= \frac{d}{dt}[E_p + E_K] \quad \text{total energy is potential plus kinetic} \\ &= \frac{d}{dt}[E_p(q) + \frac{1}{2}M(q)\dot{q}^2] \quad \text{substituting from paragraphs above} \\ \Rightarrow 0 &= \frac{d}{dq}[E_p(q)]\dot{q} + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}\dot{q}^2 + M(q)\dot{q}\ddot{q} \\ 0 &= \frac{d}{dq}[E_p(q)] + \left(\frac{1}{2}\frac{d}{dq}[M(q)]\right)\dot{q}^2 + M(q)\ddot{q} \quad \text{cancelling } \dot{q} \\ 0 &= f_1(q) + f_2(q)\dot{q}^2 + f_3(q)\ddot{q} \end{aligned} \quad (19.20)$$

$$\text{with } f_1(q) \equiv \frac{d}{dq}[E_p(q)],$$

$$f_2(q) \equiv \frac{1}{2}\frac{d}{dq}[M(q)], \quad \text{and}$$

$$f_3(q) \equiv M(q).$$

The cancellation of $\dot{q}$ above lacks mathematical rigor, but doesn't cause problems^②. The equation of motion is complicated because when we take the time derivative of a function of $M(q)$ and $E_p(q)$ we have to use the chain rule. Also, because we have products of terms, we had to use the

② The cancellation of the factor $\dot{q}$ from equation 19.20 depends on $\dot{q}$ being other than zero. While moving, $\dot{q}$ is not zero. Strictly we cannot cancel the $\dot{q}$ term from the equation at the instants when $\dot{q} = 0$. However, to say that a differential equation is true except for certain instants in time is, in practice to say that it is always true, at least if we make reasonable assumptions about the smoothness of the motions.

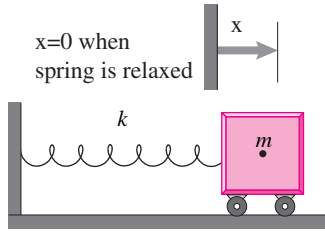


Figure 19.28: The familiar one degree of freedom spring and mass system.

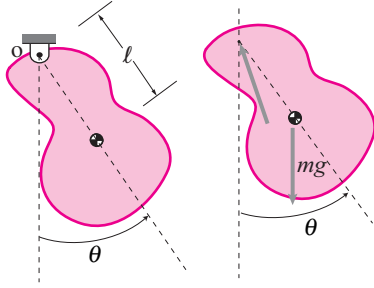


Figure 19.29: A rigid body suspended from a frictionless hinge is an energy conserving one-degree-of-freedom mechanism.

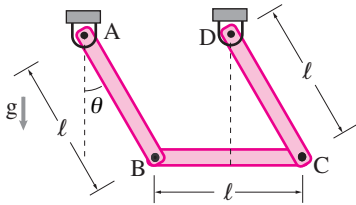


Figure 19.30: Three identical uniform bars are pinned with frictionless hinges and swing, obviously, something like a simple pendulum.

product rule. Eqn. 19.20 is the general equation of motion of a conservative one-degree-of-freedom system. It is really just a special case of the equation of motion for one-degree-of-freedom systems found from power balance. Rather than memorizing eqn. (19.20) it is probably best to look at its derivation as an algorithm to be reproduced on a problem by problem basis.

Example: Spring and mass

Although the motion of a spring and mass system can be found easily enough from linear momentum balance, it is also a good example for energy balance (see fig. 19.28). Using conservation of energy for the spring and mass system:

$$\begin{aligned}
 E_T &= \text{constant} \\
 0 &= \frac{d}{dt} E_T \\
 &= \dot{E}_K + \dot{E}_P \\
 &= \frac{d}{dt} (mv^2/2) + \frac{d}{dt} (kx^2/2) \\
 &= mv\dot{v} + kx\dot{x} \\
 v = \dot{x} \quad \Rightarrow \quad 0 &= m\ddot{x} + kx.
 \end{aligned}$$

Similarly power balance could have been used to get the same result, looking at just the mass

$$\begin{aligned}
 P &= \frac{d}{dt} E_K \\
 (-kx)(\dot{x}) &= \frac{d}{dt} (mv^2/2) = mv\dot{v} \\
 \Rightarrow 0 &= kx + m\ddot{x}
 \end{aligned}$$

as before.

Example: Pendulum

Consider a rigid body with mass m and moment of inertia I^o about a hinge which is a distance ℓ from the center-of-mass (see fig. 19.29). The familiar simple pendulum is another single degree of freedom system for which the equation of motion can be found from conservation of energy.

$$\begin{aligned}
 E_T &= \text{constant} \\
 \rightarrow 0 &= \frac{d}{dt} E_T = \dot{E}_K + \dot{E}_P \\
 &= \frac{d}{dt} (I^o \omega^2/2) + \frac{d}{dt} (-mg\ell \cos \theta) \\
 &= I^o \omega \dot{\omega} + -mg\ell (-\sin \theta) \dot{\theta} \\
 \omega = \dot{\theta} \quad \Rightarrow \quad 0 &= \ddot{\theta} + \frac{mg\ell}{I^o} \sin \theta.
 \end{aligned}$$

the pendulum equation that we have derived before by this and other means (angular momentum balance about point o).

The above examples are old friends which are handled easily with other techniques. Here is a problem which is much more difficult without the energy method.

Example: Three bars act like a simple pendulum

Assume all three bars in the structure shown in fig. 19.30 are of equal length ℓ

and have mass m uniformly distributed along their length. It is intuitively obvious that this device swings back and forth something like a simple pendulum. But how can we get the laws of mechanics to tell us this? One approach, which will work in the end, is to draw free-body diagrams of all the parts, write linear and angular momentum balance for each, and then add and subtract equations to eliminate the unknown constraint forces at the various hinges.

The more direct approach is to write the energy equation, adding up the potential and kinetic energies of the parts, all evaluated in terms of the single configuration variable θ . Taking the potential energy to be zero at $\theta = \pi/2$ (when all centers of mass are at hinge height) we have

$$\begin{aligned}
 E_T &= \text{constant} \\
 0 &= \frac{d}{dt} E_T = \dot{E}_P + \dot{E}_K \\
 &= \frac{d}{dt} \left((I^o \omega^2 / 2) + (I^o \omega^2 / 2) + (m(\ell \omega)^2 / 2) \right) \\
 &\quad + \frac{d}{dt} \left((-gm(\ell/2) \cos \theta - gm(\ell/2) \cos \theta) - gm\ell \cos \theta \right) \\
 I^o = m\ell^2/3 \quad \Rightarrow \quad 0 &= \frac{d}{dt} \left(5m\ell^2 \omega^2 / 6 \right) + \frac{d}{dt} \left(-2gm\ell \cos \theta \right) \\
 &= (5m\ell^2 \omega \dot{\omega} / 3 + 2gm\ell \omega \sin \theta) \\
 \Rightarrow \quad 0 &= 5\ell \dot{\omega} / 3 + 2g \sin \theta \\
 \Rightarrow \quad 0 &= \ddot{\theta} + \frac{6g}{5\ell} \sin \theta
 \end{aligned}$$

which is the same governing equation as for a point-mass pendulum with length $5\ell/6$. This is just half way between the following two cases. If the side links had no mass the equation would have been the same as for a point mass pendulum with length ℓ

$$0 = \ddot{\theta} + \frac{g}{\ell} \sin \theta$$

and if the bottom link had no mass the equation would be the same as a stick hanging from one end which goes back and forth like a point mass pendulum with length $2\ell/3$ according to

$$0 = \ddot{\theta} + \frac{3g}{2\ell} \sin \theta.$$

Example: One on the rim is like two on the frame.

A bicycle transmission is such that the speed of the bike relative to the ground is n times the speed of the pedal relative to the frame :

$$v_{\text{bike}} = n v_{\text{pedal/bike}}$$

Assume the kinetic energy of the relative motion of a rider's legs can be neglected, as can be the weight of the rider's leg. At the moment in question the velocity of the pedal is parallel to the direction from the seat to the leg. Thus the free-body diagram of the bike/person system, leaving out the pedaling leg is as shown in fig. 19.31. Let's assume the bike and rider have mass M and that the wheels have mass m_r and m_f concentrated on the rim (the hubs are considered part of the frame and the spokes are neglected). Neglecting air resistance *etc.* the power balance equation is:

$$P = \dot{E}_K \quad (19.21)$$

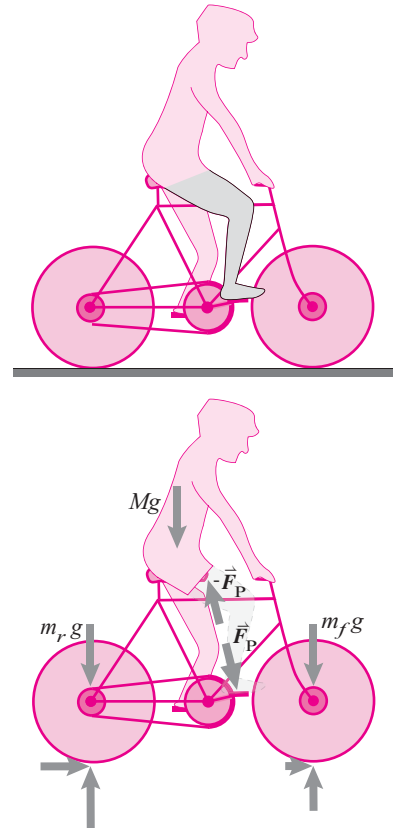


Figure 19.31: A person rides a bike. The pedaling leg is idealized as a pair of equal and opposite forces acting on the seat and pedal.

Let's do some side calculations for evaluating the terms in eqn. (19.21). First, the only forces that do work on the system as drawn are the force on the pedal and the force on the seat.

$$\begin{aligned}
 P &= -\vec{F}_p \cdot \vec{v}_{\text{seat}} + \vec{F}_p \cdot \vec{v}_{\text{pedal}} \\
 &= -\vec{F}_p \cdot \vec{v}_{\text{seat}} + \vec{F}_p \cdot (\vec{v}_{\text{bike}} + \vec{v}_{\text{pedal/bike}}) \\
 &= \underbrace{-\vec{F}_p \cdot \vec{v}_{\text{seat}} + \vec{F}_p \cdot \vec{v}_{\text{bike}}}_0 + \underbrace{\vec{F}_p \cdot \vec{v}_{\text{pedal/bike}}}_{F_p v_{\text{pedal/bike}}} \\
 &= F_p v_{\text{bike}}/n
 \end{aligned}$$

The net power of the leg is expressed by the compression it carries times its extension rate. The kinetic energy of the wheel comes from both its rotation and its translation. The moment of inertia of a hoop about its center is $I = mr^2$. For rolling contact $|\omega R| = v$ so, for *one* wheel:

$$\begin{aligned}
 E_{\text{Kwheel}} &= mv_{\text{bike}}^2/2 + I\omega^2/2 \\
 &= mv_{\text{bike}}^2/2 + (mR^2)(v/R)^2/2 \\
 &= mv_{\text{bike}}^2.
 \end{aligned}$$

The kinetic energy of a rolling hoop is twice that of a point mass moving at the same speed. Putting these results back in to eqn. (19.21) we have

$$P = \dot{E}_K \quad (19.22)$$

$$F_p v_{\text{bike}}/n = \frac{d}{dt} (Mv_{\text{bike}}^2/2 + (m_r + m_f)v_{\text{bike}}^2) \quad (19.23)$$

$$= (M\dot{v}_{\text{bike}} + 2(m_r + m_f)\dot{v}_{\text{bike}}) v_{\text{bike}} \quad (19.24)$$

$$\Rightarrow F_p/n = (M + 2(m_r + m_f))\dot{v}_{\text{bike}} \quad (19.25)$$

$$\Rightarrow \dot{v}_{\text{bike}} = \frac{F_p}{n(M + 2(m_r + m_f))}. \quad (19.26)$$

The bigger the pedal force, the bigger the acceleration, obviously. The higher the gear ratio, the less the acceleration; the faster gears let you pedal slower for a given bike speed, but demand more pedal force for a given acceleration. A heavier bike accelerates less. But the contribution to slowing a bike is twice as much for mass added to the rim as for mass added to the frame or body.

Some comments. n typically ranges from about 1.7 to 8 for a new 21 speed bike and is about 5 for an adult European, Indian or Chinese 1-speed. For a given speed of bicycle riding your feet go n times slower relative to your body than for walking or running at that speed. This calculation is for accelerating a bike on level ground with no wind and rolling resistance. The net speed of a bike in a bike race is not so dependent on weight, because the main enemy is wind resistance. To the extent that weight is a problem it is for steady uphill travel. In this case the mass on the rim makes the same contribution as mass on the frame.

Vibrations

The preponderance of systems where vibrations occur is not due to the fact that so many systems look like a spring connected to a mass, a simple pendulum, or a torsional oscillator. Instead there is a general class of systems which can be expected to vibrate sinusoidally near some equilibrium position. These systems are one-degree-of-freedom (one DOF) near an energy minimum.

In detail why this works out is explained in Box 19.3.

Examples of 1 DOF harmonic oscillators

In the previous section, we have shown that any non-dissipative one-degree-of-freedom system that is near a potential energy minimum can be expected to have simple harmonic motion. Besides the three examples we have given so far, namely,

- a spring and mass,
- a simple pendulum, and
- a rigid body and a torsional spring,

there are examples that are somewhat more complex, such as

- a cylinder rolling near the bottom of a valley,
- a cart rolling near the bottom of a valley, and a
- a four bar linkage swinging freely near its energy minimum.

The restriction of this theory to systems with only one-degree-of-freedom is not so bad as it seems at first sight. First of all, it turns out that sim-

19.3 One degree of freedom systems near a potential energy minimum are harmonic oscillators

In order to specialize to the case of oscillations, we want to look at a one degree of freedom system near a stable equilibrium point, a potential energy minimum.

At a potential energy minimum we have, as you will recall from 'max-min' problems in calculus, that $dE_p(q)/dq = 0$. To keep our notation simple, let's assume that we have defined q so that $q = 0$ at this minimum. Physically this means that q measures how far the system is from its equilibrium position. That means that if we take a Taylor series approximation of the potential energy the expression for potential energy can be expressed as follows:

$$E_p \approx \underbrace{\text{const}}_0 + \underbrace{\frac{dE_p}{dq}}_0 \cdot q + \frac{1}{2} \underbrace{\frac{d^2E_p}{dq^2}}_{K_{\text{equiv}}} \cdot q^2 + \dots$$

$$\Rightarrow \frac{dE_p}{dq} \approx K_{\text{equiv}} \cdot q \quad (19.27)$$

Applying this result to equation (19.20) we get:

$$0 = K_{\text{equiv}}q + \frac{1}{2} \frac{d}{dq} [M(q)] \dot{q}^2 + M(q) \ddot{q}. \quad (19.28)$$

We now write $M(q)$ in terms of its Taylor series. We have

$$M(q) = M(0) + dM/dq|_0 \cdot q + \dots \quad (19.29)$$

and substitute this result into equation 19.28. We have not finished using our assumption that we are only going to look at motions that are close to the equilibrium position $q = 0$ where q

is small. The nature of motion close to an equilibrium is that when the deflections are small, the rates and accelerations are also small. Thus, to be consistent in our approximation we should neglect any terms that involve products of $q, \dot{q}$, or $\ddot{q}$. Thus the middle term involving $\dot{q}^2$ is negligibly smaller than other terms. Similarly, using the Taylor series for $M(q)$, the last term is well approximated by $M(0)\ddot{q}$, where $M(0)$ is a constant which we will call M_{equiv} . Now we have for the equation of motion:

$$0 = \frac{d}{dt} E_T \Rightarrow 0 = K_{\text{equiv}}q + M_{\text{equiv}}\ddot{q}, \quad (19.30)$$

which you should recognize as the harmonic oscillator equation. So we have found that for any energy conserving one degree of freedom system near a position of stable equilibrium, the equation governing small motions is the harmonic oscillator equation. The effective stiffness is found from the potential energy by $K_{\text{equiv}} = d^2E_p/dq^2$ and the effective mass is the coefficient of $\ddot{q}^2/2$ in the expansion for the kinetic energy E_K . The displacement of any part of the system from equilibrium will thus be given by

$$A \sin(\lambda t) + B \cos(\lambda t) \quad (19.31)$$

with $\lambda^2 = K_{\text{equiv}}/M_{\text{equiv}}$, and A and B determined by the initial conditions. So we have found that *all stable non-dissipative one-degree-of-freedom systems oscillate when disturbed slightly from equilibrium* and we have found how to calculate the frequency of vibration.

ple harmonic motion is important for systems with multiple-degrees-of-freedom. We will discuss this generalization in more detail later with regard to normal modes. Secondly, one can also get a good understanding of a vibrating system with multiple-degrees-of-freedom by modeling it as if it has only one-degree-of-freedom.

Example: Cylinder rolling in a valley

Consider the uniform cylinder with radius r rolling without slip in an cylindrical ‘ideal’ valley of radius R .

For this problem we can calculate E_K and E_P in terms of θ . Briefly,

$$\begin{aligned} E_P &= -mg(R-r)\cos\theta \\ E_K &= \frac{1}{2}\left(\frac{3}{2}mr^2\right)\left(\frac{\dot{\theta}(R-r)}{r}\right)^2 \\ &= \frac{3}{4}m(R-r)^2\dot{\theta}^2 \end{aligned}$$

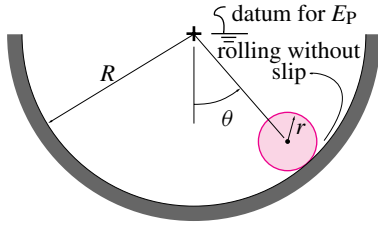


Figure 19.32: Cylinder rolling without slip in a cylinder. This gives the pendulum equation, but the effective pendulum length does not tend to R as $r \rightarrow 0$ (see text).

So we can derive the equation of motion using the fact of constant total energy.

$$\begin{aligned} 0 &= \frac{d}{dt}(E_T) = \frac{d}{dt}(E_K + E_P) \\ &= \frac{d}{dt}\left(\underbrace{-mg(R-r)\cos\theta}_{E_P} + \underbrace{\frac{3}{4}m(R-r)^2\dot{\theta}^2}_{E_K}\right) \\ &= (mg(R-r)\sin\theta)\dot{\theta} + \frac{3}{2}(R-r)^2\dot{\theta}\ddot{\theta} \\ \Rightarrow 0 &= mg(R-r)\sin\theta + \frac{3}{2}(R-r)^2m\ddot{\theta} \end{aligned}$$

Now, assuming small angles, so $\theta \approx \sin\theta$, we get

$$g(R-r)\theta + \frac{3}{2}(R-r)^2\ddot{\theta} = 0 \quad (19.32)$$

$$\ddot{\theta} + \underbrace{\left(\frac{2}{3}\frac{g}{(R-r)}\right)}_{\lambda^2}\theta = 0 \quad (19.33)$$

This is, naturally, our old friend the harmonic oscillator equation. The period is a funny combination of terms. If $r \ll R$ it looks like a point mass pendulum with length $3R/2$, more than R . That is, the rolling effect doesn't go away and make the roller act like a point mass even when the radius goes to zero. See page 889 for the angular momentum approach to this problem.

Although, in some abstract way the energy approach always works, practically speaking it has limitations for systems where the configuration is not easily found from one configuration variable.

Example: A four bar linkage analyzed with energy methods

Although probably not usually the best approach, energy methods can be used to find the motions of a 4-bar linkage. Take a four-bar linkage with one bar grounded. Assume the bars all have different lengths. This is a one-DOF system which can use the angle θ of one of the links as a configuration variable. But finding the potential energy as a single formula in terms of all of the links in terms of θ is more trigonometry than most of us like. And then finding the kinetic energy in terms of θ and $\dot{\theta}$ is close enough to impossible that people don't do it.

So, though it is true that there are functions $E_p(\theta)$ and $E_k(\theta, \dot{\theta})$ and that the equations of motion could be written in terms of them, it is really not practical to do so.

How do you find the motions of a 4-bar linkage in practice? It's more tricky. One approach is to solve the kinematics by integrating kinematic differential equations, as in Sample 18.5 on page 990. Then you set up and solve the balance equations of the separate parts as in Sample 19.21 on page 1051

SAMPLE 19.11 A plate pendulum. A $2a \times 2b$ rectangular plate of mass m hangs from two parallel, massless links EA and FD of length ℓ each. The links are hinged at both ends so that when the plate swings, its edges AD and BC remain horizontal at all times. The only driving force present is gravity. Find the equation of motion of the plate.

Solution The given system is a single DOF system. So, we need just one configuration variable and the equation of motion will be just one scalar equation in this variable. Let us take angle θ (fig. 19.34) as our configuration variable.

The free-body diagram of the plate is shown in fig. 19.35. Note that the link forces $\vec{F}_1$ and $\vec{F}_2$ act along the links because massless links are two-force bodies. Let (x, y) be the coordinates of the center-of-mass. Then the linear momentum balance for the plate gives

$$\vec{F}_1 + \vec{F}_2 - mg\vec{j} = m\vec{a} = m(\ddot{x}\vec{i} + \ddot{y}\vec{j}).$$

Now, we can eliminate both the unknown link forces from this equation by dotting the equation with $\hat{n} = \cos\theta\vec{i} + \sin\theta\vec{j}$, a unit vector normal to the links. Then, we have

$$\begin{aligned} -mg(\vec{j} \cdot \hat{n}) &= m[\ddot{x}(\vec{i} \cdot \hat{n}) + \ddot{y}(\vec{j} \cdot \hat{n})] \\ -g\sin\theta &= \ddot{x}\cos\theta + \ddot{y}\sin\theta. \end{aligned} \quad (19.34)$$

Now we need to find a relationship between x and θ , and y and θ , so that we can write $\ddot{x}$ and $\ddot{y}$ in terms of our configuration variable θ and its derivatives. From fig. 19.34, we have

$$\begin{aligned} \vec{r}_G &= \overbrace{(\ell \sin\theta + a)}^x \vec{i} + \overbrace{(-\ell \cos\theta - b)}^y \vec{j} \\ \Rightarrow \ddot{x} &= \ell(\cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2) \\ \ddot{y} &= \ell(\sin\theta \cdot \ddot{\theta} + \cos\theta \cdot \dot{\theta}^2) \end{aligned}$$

Substituting these expressions for $\ddot{x}$ and $\ddot{y}$ in eqn. (19.34), we get

$$\begin{aligned} -g\sin\theta &= \ell \ddot{\theta} \\ \Rightarrow \ddot{\theta} + \frac{g}{\ell} \sin\theta &= 0. \end{aligned}$$

$$\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0$$

This is the equation of a simple pendulum! Well, the plate does behave just like a simple pendulum in the given mechanism. From the expressions for the x and y coordinates of the center-of-mass, we have

$$\begin{aligned} x - a &= \ell \sin\theta \\ y + b &= -\ell \cos\theta \\ \Rightarrow (x - a)^2 + (y + b)^2 &= \ell^2 \end{aligned}$$

that is, the center-of-mass follows a circle of radius ℓ centered at $(-a, b)$. Since the orientation of the plate never changes (AD and BC always remain horizontal), the plate has no angular velocity. Thus the motion of the plate is equivalent to the motion of a particle of mass m going in a circle centered at $(-a, b)$ and driven by gravity. That is the simple pendulum.

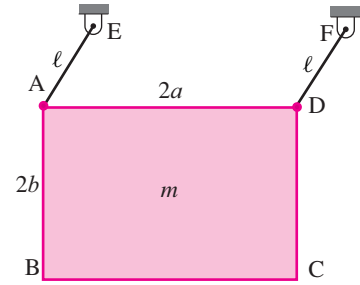


Figure 19.33

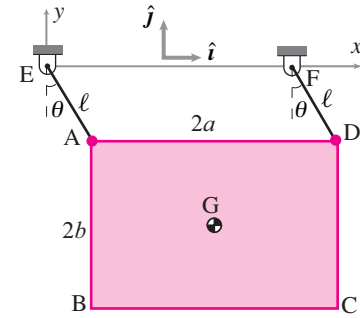


Figure 19.34

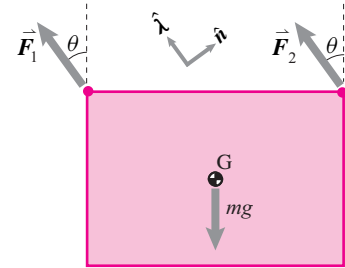


Figure 19.35

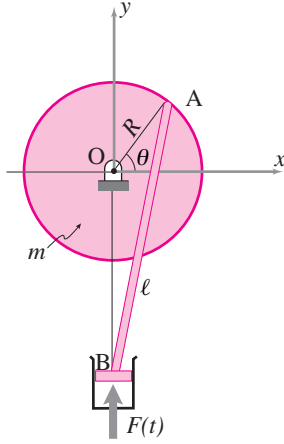


Figure 19.36

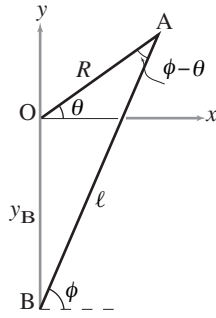


Figure 19.37

SAMPLE 19.12 Equation of motion from power balance. A slider crank mechanism is shown in fig. 19.36 where the crank is a uniform wheel of mass m and radius R anchored at the center and the connecting rod AB is a massless rod of length ℓ . The rod is driven by a piston at B with a known force $F(t) = F_0 \cos \Omega t$. There is no gravity. Find the equation of motion of the wheel.

Solution The given mechanism is a one DOF system. So, let us choose a single configuration variable, θ for specifying the configuration of the system and derive an equation that determines θ . Since the applied force is given and the point of application of the force has a simple motion (vertical), it will be easy to calculate power of this force. Also, the connecting rod is massless, so it does not enter into dynamic calculations. The wheel rotates about its center and, therefore, it is easy to calculate its kinetic energy. So, we use the power balance, $\dot{E}_K = P$, here to find the equation of motion of the wheel. Since $E_K = \frac{1}{2} I_{zz}^{\text{cm}} \dot{\theta}^2$ and $P = \vec{F} \cdot \vec{v}_B$, we have

$$I_{zz}^{\text{cm}} \ddot{\theta} = F(t) \mathbf{j} \cdot v_B \mathbf{j} = F(t) v_B \quad (19.35)$$

Now, we need to find v_B and express it using the configuration variable θ and its derivatives. There are several ways we could find v_B . Vectorially, we could write, $\vec{v}_B \equiv v_B \mathbf{j} = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$ where $\vec{\omega}_{AB} = \dot{\phi} \mathbf{k}$. Dotting both sides of this equation with $\mathbf{i}$ and $\mathbf{j}$ we can find ϕ in terms of θ and v_B in terms of θ and $\dot{\theta}$. But, for a change, let us use geometry here.

See fig. 19.37. From triangle ABO, we have

$$\begin{aligned} \frac{R}{\sin(90^\circ - \phi)} &= \frac{\ell}{\sin(90^\circ + \theta)} \\ \Rightarrow \ell \cos \phi &= R \cos \theta \\ \Rightarrow -\ell \sin \phi \cdot \dot{\phi} &= -R \sin \theta \cdot \dot{\theta} \\ \dot{\phi} &= \frac{R \sin \theta}{\ell \sin \phi} \dot{\theta} \end{aligned}$$

Now,

$$\begin{aligned} y_B &= R \sin \theta - \ell \sin \phi \\ \Rightarrow v_B &\equiv \dot{y}_B = R \cos \theta \cdot \dot{\theta} - \ell \cos \phi \cdot \dot{\phi} \\ &= R \dot{\theta} \cos \theta - R \cos \theta \cdot \frac{R \sin \theta}{\ell \sin \phi} \dot{\theta} \\ &= R \dot{\theta} \left(\cos \theta - \frac{R \sin \theta \cos \theta}{\sqrt{\ell^2 - R^2 \cos^2 \theta}} \right) \end{aligned}$$

Substituting this expression for v_B in power balance eqn. (19.35), we get

$$\begin{aligned} I_{zz}^{\text{cm}} \ddot{\theta} &= F(t) \cdot R \dot{\theta} \left(\cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right) \\ \ddot{\theta} &= \frac{RF_0 \cos \Omega t}{\frac{1}{2} m R^2} \left(\cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right). \end{aligned}$$

This is the required equation of motion. As is evident, it is a nonlinear ODE which requires numerical solution on a computer if we would like to plot $\theta(t)$.

$$\ddot{\theta} = \frac{2F_0 \cos \Omega t}{mR} \left(\cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right)$$

SAMPLE 19.13 Instantaneous dynamics of slider crank. A uniform rigid rod AB of mass m and length $\ell = 4R$ has one of its ends pinned to the rim of a disk of radius R . The other end of the bar is free to slide on a frictionless horizontal surface. A motor, connected to the center of the disk at O, keeps the disk rotating at a constant angular speed ω_D . At the instant shown, end B of the rod is directly above the center of the disk making θ to be 30° .

1. Find all the forces acting on the rod.
2. Is there a value of ω_D which makes end A of the rod lift off the horizontal surface when $\theta = 30^\circ$?

Solution The disk is rotating at constant speed. Since end B of the rod is pinned to the disk, end B is going in circles at constant rate. The motion of end B of the rod is completely prescribed. Since end A can only move horizontally (assuming it has not lifted off yet), the orientation (and hence the position of each point) of the rod is completely determined at any instant during the motion. Therefore, the rod represents a zero degree of freedom system.

1. **Forces on the rod:** The free-body diagram of the rod is shown in Fig. 19.39. The pin at B exerts two forces B_x and B_y while the surface in contact at A exerts only a normal force N because there is no friction. Now, we can write the momentum balance equations for the rod. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the rod gives

$$B_x \hat{i} + (B_y + N - mg) \hat{j} = m \vec{a}_G. \quad (19.36)$$

The angular momentum balance about the center-of-mass G ($\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$) of the rod gives

$$\vec{r}_{A/G} \times N \hat{j} + \vec{r}_{B/G} \times (B_x \hat{i} + B_y \hat{j}) = I_{zz/G} \alpha_{rod} \hat{k}. \quad (19.37)$$

From these two vector equations we can get three scalar equations (the Angular Momentum Balance gives only one scalar equation in 2-D since the quantities on both sides of the equation are only in the $\hat{k}$ direction), but we have six unknowns — B_x , B_y , N , $\vec{a}_G$ (counts as two unknowns), and α_{rod} . Therefore, we need more equations. We have already used the momentum balance equations, hence, the extra equations have to come from kinematics.

$$\begin{aligned} \vec{v}_A &= \vec{v}_B + \overbrace{\vec{\omega}_{rod} \times \vec{r}_{A/B}}^{\vec{v}_{A/B}} \\ \text{or } v_A \hat{i} &= \omega_D R \hat{i} + \omega_{rod} \hat{k} \times \ell (-\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= (\omega_D R + \omega_{rod} \ell \sin \theta) \hat{i} - \omega_{rod} \ell \cos \theta \hat{j} \end{aligned}$$

Dotting both sides of the equation with $\hat{j}$ we get

$$0 = \omega_{rod} \ell \cos \theta \quad \Rightarrow \quad \omega_{rod} = 0.$$

Also,

$$\begin{aligned} \vec{a}_A &= \vec{a}_B + \overbrace{\vec{\omega} \times \vec{r}_{A/B} + \underbrace{\vec{\omega}_{rod} \times (\vec{\omega}_{rod} \times \vec{r}_{A/B})}_{\vec{0}}}^{\vec{a}_{A/B}} \\ \text{or } a_A \hat{i} &= -\omega_D^2 R \hat{j} + \dot{\omega}_{rod} \hat{k} \times \ell (-\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= -(\omega_D^2 R + \dot{\omega}_{rod} \ell \cos \theta) \hat{j} + \dot{\omega}_{rod} \ell \sin \theta \hat{i}. \end{aligned}$$

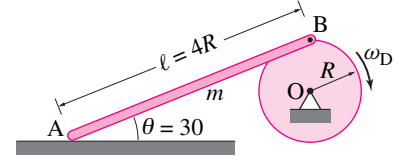


Figure 19.38: End A of bar AB is free to slide on the frictionless horizontal surface while end B is going in circles with a disk rotating at a constant rate.

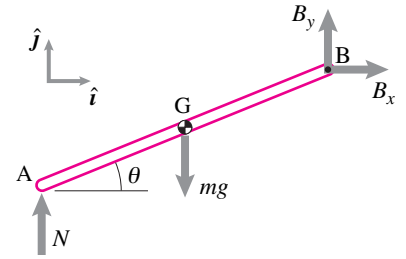


Figure 19.39: Free-body diagram of the bar.

Dotting both sides of this equation by $\mathbf{j}$ we get

$$\dot{\omega}_{rod} = -\frac{\omega_D^2 R}{\ell \cos \theta}. \quad (19.38)$$

Now, we can find the acceleration of the center-of-mass:

$$\begin{aligned} \vec{\mathbf{a}}_G &= \vec{\mathbf{a}}_B + \overbrace{\dot{\vec{\omega}} \times \vec{\mathbf{r}}_{G/B} + \underbrace{\vec{\omega}_{rod}}_{\vec{0}} \times (\vec{\omega}_{rod} \times \vec{\mathbf{r}}_{G/B})}^{\vec{\mathbf{a}}_{G/B}} \\ &= -\omega_D^2 R \mathbf{j} + \dot{\omega}_{rod} \hat{\mathbf{k}} \times \frac{1}{2} \ell (-\cos \theta \hat{\mathbf{i}} - \sin \theta \mathbf{j}) \\ &= -(\omega_D^2 R + \frac{1}{2} \dot{\omega}_{rod} \ell \cos \theta) \mathbf{j} + \frac{1}{2} \dot{\omega}_{rod} \ell \sin \theta \hat{\mathbf{i}}. \end{aligned}$$

Substituting for $\dot{\omega}_{rod}$ from eqn. (19.38) and 30° for θ above, we obtain

$$\vec{\mathbf{a}}_G = -\frac{1}{2} \omega_D^2 R \left(\frac{1}{\sqrt{3}} \hat{\mathbf{i}} + \mathbf{j} \right).$$

Substituting this expression for $\vec{\mathbf{a}}_G$ in eqn. (19.36) and dotting both sides by $\hat{\mathbf{i}}$ and then by $\mathbf{j}$ we get

$$\begin{aligned} B_x &= -\frac{1}{2\sqrt{3}} m \omega_D^2 R, \\ B_y + N &= -\frac{1}{2} m \omega_D^2 R + mg \end{aligned} \quad (19.39)$$

From eqn. (19.37)

$$\begin{aligned} \frac{1}{2} \ell [(B_y - N) \cos \theta - B_x \sin \theta] \hat{\mathbf{k}} &= \frac{1}{12} m \ell^2 \left(-\frac{\omega_D^2 R}{\ell \cos \theta} \right) \hat{\mathbf{k}} \\ \text{or } B_y - N &= -\frac{1}{6} \frac{m \omega_D^2 R}{\cos^2 \theta} + B_x \tan \theta \\ &= -\frac{2}{9} m \omega_D^2 R - \frac{1}{6} m \omega_D^2 R \\ &= -\frac{7}{18} m \omega_D^2 R \end{aligned} \quad (19.40)$$

From eqns. (19.39) and (19.40)

$$B_y = \frac{1}{2} \left(mg - \frac{8}{9} m \omega_D^2 R \right)$$

and

$$N = \frac{1}{2} \left(mg - \frac{1}{9} m \omega_D^2 R \right).$$

2. **Lift off of end A:** End A of the rod loses contact with the ground when normal force N becomes zero. From the expression for N from above, this condition is satisfied when

$$\begin{aligned} \frac{2}{9} m \omega_D^2 R &= mg \\ \Rightarrow \omega_D &= 3 \sqrt{\frac{g}{R}}. \end{aligned}$$

SAMPLE 19.14 A bar sliding on a sliding wedge. A bar AB of mass m and length ℓ is hinged at end A and rests on a wedge of mass M at the other end B. The contact at B is frictionless. The wedge is free to slide horizontally without any friction. The motion of the system is driven only by gravity. Find the equation of motion of the bar using

1. momentum balance
2. energy (or power) balance.

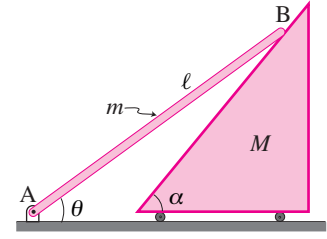


Figure 19.40

Solution (a) **Momentum Balance:** The bar and the wedge make up a single DOF system. To derive the equations of motion of the bar, let us choose θ as the configuration variable. The free-body diagram of the bar and the wedge are shown in fig. 19.41. Note that the normal reaction at B is normal to the wedge surface, i.e., $\vec{N} = N\hat{n}$. Now, the angular momentum balance for the bar about point A gives,

$$\begin{aligned}\dot{\vec{H}}_A &= \sum \vec{M}_A \\ I_{zz}^A \ddot{\theta} \hat{k} &= \frac{\ell}{2} \hat{e}_r \times (-mg\hat{j}) + \ell \hat{e}_r \times N\hat{n} \\ &= -\frac{1}{2}mg\ell \cos\theta \hat{k} + N\ell \cos(\alpha - \theta) \hat{k}\end{aligned}\quad (19.41)$$

where the last line follows from the fact that $\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$, the unit normal $\hat{n} = -\sin\alpha\hat{i} + \cos\alpha\hat{j}$, and so, $\hat{e}_r \times \hat{j} = \cos\theta\hat{k}$ and $\hat{e}_r \times \hat{n} = \cos(\alpha - \theta)\hat{k}$. We now need to eliminate the unknown normal reaction N from the above equation. Since the wedge is constrained to move only horizontally, we can write the linear momentum balance for the wedge as

$$N\hat{n} \cdot \hat{i} = M\ddot{x} \quad \Rightarrow \quad N = \frac{M\ddot{x}}{\hat{n} \cdot \hat{i}} = \frac{M\ddot{x}}{\sin\alpha} \quad (19.42)$$

Thus, we have found N in terms of $\ddot{x}$ that we can use in eqn. (19.41) to get rid of N . But, we now need to express $\ddot{x}$ in terms of our configuration variable θ and its derivatives. Consider the triangle ABC formed by the bar and the slanted edge of the wedge. Let $x = AC$ denote the horizontal position of the wedge. Then, from the law of sines, we have $\frac{x}{\sin(\alpha - \theta)} = \frac{\ell}{\sin\alpha}$, so that

$$\begin{aligned}x &= \frac{\ell}{\sin\alpha} \sin(\alpha - \theta) \\ \Rightarrow \dot{x} &= \frac{\ell}{\sin\alpha} \cos(\alpha - \theta) \cdot (-\dot{\theta}) \\ \Rightarrow \ddot{x} &= \frac{\ell}{\sin\alpha} [\ddot{\theta} \cos(\alpha - \theta) + \dot{\theta}^2 \sin(\alpha - \theta)].\end{aligned}\quad (19.43)$$

$$\Rightarrow \ddot{x} = \frac{\ell}{\sin\alpha} [\ddot{\theta} \cos(\alpha - \theta) + \dot{\theta}^2 \sin(\alpha - \theta)]. \quad (19.44)$$

Now, substituting for N in eqn. (19.41) from eqn. (19.42), using the expression for $\ddot{x}$ from above, and dotting the resulting equation with $\hat{k}$, we get

$$\begin{aligned}\frac{1}{3}m\ell^2\ddot{\theta} &= -\frac{mg\ell}{2}\cos\theta + \frac{M\ell^2\cos(\alpha - \theta)}{\sin^2\alpha} [\ddot{\theta} \cos(\alpha - \theta) + \dot{\theta}^2 \sin(\alpha - \theta)] \\ \Rightarrow \ddot{\theta} &= -\frac{3mg\sin^2\alpha\cos\theta + 3M\ell\dot{\theta}^2\sin 2(\alpha - \theta)}{2m\ell\sin^2\alpha + 6M\ell\cos^2(\alpha - \theta)} \\ &= -\frac{3[(g/\ell)\sin^2\alpha\cos\theta + (M/m)\dot{\theta}^2\sin 2(\alpha - \theta)]}{2[\sin^2\alpha + 3(M/m)\cos^2(\alpha - \theta)]}\end{aligned}\quad (19.45)$$

$$\ddot{\theta} = -\frac{3[(g/\ell)\sin^2\alpha\cos\theta + (M/m)\dot{\theta}^2\sin 2(\alpha - \theta)]}{2[\sin^2\alpha + 3(M/m)\cos^2(\alpha - \theta)]}$$

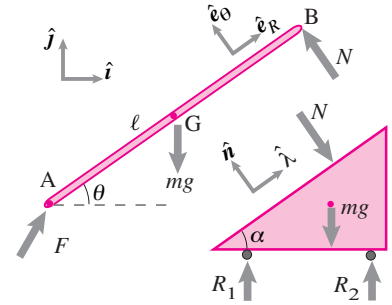


Figure 19.41

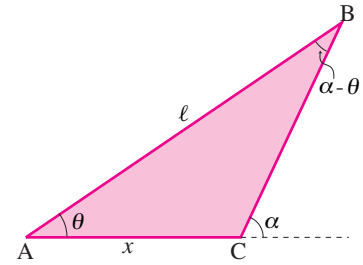


Figure 19.42

(b) **Power balance:** Now we derive the equation of motion for the bar using power balance $\dot{E}_K = P$. For power balance, we have to consider both the bar and the wedge. The bar rotates about the fixed point A, therefore, its kinetic energy is $(1/2)I_z^A \dot{\theta}^2$. The wedge moves with horizontal speed $\dot{x}$, therefore, its kinetic energy is $(1/2)M\dot{x}^2$. Thus, $E_K = (1/2)I_z^A \dot{\theta}^2 + (1/2)M\dot{x}^2$. The only force that contributes to power is the force of gravity on the rod because $\vec{v}_G \cdot (-mg\mathbf{j})$ is non-zero. The sliding contact at B is frictionless and hence the net power due to the contact force there is zero. Now, $\vec{v}_G = \frac{1}{2}\ell \dot{\theta} \hat{\mathbf{e}}_\theta$. So, $P = -\frac{1}{2}mg\ell \dot{\theta}(\hat{\mathbf{e}}_\theta \cdot \mathbf{j}) = -\frac{1}{2}mg\ell \dot{\theta} \cos \theta$. Thus the power balance for the system gives

$$I_{zz}^A \ddot{\theta} + M\ddot{x} = -\frac{1}{2}mg\ell \dot{\theta} \cos \theta. \quad (19.46)$$

We can simplify this equation further. Note that, $\ddot{\theta} = \frac{d}{dt}(\frac{1}{2}\dot{\theta}^2)$ and $\ddot{x} = \frac{d}{dt}(\frac{1}{2}\dot{x}^2)$. So that we can write the above equation as

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} I_{zz}^A \dot{\theta}^2 \right) + \frac{d}{dt} \left(\frac{1}{2} M \dot{x}^2 \right) &= -\frac{1}{2} mg\ell \dot{\theta} \cos \theta \\ \text{or} \quad \int d(I_{zz}^A \dot{\theta}^2 + M \dot{x}^2) &= -mg\ell \int \cos \theta d\theta \\ \Rightarrow I_{zz}^A \dot{\theta}^2 + M \dot{x}^2 &= C - mg\ell \sin \theta \end{aligned}$$

where C is a constant of integration to be determined from initial conditions. For example, when the bar begins to slide from rest, we have, at $t = 0$, $\dot{x} = 0$ and $\dot{\theta} = 0$. At that instant, if $\theta(0) = \theta_0$, then $C = mg\ell \sin \theta_0$. So, we can write

$$I_{zz}^A \dot{\theta}^2 + M \dot{x}^2 = mg\ell (\sin \theta_0 - \sin \theta).$$

Now, replacing $\dot{x}$ in this equation with the expression we obtained in eqn. (19.43), we have

$$\begin{aligned} I_{zz}^A \dot{\theta}^2 + M\ell^2 \dot{\theta}^2 \frac{\cos^2(\alpha - \theta)}{\sin^2 \alpha} &= mg\ell (\sin \theta_0 - \sin \theta) \\ \Rightarrow \dot{\theta}^2 &= \frac{mg\ell (\sin \theta_0 - \sin \theta)}{I_{zz}^A + M\ell^2 \frac{\cos^2(\alpha - \theta)}{\sin^2 \alpha}} \\ &= \frac{mg\ell \sin^2 \alpha (\sin \theta_0 - \sin \theta)}{(1/3)m\ell^2 \sin^2 \alpha + M\ell^2 \cos^2(\alpha - \theta)} \\ \Rightarrow \dot{\theta} &= \sqrt{\frac{3g}{\ell}} \sin \alpha \sqrt{\frac{\sin \theta_0 - \sin \theta}{\sin^2 \alpha + 3(M/m) \cos^2(\alpha - \theta)}}. \end{aligned}$$

This is a first order, nonlinear, ODE compared to the second order equation we got in eqn. (19.45). However, again, we need to resort to numerical solution if we wish to solve for $\theta(t)$, in which case, this reduction to the first order equation does not save much work.

$$\dot{\theta} = \sqrt{\frac{3g}{\ell}} \sin \alpha \sqrt{\frac{\sin \theta_0 - \sin \theta}{\sin^2 \alpha + 3(M/m) \cos^2(\alpha - \theta)}}$$

19.2 One-degree-of-freedom 2-D mechanisms

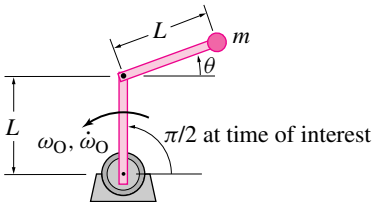
19.2.1 A conservative vibratory system has the following equation of conservation of energy.

$$m(\ell \dot{\theta})^2 - mg\ell(1 - \cos \theta) + K(a\theta)^2 = E_0$$

where E_0 is a constant.

- Obtain the differential equation of motion of this system by differentiating this energy equation with respect to t .
- Determine the circular frequency of small oscillations of the system in part (a). HINT: (Let $\sin \theta \cong \theta$ and $\cos \theta \cong 1 - \theta^2/2$).

19.2.2 A motor at O turns at rate ω_o whose rate of change is $\dot{\omega}_o$. At the end of a stick connected to this motor is a frictionless hinge attached to a second massless stick. Both sticks have length L . At the end of the second stick is a mass m . For the configuration shown, what is $\dot{\theta}$? Answer in terms of ω_o , $\dot{\omega}_o$, L , m , θ , and $\dot{\theta}$. Ignore gravity.

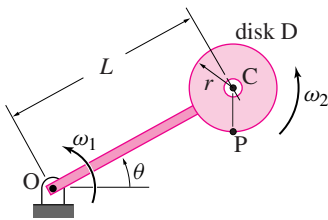


Problem 19.2.2

19.2.3 In problem 18.3.4, find

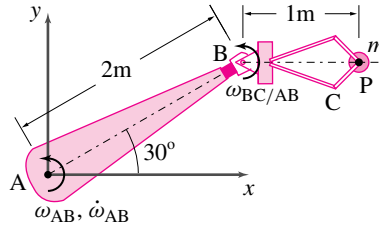
- the angular momentum about point O, *
- the rate of change of angular momentum of the disk about point O, *
- the angular momentum about point C, *
- the rate of change of angular momentum of the disk about point C. *

Assume the rod is massless and the disk has mass m .



Problem 19.2.3

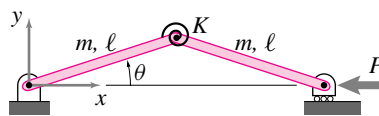
19.2.4 Robot arm, 2-D. The robot arm AB is rotating about point A with $\omega_{AB} = 5 \text{ rad/s}$ and $\dot{\omega}_{AB} = 2 \text{ rad/s}^2$. Meanwhile the forearm BC is rotating at a constant angular speed with respect to AB of $\omega_{BC/AB} = 3 \text{ rad/s}$. Gravity cannot be neglected. At the instant shown, find the net force acting on the object P which has mass $m = 1 \text{ kg}$. *



Problem 19.2.4

19.2.5 A crude model for a column, shown in the figure, consists of two identical rods of mass m and length ℓ with hinge connections, a linear torsional spring of stiffness K attached to the center hinge (the spring is relaxed when $\theta = 0$), and a load P applied at the top end.

- Obtain the exact nonlinear equation of motion.
- Obtain the squared natural frequency for small motions θ .
- Check the stability of the straight equilibrium state $\theta = 0$ for all $P \geq 0$ via minimum potential energy. How do these results compare with those from part(b)?

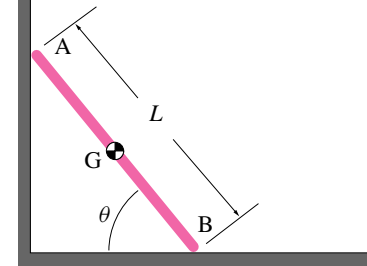


Problem 19.2.5

19.2.6 A thin uniform rod of mass m rests against a frictionless wall and on a frictionless floor. There is gravity.

- Draw a free-body diagram of the rod.
- The rod is released from rest at $\theta = \theta_0 \neq 0$. Write the equation of motion of the rod. *
- Using the equation of motion, find the initial angular acceleration, $\dot{\omega}_{AB}$, and the acceleration of the center of mass, $\vec{a}_G$, of the rod. *

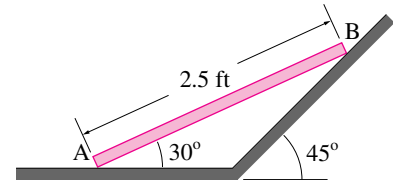
- Find the reactions on the rod at points A and B. *
- Find the acceleration of point B. *
- When $\theta = \frac{\theta_0}{2}$, find $\ddot{\omega}_{AB}$ and the acceleration of point A. *



Problem 19.2.6

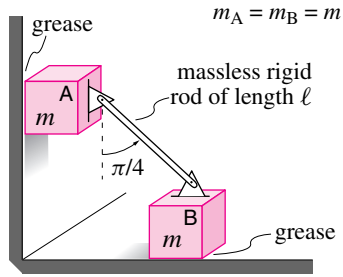
19.2.7 Bar leans on a crooked wall. A uniform 3 lbf bar leans on a wall and floor and is let go from rest. Gravity pulls it down.

- Draw a Free-Body Diagram of the bar.
- Kinematics: find the velocity and acceleration of point B in terms of the velocity and acceleration of point A
- Using equations of motion, find the acceleration of point A.
- Do the reaction forces at A and B add up to the weight of the bar? Why or why not? (You do not need to solve for the reaction forces in order to answer this part.)



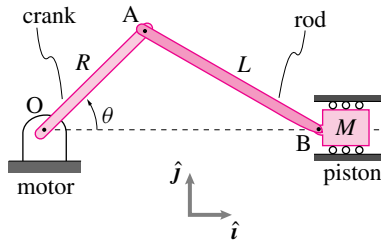
Problem 19.2.7

19.2.8 Two blocks, each with mass m , slide without friction on the wall and floor shown. They are attached with a rigid massless rod of length ℓ that is pinned at both ends. The system is released from rest when the rod makes an angle of 45° . What is the acceleration of the block B immediately after the system is released?



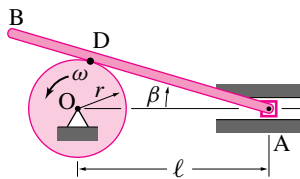
Problem 19.2.8

19.2.9 Slider Crank. 2-D . No gravity. Refer to the figure in problem 18.4.2. What is the tension in the massless rod AB (with length L) when the slider crank is in the position with $\theta = 0$ (piston is at maximum extent)? Assume the crankshaft has constant angular velocity ω , that the connecting rod AB is massless, that the cylinder walls are frictionless, that there is no gas pressure in the cylinder, that the piston has mass M and the crank has radius R .



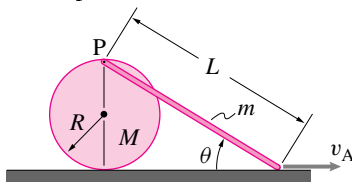
Problem 19.2.9

19.2.10 In problem 18.4.8, find the force on the wheel at point D due to the rod.



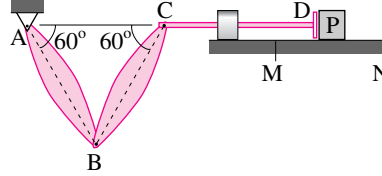
Problem 19.2.10

19.2.11 In problem 18.4.9, find the force on the rod at point P .



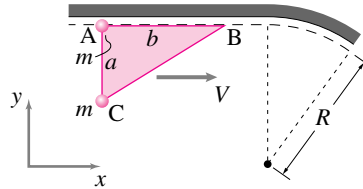
Problem 19.2.11

19.2.12 Slider-Crank mechanism. The slider crank mechanism shown is used to push a 2 lbf block P . Arm AB and BC are each 0.5 ft long. Given that arm AB rotates counterclockwise at a constant 2 revolutions per second, what is the force on P at the instant shown? *



Problem 19.2.12

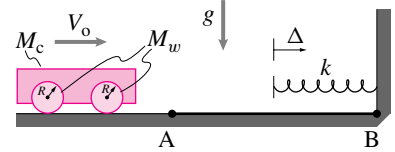
19.2.13 The large masses m at C and A were supported by the light triangular plate ABC , whose corners follow the guide. B enters the curved guide with velocity V . Neglecting gravity, find the vertical reactions (in the y direction) at A and B . (Hint: for the rigid planar body ABC , find $\ddot{y}_A$ and $\ddot{y}_B$ in terms of a_y^C and $\ddot{\theta}$, assuming $\dot{\theta} = 0$ initially.)



Problem 19.2.13

19.2.14 An idealized model for a car comprises a rigid chassis of mass M_c and four identical rigid disks (wheels) of mass M_w and radius R , as shown in the figure. Initially in motion with speed v_0 , the car is momentarily brought to rest by compressing an initially uncompressed spring of stiffness k . Assume no frictional losses.

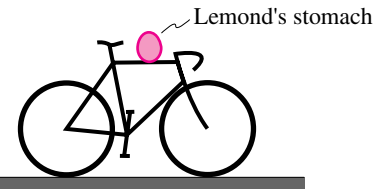
- Assuming no wheel slip, determine the compression Δ of the spring required to stop the car. *
- While the car is in contact with the spring but still moving forward, in which direction is the *tangential* force on any one of the wheels (due to contact with the ground) pointing? Why? (Illustrate with a sketch). *
- Repeat part (a) assuming now that the ground is perfectly frictionless from points A to B shown in the figure. *



Problem 19.2.14

19.2.15 Assume Greg Lemond's riding "tuck" was so good that you can neglect air resistance when you think about him and his bike. Further, you can regard his and the bike's combined mass (all 70 kg) as concentrated at a point in his stomach somewhere. Greg's left foot has just fallen off the pedal so he is only pedaling with his right foot, which at the moment in question is at its lowest point in the motion. You note that, relative to the ground the right foot is only going $3/4$ as fast as the bike (since it is going backwards relative to the bike), though you can't make out all the radii of his frictionless gears and rigid round wheels. Greg, ever in touch with his body, tells you he is pushing back on the pedal with a force of 70 N. You would like to know Greg's acceleration.

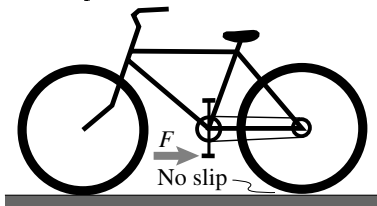
- In your first misconceived experiment you set up a 70 kg bicycle in your laboratory and balance it with strings that cause no fore or aft forces. You tie a string to the vertically down right pedal and pull back with a force of 70 N. What acceleration do you measure?
- What is Greg's actual acceleration? [Hint: Greg's massless leg is pushing forward on his body with a force of 70 N] You can neglect the mass of the wheels and other transmission parts (chain, crank, etc).



Problem 19.2.15

19.2.16 Which way does the bike accelerate? A bicycle with all frictionless bearings is standing still on level ground. A horizontal force F is applied on one of the pedals as shown. There is no slip between the wheels and the ground. The

bicycle is gently balanced from falling over sideways. It is heavy enough so that both wheels stay on the ground. Does the bicycle accelerate forward, backward, or not at all? Make any reasonable assumptions about the dimensions and mass. Justify your answer as clearly as you can, clearly enough to convince a person similar to yourself but who has not seen the experiment performed.



Problem 19.2.16

19.2.1 A conservative vibratory system has the following equation of conservation of energy.

$$m(\ell\dot{\theta})^2 - mg\ell(1 - \cos\theta) + K(a\theta)^2 = E_0$$

where E_0 is a constant.

- a) Obtain the differential equation of motion of this system by differen-

tiating this energy equation with respect to t .

- b) Determine the circular frequency of small oscillations of the system in part (a). HINT: (Let $\sin\theta \cong \theta$ and $\cos\theta \cong 1 - \theta^2/2$).

18.2.1

Given, the equation of conservation of energy:

$$m(\ell\dot{\theta})^2 - mg\ell(1 - \cos\theta) + K(a\theta)^2 = E_0$$

Differentiating w.r.t. time

$$2m\ell\dot{\theta}\ddot{\theta} - mg\ell(+\sin\theta)\dot{\theta} + 2K(a\theta)a\dot{\theta} = 0$$

$$\Rightarrow \ddot{\theta} = \frac{mg\ell\sin\theta - 2Ka^2\theta}{2m\ell}$$

Differential equation of motion is

$$\ddot{\theta} + \frac{K}{m} \left(\frac{a^2}{\ell} \right) \theta - \frac{g}{2\ell} \sin\theta$$

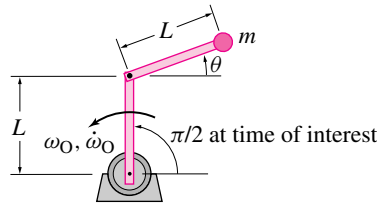
- b) As $\sin\theta \approx \theta$ for small oscillations, equation of motion can be modified as follows.

$$\ddot{\theta} + \left\{ \frac{K}{m} \left(\frac{a^2}{\ell} \right) - \frac{g}{2\ell} \right\} \theta = 0$$

Drawing analogy with $\ddot{\theta} + \tilde{\omega}^2\theta = 0$, the circular frequency,

of small oscillations, $\lambda = \sqrt{\frac{K}{m} \left(\frac{a^2}{\ell} \right) - \frac{g}{2\ell}}$ Provided, $\frac{Ka^2}{m\ell} > \frac{g}{2\ell}$

19.2.2 A motor at O turns at rate ω_0 , whose rate of change is $\dot{\omega}_0$. At the end of a stick connected to this motor is a frictionless hinge attached to a second massless stick. Both sticks have length L . At the end of the second stick is a mass m . For the configuration shown, what is $\ddot{\theta}$? Answer in terms of ω_0 , $\dot{\omega}_0$, L , m , θ , and $\dot{\theta}$. Ignore gravity.



Problem 19.2

18.2.2

Applying LMB on 'mass, m': $\sum \underline{F} = m \underline{a}$

$$-T \hat{e}_r = m [\underline{a}_{O'} + \underline{\dot{\theta}} \times \underline{r}_{A/O'} + \underline{\dot{\theta}} \times (\underline{\dot{\theta}} \times \underline{r}_{A/O'})]$$

$$\underline{a} = \underline{a}_{O'} + \ddot{\theta} L \hat{e}_\theta - L \dot{\theta}^2 \hat{e}_r$$

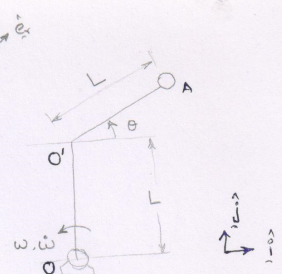
$$\underline{a}_{O'} = \underline{a}_O + \dot{\omega}_0 \times \underline{r}_{O'/O} + \omega_0 \times (\omega_0 \times \underline{r}_{O'/O})$$

$$= -\dot{\omega}_0 L \hat{i} - L \omega_0^2 \hat{j}$$

$$\therefore \{-T \hat{e}_r = m [-\dot{\omega}_0 L \hat{i} - L \omega_0^2 \hat{j} + \ddot{\theta} L \hat{e}_\theta - L \dot{\theta}^2 \hat{e}_r]\}$$

$$\{ \} \cdot \hat{e}_\theta \Rightarrow 0 = m [-\dot{\omega}_0 L \sin \theta - L \omega_0^2 \cos \theta + \ddot{\theta} L]$$

$$\therefore \ddot{\theta} = \omega_0^2 \cos \theta - \dot{\omega}_0 \sin \theta //$$



$$\hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

18.2.4

Let $\underline{T}$ be the net force acting on the mass at 'P'.Using L.M.B. $\sum \underline{F} = m \underline{a}$

$$\underline{T} = \underline{F}(-\hat{i}) + mg(-\hat{j}) = m \underline{a}_P$$

$$\underline{a}_P = \underline{a}_B + \underline{\dot{\omega}}_{BC} \times \underline{r}_2 + \underline{\omega}_{BC} \times (\underline{\omega}_{BC} \times \underline{r}_2)$$

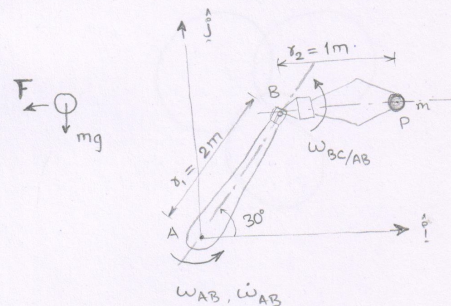
$$= \underline{\dot{\omega}}_{AB} \times \underline{r}_1 + \underline{\omega}_{AB} \times (\underline{\omega}_{AB} \times \underline{r}_1) + \underline{\dot{\omega}}_{AB} \hat{j} - \omega_{BC}^2 \hat{i}$$

$$\underline{a}_P = 2\dot{\omega}_{AB} r_1 \hat{e}_\theta - r_1 \omega_{AB}^2 \hat{e}_r - \omega_{BC}^2 \hat{i} + \dot{\omega}_{AB} \hat{j}$$

$$\therefore \underline{T} = m(-r_1 \dot{\omega}_{AB} \sin 30 - \omega_{BC}^2 - r_1 \omega_{AB}^2 \cos 30) \hat{i} + m(r_1 \dot{\omega}_{AB} \cos 30 - r_1 \omega_{AB}^2 \sin 30 + \dot{\omega}_{AB}) \hat{j}$$

$$\text{Given, } r_1 = 2\text{ m, } \omega_{AB} = 5 \text{ rad/s, } \dot{\omega}_{AB} = 2 \text{ rad/s}^2; m = 1\text{ kg, } \omega_{BC} = \omega_{BC/AB} + \omega_{AB} = 8 \text{ rad/s}$$

$$\Rightarrow \underline{T} = -109.3 \hat{i} - 19.5 \hat{j} //$$



$$\hat{e}_r = \cos 30 \hat{i} + \sin 30 \hat{j}$$

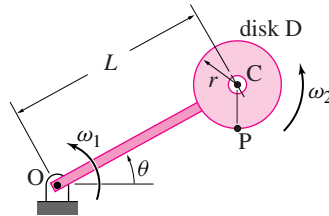
$$\hat{e}_\theta = -\sin 30 \hat{i} + \cos 30 \hat{j}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

19.2.3 In problem ??, find

- the angular momentum about point O ,
- the rate of change of angular momentum of the disk about point O ,
- the angular momentum about point C ,
- the rate of change of angular momentum of the disk about point C .

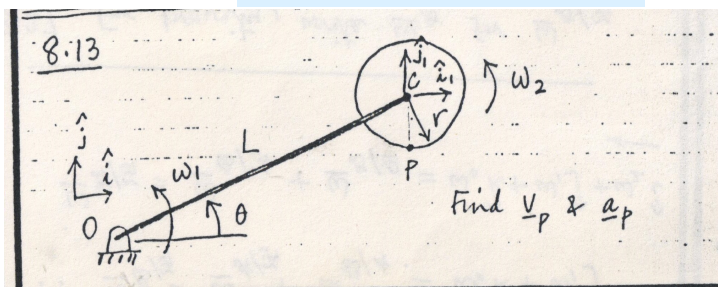
Assume the rod is massless and the disk has mass m .



Problem 19.3

Answer:

- $\dot{\mathbf{H}}_O = [mL^2\omega_1 + \frac{1}{2}mr^2(\omega_1 + \omega_2)]\hat{\mathbf{k}}$.
- $\dot{\mathbf{H}}_O = \mathbf{0}$.
- $\dot{\mathbf{H}}_C = \frac{1}{2}mr^2(\omega_1 + \omega_2)\hat{\mathbf{k}}$.
- $\dot{\mathbf{H}}_C = \mathbf{0}$.



Page 3

Attach a moving frame at point C , with unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, called $\mathcal{B}$.

a) First, assume that the moving frame does not rotate (it just translates; unit vectors don't change direction).

$$\omega_{\text{disk}/\mathcal{B}} = (\omega_1 + \omega_2)\hat{\mathbf{k}}$$

$$\therefore \mathbf{v}_{P/\mathcal{B}} = (\omega_1 + \omega_2)\hat{\mathbf{k}} \times \mathbf{r}_{CP}$$

$$= (\omega_1 + \omega_2)r\hat{\mathbf{k}} \times (-\hat{\mathbf{j}}_1)$$

$$= (\omega_1 + \omega_2)r\hat{\mathbf{i}} \quad \leftarrow$$

$$\mathbf{v}_C = \omega_1\hat{\mathbf{k}} \times L(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})$$

$$= \omega_1 L(\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{i}})$$

$$\mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{P/\mathcal{B}} + \omega_{\mathcal{B}} \times \mathbf{r}_{CP}$$

$$\mathbf{v}_P = \omega_1 L(\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{i}}) + (\omega_1 + \omega_2)r\hat{\mathbf{i}}$$

$$\quad \quad \quad \omega_1 L \cos\theta \hat{\mathbf{j}} - \omega_1 L \sin\theta \hat{\mathbf{i}} + (\omega_1 + \omega_2)r\hat{\mathbf{i}}$$

$$\mathbf{a}_{P/\mathcal{B}} = (\omega_1 + \omega_2)\hat{\mathbf{k}} \times [(\omega_1 + \omega_2)\hat{\mathbf{k}} \times \mathbf{r}_{CP}]$$

$$= (\omega_1 + \omega_2)^2 r\hat{\mathbf{j}}$$

$$\mathbf{a}_C = \omega_1\hat{\mathbf{k}} \times [\omega_1\hat{\mathbf{k}} \times L(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})]$$

$$\mathbf{a}_P = \mathbf{a}_C + \mathbf{a}_{P/\mathcal{B}} + \text{bunch of zeros}$$

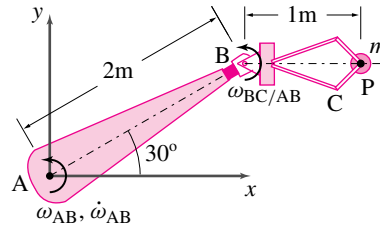
$$\mathbf{a}_P = -\omega_1^2 L(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) + (\omega_1 + \omega_2)^2 r\hat{\mathbf{j}}$$

b) Next, consider the situation where moving frame rotates with the rod, i.e., $\omega_{\mathcal{B}} = \omega_1\hat{\mathbf{k}}$ and $\omega_{\text{disk}/\mathcal{B}} = \omega_2\hat{\mathbf{k}}$

$$\therefore \mathbf{v}_{P/\mathcal{B}} = \omega_2\hat{\mathbf{k}} \times \mathbf{r}_{CP} = \omega_2 r\hat{\mathbf{i}}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

19.2.4 Robot arm, 2-D. The robot arm AB is rotating about point A with $\omega_{AB} = 5 \text{ rad/s}$ and $\dot{\omega}_{AB} = 2 \text{ rad/s}^2$. Meanwhile the forearm BC is rotating at a constant angular speed with respect to AB of $\omega_{BC/AB} = 3 \text{ rad/s}$. Gravity cannot be neglected. At the instant shown, find the net force acting on the object P which has mass $m = 1 \text{ kg}$.



Problem 19.4

Answer:

$$\vec{F} = m\vec{a} = -109.3 \text{ N}\hat{i} - 19.54 \text{ N}\hat{j}.$$

18.2.2

Applying LMB on 'mass, m': $\sum \vec{F} = m\vec{a}$

$$-T\hat{e}_r = m[\ddot{r}_{A/O'} + \ddot{\theta} \times r_{A/O'} + \dot{\theta} \times (\dot{\theta} \times r_{A/O'})]$$

$$\ddot{r} = \ddot{r}_{O'} + \ddot{\theta} L \hat{e}_\theta - L \ddot{\theta} \hat{e}_r$$

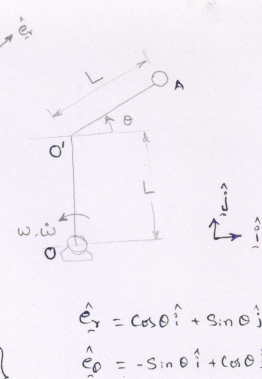
$$\ddot{r}_{O'} = \ddot{r}_{O'} + \dot{\omega}_0 \times r_{O'/O} + \omega_0 \times (\omega_0 \times r_{O'/O})$$

$$= -\dot{\omega}_0 L \hat{i} - L \dot{\omega}_0 \hat{j}$$

$$\therefore \{-T\hat{e}_r = m[-\dot{\omega}_0 L \hat{i} - L \dot{\omega}_0 \hat{j} + \ddot{\theta} L \hat{e}_\theta - L \ddot{\theta} \hat{e}_r]\}$$

$$\{ \} \cdot \hat{e}_\theta \Rightarrow 0 = m[-\dot{\omega}_0 L (\sin \theta) - L \dot{\omega}_0 \cos \theta + \ddot{\theta} L]$$

$$\therefore \ddot{\theta} = \dot{\omega}_0 \cos \theta - \dot{\omega}_0 \sin \theta //$$



$$\hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

18.2.4

Let $\vec{T}$ be the net force acting on the mass at 'P'.Using L.M.B. $\sum \vec{F} = m\vec{a}$

$$\vec{T} = \vec{F}(-\hat{i}) + mg(-\hat{j}) = m\vec{a}_P$$

$$\vec{a}_P = \vec{a}_B + \dot{\omega}_{BC} \times \vec{r}_2 + \dot{\omega}_{BC} \times (\dot{\omega}_{BC} \times \vec{r}_2)$$

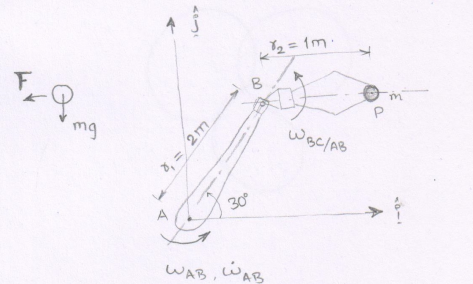
$$= \dot{\omega}_{AB} \times \vec{r}_1 + \dot{\omega}_{AB} \times (\dot{\omega}_{AB} \times \vec{r}_1) + \dot{\omega}_{AB} \hat{j} - \dot{\omega}_{BC} \hat{i}$$

$$\vec{a}_P = 2\dot{\omega}_{AB} \vec{r}_1 \hat{e}_\theta - \vec{r}_1 \dot{\omega}_{AB} \hat{e}_r - \dot{\omega}_{BC} \hat{i} + \dot{\omega}_{AB} \hat{j}$$

$$\therefore \vec{T} = m(-\vec{r}_1 \dot{\omega}_{AB} \sin 30^\circ - \dot{\omega}_{BC} - \vec{r}_1 \dot{\omega}_{AB} \cos 30^\circ) \hat{i} + m(\vec{r}_1 \dot{\omega}_{AB} \cos 30^\circ - \vec{r}_1 \dot{\omega}_{AB} \sin 30^\circ + \dot{\omega}_{AB}) \hat{j}$$

$$\text{Given: } \vec{r}_1 = 2\text{m}, \omega_{AB} = 5 \text{ rad/s}, \dot{\omega}_{AB} = 2 \text{ rad/s}^2; m = 1\text{kg}, \omega_{BC} = \omega_{BC/AB} + \omega_{AB} = 8 \text{ rad/s}$$

$$\Rightarrow \vec{T} = -109.3 \hat{i} - 19.5 \hat{j}$$



$$\hat{e}_r = \cos 30^\circ \hat{i} + \sin 30^\circ \hat{j}$$

$$\hat{e}_\theta = -\sin 30^\circ \hat{i} + \cos 30^\circ \hat{j}$$

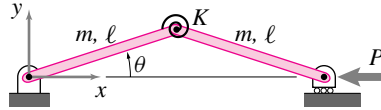
Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

19.2.5 A crude model for a column, shown in the figure, consists of two identical rods of mass m and length ℓ with hinge connections, a linear torsional spring of stiffness K attached to the center hinge (the spring is relaxed when $\theta = 0$), and a load P applied at the top end.

- Obtain the exact nonlinear equation of motion.
- Obtain the squared natural fre-

quency for small motions θ .

- Check the stability of the straight equilibrium state $\theta = 0$ for all $P \geq 0$ via minimum potential energy. How do these results compare with those from part(b)?



Problem 19.5

18.2.5

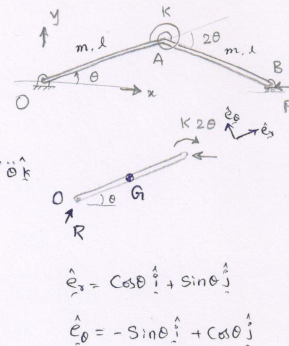
Applying A.M.B about 'O': $\sum \vec{M}_O = \vec{H}/O$

$$\vec{r}_A \times \vec{P} - K(2\theta)\hat{k} = \vec{r}_G \times m\vec{a}_G + \frac{1}{12}m\ell^2\ddot{\theta}\hat{k}$$

$$1\dot{\vec{e}}_2 \times P(-\hat{i}) - 2K\theta\hat{k} = \frac{\ell}{2}\vec{e}_2 \times m\left[-\omega^2\frac{\ell}{2}\vec{e}_2 + \ddot{\theta}\frac{\ell}{2}\vec{e}_2\right] + \frac{1}{12}m\ell^2\ddot{\theta}\hat{k}$$

$$\Rightarrow \left\{ (Pl \sin\theta - 2K\theta)\hat{k} = \left(\frac{m\ell^2}{4}\hat{k} + \frac{m\ell^2}{12}\hat{k} \right) \ddot{\theta} \right\}$$

$$\left\{ \right\} \cdot \hat{k} \Rightarrow \frac{1}{3}m\ell^2\ddot{\theta} + 2K\theta - Pl \sin\theta = 0$$



$$\begin{aligned}\hat{e}_2 &= \cos\theta\hat{i} + \sin\theta\hat{j} \\ \dot{\hat{e}}_2 &= -\sin\theta\dot{\theta}\hat{i} + \cos\theta\dot{\theta}\hat{j}\end{aligned}$$

b) For small oscillations, $\sin\theta \approx \theta$

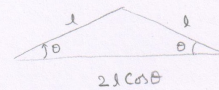
$$\Rightarrow \frac{1}{3}m\ell^2\ddot{\theta} + (2K - Pl)\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{3(2K - Pl)}{m\ell^2}\theta = 0$$

$$\therefore \text{Squared natural frequency, } \lambda = \frac{3(2K - Pl)}{m\ell^2}$$

c) Energy stored in the spring, E_{spring}

$$E_{\text{spring}} = \frac{1}{2}K(2\theta)^2 = 2K\theta^2$$



Work done by the load 'P' in moving a distance, d , is

$$W = P \cdot d = P(2\ell - 2\ell \cos\theta)$$

$$\Rightarrow W = 2Pl(1 - \cos\theta) = 2Pl\left(1 - \left(1 - \frac{\theta^2}{2} + \dots\right)\right) \approx Pl\theta^2$$

$$\therefore \text{Potential Energy} = E_{\text{spring}} - W$$

System becomes unstable if $W - E_{\text{spring}} > 0$.

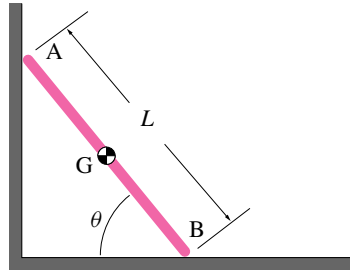
For $P \geq 0$, $E_{\text{spring}} - W > 0$ results in a stable system.

Also, $2K - Pl > 0$ satisfies the requirement in part(b)

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

19.2.6 A thin uniform rod of mass m rests against a frictionless wall and on a frictionless floor. There is gravity.

- Draw a free-body diagram of the rod.
- The rod is released from rest at $\theta = \theta_0 \neq 0$. Write the equation of motion of the rod.
- Using the equation of motion, find the initial angular acceleration, $\dot{\omega}_{AB}$, and the acceleration of the center of mass, $\ddot{\mathbf{a}}_G$, of the rod.
- Find the reactions on the rod at points A and B.
- Find the acceleration of point B.
- When $\theta = \frac{\theta_0}{2}$, find $\ddot{\omega}_{AB}$ and the acceleration of point A.



Problem 19.6

Answer:

(b) $\ddot{\theta} = -\frac{3g}{2L} \cos \theta$.

(c) From the diagram, we see $\dot{\omega}_{AB} = -\ddot{\theta}(t=0)\hat{\mathbf{k}} = \frac{3g}{2L} \cos \theta_0$, $\ddot{\mathbf{a}}_G = \frac{3}{4}g \cos \theta_0 [\sin \theta_0 \hat{\mathbf{i}} - \cos \theta_0 \hat{\mathbf{j}}]$.

(d) $\vec{\mathbf{R}}_A = \frac{3}{4}mg \sin \theta_0 \cos \theta_0 \hat{\mathbf{i}}$, $\vec{\mathbf{R}}_B = mg[1 - \frac{3}{4} \cos^2 \theta_0] \hat{\mathbf{j}}$.

(e) $\ddot{\mathbf{a}}_B = \frac{3}{2}g \sin \theta_0 \cos \theta_0 \hat{\mathbf{i}}$.

(f) At $\theta = \frac{\theta_0}{2}$, $\ddot{\omega}_{AB} = \sqrt{\frac{3g}{L}(\sin \theta_0 - \sin \frac{\theta_0}{2})} \hat{\mathbf{k}}$, $\ddot{\mathbf{a}}_A = -3g \left[\frac{1}{2} \cos^2 \frac{\theta_0}{2} + \sin \frac{\theta_0}{2} (\sin \theta_0 - \sin \frac{\theta_0}{2}) \right] \hat{\mathbf{j}}$.

19.2.6

a) Free body diagram of rod:

b) Given, initial $\theta = \theta_0 \neq 0$; $\dot{\mathbf{r}}_G = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{G/B}$

$\dot{\mathbf{r}}_B = L \cos \theta \hat{\mathbf{i}}$ and $\dot{\mathbf{r}}_{G/B} = \frac{L}{2} \cos \theta (-\dot{\theta}) \hat{\mathbf{i}} + \frac{L}{2} \sin \theta \dot{\theta} \hat{\mathbf{j}}$

Apply L.M.B., $\Sigma \mathbf{F} = m \mathbf{a}_G$

$\Rightarrow \{ R_A \hat{\mathbf{i}} + R_B \hat{\mathbf{j}} - mg \hat{\mathbf{j}} = m \dot{\mathbf{r}}_G \}$

$\{ \dot{\mathbf{r}}_G \} \Rightarrow R_A = -\frac{mL}{2} (\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2)$

$\{ \dot{\mathbf{r}}_G \} \Rightarrow R_B = mg + \frac{mL}{2} (\cos \theta \ddot{\theta} - \sin \theta \dot{\theta}^2)$

Apply A.M.B. about G , $\Sigma M_G = I_G \dot{\omega}_G$

$-R_A \frac{L}{2} \sin \theta \hat{\mathbf{k}} + R_B \frac{L}{2} \cos \theta \hat{\mathbf{k}} = I_G \dot{\omega}_G \hat{\mathbf{k}}$

$\left\{ \frac{mL}{4} (\sin \theta \ddot{\theta} + \cos \theta \sin \theta \dot{\theta}^2) + \frac{mL}{4} (\cos \theta \ddot{\theta} - \sin \theta \sin \theta \dot{\theta}^2) + \frac{mgL}{2} \cos \theta \hat{\mathbf{k}} = \frac{mL}{12} (\ddot{\theta}) \hat{\mathbf{k}} \right\}$

$\{ \dot{\mathbf{r}}_G \} \Rightarrow \frac{mL}{4} \ddot{\theta} + \frac{mgL}{2} \cos \theta = -\frac{mL}{12} \ddot{\theta}$

$\Rightarrow \ddot{\theta} + \frac{3g}{2L} \cos \theta = 0$

c) $\dot{\omega}_{AB} = -\ddot{\theta} \hat{\mathbf{k}} = \frac{3g}{2L} \cos \theta_0 \hat{\mathbf{k}}$ and $\mathbf{a}_G = \ddot{\mathbf{r}}_G = \frac{3g}{4} \cos \theta_0 (\sin \theta_0 \hat{\mathbf{i}} - \cos \theta_0 \hat{\mathbf{j}})$

d) Reactions, $R_A \hat{\mathbf{i}} = \frac{3mg}{4} \sin \theta_0 \cos \theta_0 \hat{\mathbf{i}}$ and $R_B \hat{\mathbf{j}} = mg(1 - \frac{3}{4} \cos^2 \theta_0) \hat{\mathbf{j}}$

e) $\dot{\mathbf{r}}_B = L \sin \theta \dot{\theta} \hat{\mathbf{i}} \Rightarrow \dot{\mathbf{r}}_B = (-L \cos \theta \dot{\theta} - L \sin \theta \ddot{\theta}) \hat{\mathbf{i}} \Rightarrow \dot{\mathbf{r}}_B = \frac{3}{2}g \sin \theta_0 \cos \theta_0 \hat{\mathbf{i}} \quad (\dot{\theta} = 0)$

f) $\dot{\omega}_{AB} = ?$ $\dot{\mathbf{r}}_A = ?$ at $\theta = \frac{\theta_0}{2}$

Re writing the equation of motion, $\int \ddot{\theta} d\theta = \int_0^{\theta_0} -\frac{3g}{2L} \cos \theta d\theta$

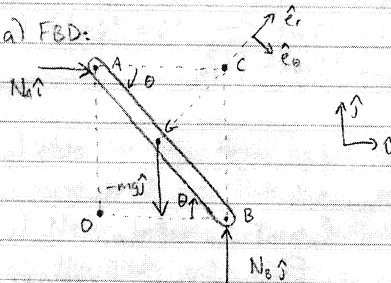
$\therefore \dot{\theta} = -\sqrt{\frac{3g}{L} [\sin \theta_0 - \sin \theta_0/2]}$

$\dot{\mathbf{r}}_A = L \cos \theta \dot{\theta} \hat{\mathbf{j}} \Rightarrow \dot{\mathbf{r}}_A = (L \cos \theta \dot{\theta} - L \sin \theta \ddot{\theta}) \hat{\mathbf{j}} \quad (\text{or}) \quad \dot{\mathbf{r}}_A = -3g \left[\frac{1}{2} \cos^2 \frac{\theta_0}{2} + \sin \frac{\theta_0}{2} (\sin \theta_0 - \sin \frac{\theta_0}{2}) \right] \hat{\mathbf{j}}$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

Johannes Ferry
 HW problems 16.30, 16.39,
 16.41, 16.43, 16.48
 Due 12/4/98
 TAM 2030
 Section 2-1-25
 TA: Rung Long

(6.30) a) FBD:



b) $\theta(t=0) = \theta_0$

$$AMB_{/C}: \sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$$\vec{r}_{G/C} \times (-mg\hat{j}) = \vec{r}_{G/C} \times m\vec{a}_G + I_G \dot{\omega} \hat{k}$$

$$\left(-\frac{L}{2} \cos \theta \hat{i} - \frac{L}{2} \sin \theta \hat{j}\right) \times (-mg\hat{j}) = \left(-\frac{L}{2} \cos \theta \hat{i} - \frac{L}{2} \sin \theta \hat{j}\right) \times m\vec{a}_G - \frac{mL^2}{12} \ddot{\theta} \hat{k}$$

$$\frac{1}{2} mgL \cos \theta \hat{k} = \left(-\frac{L}{2} \cos \theta \hat{i} - \frac{L}{2} \sin \theta \hat{j}\right) \times m\vec{a}_G - \frac{mL^2}{12} \ddot{\theta} \hat{k}$$

Need $\vec{a}_G$: use circular motion of G

$$\begin{aligned} \vec{a}_G &= -\dot{\theta}^2 \frac{L}{2} \hat{e}_r + \ddot{\theta} \frac{L}{2} \hat{e}_\theta = -\dot{\theta}^2 \frac{L}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) + \ddot{\theta} \frac{L}{2} (\sin \theta \hat{i} - \cos \theta \hat{j}) \\ &= (-\dot{\theta}^2 \frac{L}{2} \cos \theta + \ddot{\theta} \frac{L}{2} \sin \theta) \hat{i} + (-\dot{\theta}^2 \frac{L}{2} \sin \theta - \ddot{\theta} \frac{L}{2} \cos \theta) \hat{j} \end{aligned}$$

Plug $\vec{a}_G$ into AMB:

$$\left\{ \frac{1}{2} mgL \cos \theta \hat{k} = -\left(\frac{mL^2}{4} (\ddot{\theta} \sin \theta \cos \theta + \dot{\theta}^2 \cos^2 \theta) + \frac{mL^2}{4} (-\dot{\theta}^2 \sin \theta \cos \theta + \ddot{\theta} \sin^2 \theta) \right) \hat{k} - \frac{mL^2}{12} \ddot{\theta} \hat{k} \right\}$$

$$\left\{ \hat{i} \cdot \hat{k} \rightarrow \frac{1}{2} mgL \cos \theta = -\frac{mL^2}{4} \ddot{\theta} (\cos^2 \theta + \sin^2 \theta) - \frac{mL^2}{12} \ddot{\theta} \right\}$$

$$6g \cos \theta = -3L\ddot{\theta} - L\ddot{\theta} = -4L\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3g}{4L} \cos \theta$$

c) From part (b): $\dot{\theta}(t=0) = \frac{-3g}{4L} \cos \theta_0$ (released from rest)

$$\vec{a}_G(t=0) = (-\dot{\theta}_0^2 \frac{L}{2} \cos \theta_0 + \ddot{\theta}_0 \frac{L}{2} \sin \theta_0) \hat{i} + (-\dot{\theta}_0^2 \frac{L}{2} \sin \theta_0 - \ddot{\theta}_0 \frac{L}{2} \cos \theta_0) \hat{j}$$

$$\vec{a}_G(t=0) = -\frac{3g}{4} \cos \theta_0 (\sin \theta_0 \hat{i} - \cos \theta_0 \hat{j})$$

d) LMB: $\sum \vec{F} = m\vec{a}_G$

$$\{ N_A \hat{i} + (N_B - mg) \hat{j} = m\vec{a}_G \}$$

$$\{ \hat{i} \cdot \rightarrow N_A = \frac{mL}{2} (-\dot{\theta}^2 \cos \theta + \ddot{\theta} \sin \theta) \}$$

$$\{ \hat{j} \cdot \rightarrow N_B - mg = \frac{mL}{2} (-\dot{\theta}^2 \sin \theta - \ddot{\theta} \cos \theta) \}$$

$$\{ N_B = m \left(\frac{L}{2} (-\dot{\theta}^2 \sin \theta - \ddot{\theta} \cos \theta) + g \right) \}$$

$$\begin{aligned} \text{e) } \vec{a}_B &= \vec{a}_G + \vec{a}_{G/B} = (-\dot{\theta}^2 \frac{L}{2} \cos \theta + \ddot{\theta} \frac{L}{2} \sin \theta) \hat{i} + (-\dot{\theta}^2 \frac{L}{2} \sin \theta - \ddot{\theta} \frac{L}{2} \cos \theta) \hat{j} - \dot{\theta}^2 \frac{L}{2} \hat{e}_\theta + \ddot{\theta} \frac{L}{2} \hat{e}_r \\ &= (-\dot{\theta}^2 \frac{L}{2} \cos \theta + \ddot{\theta} \frac{L}{2} \sin \theta) \hat{i} + (-\dot{\theta}^2 \frac{L}{2} \sin \theta - \ddot{\theta} \frac{L}{2} \cos \theta) \hat{j} - \dot{\theta}^2 \frac{L}{2} (\sin \theta \hat{i} - \cos \theta \hat{j}) + \ddot{\theta} \frac{L}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= \left[-\dot{\theta}^2 \frac{L}{2} (\cos \theta + \sin \theta) + \ddot{\theta} \frac{L}{2} (\sin \theta + \cos \theta) \right] \hat{i} + \left[-\dot{\theta}^2 \frac{L}{2} (\sin \theta + \cos \theta) + \ddot{\theta} \frac{L}{2} (\cos \theta + \sin \theta) \right] \hat{j} \end{aligned}$$



f) Point A moves in circles around Point G

$$\vec{V}_A = -\omega \frac{L}{2} \hat{e}_r = -\omega \frac{L}{2} (\cos\theta \hat{i} + \sin\theta \hat{j})$$

$$\vec{V}_A = \vec{V}_B \text{ (rigid object)}$$

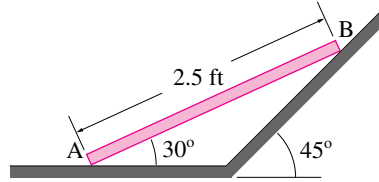
$$|\vec{a}_A| = |\vec{a}_B|$$

19.2.7 Bar leans on a crooked wall.

A uniform 3 lbm bar leans on a wall and floor and is let go from rest. Gravity pulls it down.

- Draw a Free-Body Diagram of the bar.
- Kinematics: find the velocity and acceleration of point B in terms of the velocity and acceleration of point A
- Using equations of motion, find the acceleration of point A.

- Do the reaction forces at A and B add up to the weight of the bar? Why or why not? (You do not need to solve for the reaction forces in order to answer this part.)



Problem 19.7

19.2.7

a) Free body diagram:

b) Velocity and acceleration of point B?

Velocity of point B: $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$

$\vec{v}_A = (l \cos \theta - l \sin \theta) (-\dot{\theta} \hat{i}) \Rightarrow \dot{\vec{v}}_A = (l \sin \theta \dot{\theta} + l \cos \theta \ddot{\theta}) \hat{i}$

$\vec{v}_{B/A} = l \cos \theta \dot{\theta} \hat{i} + l \sin \theta \dot{\theta} \hat{j} \Rightarrow \dot{\vec{v}}_{B/A} = -l \sin \theta \dot{\theta} \hat{i} + l \cos \theta \dot{\theta} \hat{j}$

Acceleration of point B: $\ddot{\vec{v}}_B = \ddot{\vec{v}}_A + \ddot{\vec{v}}_{B/A}$

$\ddot{\vec{v}}_A = l (\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2 + \cos \theta \ddot{\theta} - \sin \theta \dot{\theta}^2) \hat{i}$

$\ddot{\vec{v}}_{B/A} = (-l \sin \theta \ddot{\theta} - l \cos \theta \dot{\theta}^2) \hat{i} + (l \cos \theta \ddot{\theta} - l \sin \theta \dot{\theta}^2) \hat{j}$

$\ddot{\vec{v}}_B = +l [\cos \theta \ddot{\theta} - \sin \theta \dot{\theta}^2] (\hat{i} + \hat{j})$

c) Using AMB about I: $\sum \vec{M}_I = \dot{\vec{H}}_I$

$\Rightarrow \vec{r}_{G/I} \times mg(-\hat{j}) = \vec{I} \dot{\omega} + \vec{r}_{G/I} \times m \vec{a}_G$

$\vec{r}_{G/I} = \frac{l}{2} \cos \theta \hat{i} + (l \cos \theta + \frac{l}{2} \sin \theta) (-\hat{j})$

$\vec{a}_G = \dot{\omega} \times \vec{r}_{G/I} + \omega \times (\omega \times \vec{r}_{G/I}) = \dot{\omega} \times \vec{r}_{G/I} - \omega^2 \vec{r}_{G/I}$

$\Rightarrow -\frac{mgl}{2} \cos \theta \hat{k} = (-\dot{\theta}) \frac{1}{12} m l^2 \hat{k} + \left[(-\dot{\theta}) \left\{ \frac{1}{2} \cos \theta \hat{k} + (-\dot{\theta}) \left[-l \cos \theta - \frac{l}{2} \sin \theta \right] \hat{k} \right\} \right] m$

$\Rightarrow \ddot{\theta} = \frac{3g \cos \theta}{l(2 + 6 \cos^2 \theta + 3 \sin 2\theta)}$

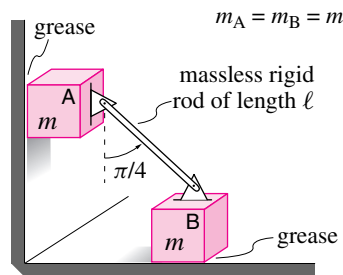
As the bar starts from rest, $\dot{\theta} = 0$

$\therefore \ddot{\vec{v}}_A = -l (\sin \theta_0 + \cos \theta_0) \frac{3g \cos \theta_0}{l(2 + 6 \cos^2 \theta_0 + 3 \sin 2\theta_0)} \hat{i} = -\frac{3g}{2} \frac{\sqrt{3} + 3}{3\sqrt{3} + 13} \hat{i}$

d) Reaction forces at A and B do not add up to the weight of the bar as non zero angular acceleration of the bar would modify them.

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19.2.8 Two blocks, each with mass m , slide without friction on the wall and floor shown. They are attached with a rigid massless rod of length ℓ that is pinned at both ends. The system is released from rest when the rod makes an angle of 45° . What is the acceleration of the block B immediately after the system is released?



Problem 19.8

18.2.8

Acceleration of Block B: ?

$$\vec{r}_{G/I} = \vec{r}_A + \vec{r}_{G/A}$$

$$= -l \sin \theta \hat{i} + \frac{1}{2} l \sin \theta \hat{i} - \frac{1}{2} l \cos \theta \hat{j}$$

$$\Rightarrow \vec{r}_{G/I} = -\frac{1}{2} l \sin \theta \hat{i} - \frac{1}{2} l \cos \theta \hat{j}$$

Applying AMB about I: $\Sigma \vec{M}_I = \dot{\vec{H}}_I$

$$\Rightarrow \vec{r}_{A/I} \times m \vec{g} (-\hat{j}) = \vec{r}_G \times m \vec{a}_G = 2m \left(\frac{1}{2} \right) \ddot{\theta} \hat{k} + \vec{r}_G \times m [\ddot{\theta} \hat{k} \times \vec{r}_G - \dot{\theta}^2 \vec{r}_G]$$

$$\Rightarrow \left\{ -l \sin \theta \hat{i} \times m g (-\hat{j}) = \frac{m l}{2} \ddot{\theta} \hat{k} + m \ddot{\theta} \left[\left(-\frac{1}{2} l \sin \theta \right) \hat{i} + \left(-\frac{1}{2} l \cos \theta \right) \hat{j} \right] \times \hat{k} \right\}$$

$$\left\{ \right\} \cdot \hat{k} \Rightarrow m g l \sin \theta = \frac{m l}{2} (1 + \frac{1}{2}) \ddot{\theta} = \frac{3}{4} m l \ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = \frac{4g}{3l} \sin \theta$$

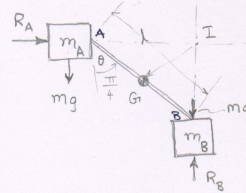
$$\vec{r}_B = l \sin \theta \hat{i}$$

$$\dot{\vec{r}}_B = l \cos \theta \dot{\theta} \hat{i}$$

$$\ddot{\vec{r}}_B = (-l \sin \theta \ddot{\theta} + l \cos \theta \dot{\theta}^2) \hat{i}$$

Immediately after release, $\theta = \frac{\pi}{4}$; $\dot{\theta} = 0$

$$\Rightarrow \ddot{\vec{r}}_B = \frac{4g}{3} \sin \theta \cos \theta = \frac{2g}{3}$$



18.2.9

Tension in AB = ?

$$\vec{r}_B = (R \cos \theta + L \cos \phi) \hat{i}$$

$$\dot{\vec{r}}_B = (-R \sin \theta \dot{\theta} - L \sin \phi \dot{\phi}) \hat{i}$$

$$\ddot{\vec{r}}_B = \left\{ -R [\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2] - L [\sin \phi \ddot{\phi} + \cos \phi \dot{\phi}^2] \right\} \hat{i}$$

When piston is at its maximum extent, $\theta = \phi = 0 \Rightarrow \dot{\phi} = \frac{R}{L} \dot{\theta}$

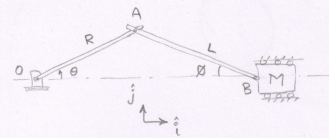
$$\Rightarrow \ddot{\vec{r}}_B = \left\{ -R \ddot{\theta} - L \frac{R}{L} \ddot{\theta} \right\} \hat{i} = -R \ddot{\theta} \left(1 + \frac{R}{L} \right) \hat{i}$$

Apply LMB on mass: $\Sigma \vec{F} = m \vec{a}$

$$\Rightarrow \left\{ -T \hat{i} + R \hat{j} = m \ddot{\vec{r}}_B = -m R \ddot{\theta} \left(1 + \frac{R}{L} \right) \hat{i} \right\}$$

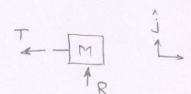
$$\left\{ \right\} \cdot \hat{i} \Rightarrow -T = -m R \ddot{\theta} \left(1 + \frac{R}{L} \right)$$

$$\therefore \text{Tension in the rod AB, } T = m(L+R) \ddot{\theta} \left(\frac{R}{L} \right)$$



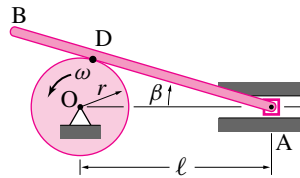
$$R \sin \theta = L \sin \phi$$

$$\cos \phi \dot{\phi} = \frac{R}{L} \cos \theta \dot{\theta}$$



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19.2.10 In problem ??, find the force on the wheel at point D due to the rod.



Problem 19.10

18.2.10

Force on the wheel at 'D' due to the rod = ?

To find ω_{AB} , $\mathbf{v}_A = \mathbf{v}_D + \mathbf{v}_{A/D}$

$$\mathbf{v}_D = \omega \hat{k} \times \mathbf{r}_D = r\omega \hat{e}_\theta$$

$$\mathbf{v}_{A/D} = \omega_{AB} \hat{k} \times \mathbf{r}_{A/D} = \omega_{AB} \hat{k} \times \sqrt{l^2 - r^2} (-\hat{e}_\theta)$$

$$\left\{ \begin{array}{l} \mathbf{v}_A = -\omega r \cos\beta \hat{i} + \omega r \sin\beta \hat{j} + \omega_{AB} \sqrt{l^2 - r^2} [\sin\beta \hat{i} + \cos\beta \hat{j}] \end{array} \right\}$$

$$\left\{ \cdot \right\} \cdot \hat{j} \Rightarrow \omega_{AB} = \frac{-\omega r^2}{(l^2 - r^2)} \Rightarrow \mathbf{v}_{A/D} = \frac{-\omega r l}{\sqrt{l^2 - r^2}} \hat{i}$$

To find $\dot{\omega}_{AB}$, consider $\mathbf{a}_A = \mathbf{a}_D + \mathbf{a}_{A/D}$

$$\mathbf{a}_D = \dot{\omega} \hat{k} \times \mathbf{r}_D + \omega \times (\omega \times \mathbf{r}_D) \Rightarrow \mathbf{a}_D = -r\dot{\omega} \hat{e}_r$$

$$\mathbf{a}_{A/D} = \dot{\omega}_{AB} \times \mathbf{r}_{A/D} + \omega_{AB} \times (\omega_{AB} \times \mathbf{r}_{A/D})$$

$$\left\{ \begin{array}{l} \mathbf{a}_A = \mathbf{a}_D + \mathbf{a}_{A/D} = -r\dot{\omega} (\sin\beta \hat{i} + \cos\beta \hat{j}) + \sqrt{l^2 - r^2} [\dot{\omega}_{AB} (\cos\beta \hat{j} + \sin\beta \hat{i}) - \omega_{AB}^2 (\cos\beta \hat{i} - \sin\beta \hat{j})] \end{array} \right\}$$

$$\left\{ \cdot \right\} \cdot \hat{j} \Rightarrow 0 = -\dot{\omega} r \cos\beta + \sqrt{l^2 - r^2} \dot{\omega}_{AB} \cos\beta + \sqrt{l^2 - r^2} \omega_{AB}^2 \sin\beta$$

$$\Rightarrow \dot{\omega}_{AB} = \frac{r\dot{\omega}(1 - 2r^2)}{(l^2 - r^2)^{5/2}} \quad \text{and} \quad \mathbf{a}_A = \frac{-l r^4 \dot{\omega}}{(l^2 - r^2)^2}$$

$$\mathbf{a}_G = \mathbf{a}_D + \mathbf{a}_{G/D}$$

$$= -r\dot{\omega} \hat{e}_r + \dot{\omega}_{AB} \times (\omega_{AB} \times \mathbf{r}_{G/D}) + \omega_{AB} \times \mathbf{r}_{G/D}$$

$$= (\dot{\omega}_{AB} |\mathbf{r}_{G/D}| - r\dot{\omega}) \hat{e}_x - \omega_{AB}^2 |\mathbf{r}_{G/D}| \hat{e}_\theta$$

Applying L.M.B on rod: $\Sigma \mathbf{F} = m \mathbf{a}_G$

$$\Rightarrow \left\{ \begin{array}{l} F_x \hat{i} + F_y \hat{j} - mg \hat{j} + N \hat{j} = m a_{Gx} \hat{i} + m a_{Gy} \hat{j} \end{array} \right\}$$

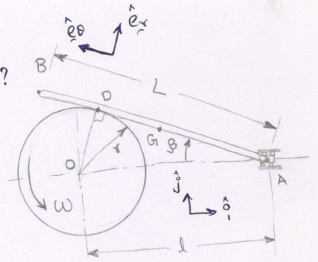
$$\left\{ \cdot \right\} \cdot \hat{i} \Rightarrow F_x = -m r \dot{\omega} \left\{ \frac{L(l^2 - r^2)^2 - l r^2 (L - 2\sqrt{l^2 - r^2})}{2l(l^2 - r^2)^{5/2}} \right\}$$

$$\left\{ \cdot \right\} \cdot \hat{j} \Rightarrow N = mg - F_y - m r \dot{\omega} \frac{L}{2l}$$

Applying A.M.B about 'D': $\Sigma \mathbf{M}_D = \dot{\mathbf{H}}_D$

$$\Rightarrow \left\{ \begin{array}{l} \mathbf{r}_{G/D} \times mg(-\hat{j}) + \mathbf{r}_{A/D} \times N \hat{j} = \mathbf{I}_G \dot{\omega}_{AB} \hat{k} + \mathbf{r}_{G/D} \times m \mathbf{a}_G \end{array} \right\}$$

$$\left\{ \cdot \right\} \cdot \hat{k} \Rightarrow F_y = \left[\frac{L}{2} \sqrt{l^2 - r^2} (mg - m r \dot{\omega} \frac{L}{2l}) - r \left(\frac{L}{2} - \sqrt{l^2 - r^2} \right) F_x + \frac{m r \dot{\omega} L (1 - 2r^2)}{12(l^2 - r^2)^{5/2}} \right] / \left[\sqrt{l^2 - r^2} (L - \sqrt{l^2 - r^2}) \right]$$

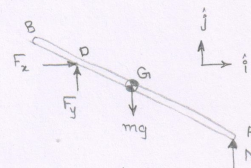
 $\therefore$ Force on wheel, $\mathbf{F} = -F_x \hat{i} - F_y \hat{j}$ 

$$\hat{e}_r = \sin\beta \hat{i} + \cos\beta \hat{j}$$

$$\hat{e}_\theta = -\cos\beta \hat{i} + \sin\beta \hat{j}$$

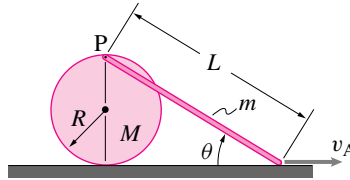
$$\sin\beta = \frac{r}{l}, \cos\beta = \frac{\sqrt{l^2 - r^2}}{l}$$

$$\mathbf{r}_{G/D} = \left(\sqrt{l^2 - r^2} - \frac{L}{2} \right) (-\hat{e}_\theta)$$



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19.2.11 In problem ??, find the force on the rod at point P.



Problem 19.11

18.2.11:

Force on rod at point 'P' = ?

$$\vec{V}_A = \vec{V}_A \hat{i} = \vec{V}_P + \vec{V}_{A/P}$$

$$\vec{V}_P = \omega_D \hat{k} \times 2R \hat{j}$$

$$\vec{V}_{A/P} = \omega_r \hat{k} \times -L \hat{\lambda}$$

$$\Rightarrow \left\{ \vec{V}_A = (-2R\omega_D + \omega_r L \sin\theta) \hat{i} + \omega_r L \cos\theta \hat{j} \right\}$$

$$\{ \} \cdot \hat{j} \Rightarrow \omega_r = 0 \text{ and } V_A = -2R\omega_D //$$

$$\vec{a}_A = \vec{a}_P + \vec{a}_{A/P}$$

$$\vec{a}_P = \dot{\omega}_D (-R \hat{i}) \times 2R \hat{j} + \omega_D \times (\omega_D \times 2R \hat{j}) = 2R\dot{\omega}_D \hat{i} - 2R\omega_D^2 \hat{j}$$

$$\vec{a}_{A/P} = \dot{\omega}_r \hat{k} \times L(-\hat{\lambda}) + \omega_r^2 \hat{k} = L\dot{\omega}_r \sin\theta \hat{i} + L\dot{\omega}_r \cos\theta \hat{j}$$

$$\Rightarrow \vec{a}_A = (2R\dot{\omega}_D + L\dot{\omega}_r \sin\theta) \hat{i} + \left(-\frac{V_A}{2R} + L\dot{\omega}_r \cos\theta \right) \hat{j} = 0$$

$$\Rightarrow \dot{\omega}_r = \frac{V_A}{2RL \cos\theta} \text{ and } \dot{\omega}_D = -\frac{V_A}{4R} \tan\theta ; \vec{a}_P = \frac{-V_A}{2R} \tan\theta \hat{i} - \frac{V_A}{2R} \hat{j}$$

$$\vec{a}_G = \vec{a}_P + \vec{a}_{G/P}$$

$$\vec{a}_{G/P} = \dot{\omega}_r \times \vec{r}_{G/P} + \omega_r \times (\omega_r \times \vec{r}_{G/P}) = \frac{V_A}{4R} \hat{j} + \frac{V_A}{4R} \tan\theta \hat{i}$$

$$\Rightarrow \vec{a}_G = \frac{-V_A}{4R} \tan\theta \hat{i} - \frac{V_A}{4R} \hat{j}$$

Applying L.M.B on rod: $\sum \vec{F} = m \vec{a}_G$

$$\Rightarrow \left\{ F_x \hat{i} + F_y \hat{j} - mg \hat{j} + N \hat{j} = -\frac{mV_A}{4R} \tan\theta \hat{i} - \frac{mV_A}{4R} \hat{j} \right\}$$

$$\{ \} \cdot \hat{i} \Rightarrow F_x = -\frac{mV_A}{4R} \tan\theta // \text{ and } \{ \} \cdot \hat{j} \Rightarrow N = -F_y + mg - \frac{mV_A}{4R}$$

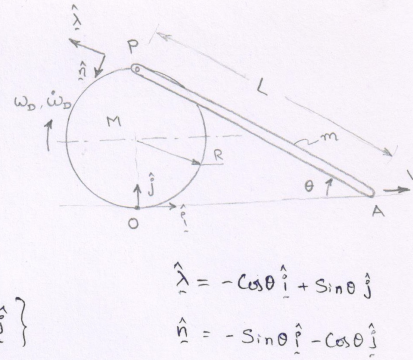
Applying A.M.B about G: $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$

$$\Rightarrow \left\{ \frac{L}{2} \hat{\lambda} \times (F_x \hat{i} + F_y \hat{j}) - \frac{L}{2} \hat{\lambda} \times N \hat{j} = I_G \dot{\omega}_r \hat{k} \right\}$$

$$\{ \} \cdot \hat{k} \Rightarrow -F_x \sin\theta - F_y \cos\theta + N \cos\theta = \frac{mV_A}{12R \cos\theta}$$

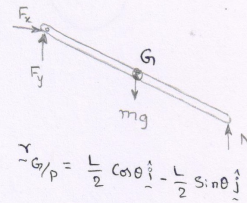
Upon simplification, we get $F_y = \frac{mg}{2} - \frac{mV_A}{8R} + \frac{mV_A}{24R} (2 \tan^2\theta - 1)$

$\therefore$ Force on rod at P, $= F_x \hat{i} + F_y \hat{j}$



$$\hat{\lambda} = -\cos\theta \hat{i} + \sin\theta \hat{j}$$

$$\hat{n} = -\sin\theta \hat{i} - \cos\theta \hat{j}$$



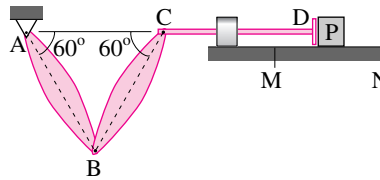
$$\vec{r}_{G/P} = \frac{L}{2} \cos\theta \hat{i} - \frac{L}{2} \sin\theta \hat{j}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

19.2.12

Slider-Crank

mechanism. The slider crank mechanism shown is used to push a 2 lbm block P. Arm AB and BC are each 0.5 ft long. Given that arm AB rotates counterclockwise at a constant 2 revolutions per second, what is the force on P at the instant shown?



Problem 19.12

Answer:

There are many solution methods.
 $-8\pi^2$ poundals to the left. (Note: 1 poundal = $1 \text{ ft} \cdot \text{lbm} \cdot \text{s}^{-2}$.)

18.2.12

Slider Crank Mechanism

Mass of block, $m = 2 \text{ lbm}$ Arm length, $AB = BC = 0.5 \text{ ft}$ Angular velocity of link AB, $\omega_{AB} = 2 \text{ rev/s (ccw)}$

What is the force on P?

Link CD is rigid constrained to move in $\pm \hat{j}$ direction. $\therefore \underline{v}_c = \underline{v}_D$ and $\underline{a}_c = \underline{a}_D$ Angular velocity of link BC, $\underline{\omega}_{BC} = ?$

$$\underline{v}_D = \underline{v}_c = \underline{v}_B + \underline{v}_{c/B}$$

$$\Rightarrow \left\{ \underline{v}_c = \underline{v}_c \hat{j} = \omega_{AB} \hat{k} \times [1 \sin 60^\circ \hat{i} + 1 \cos 60^\circ \hat{j}] + \omega_{c/B} \hat{k} \times [-1 \sin 60^\circ \hat{i} + 1 \cos 60^\circ \hat{j}] \right\}$$

$$\left\{ \right\} \cdot \hat{i} \quad \underline{v}_c \hat{j} = \omega_{AB} 1 \sin 60^\circ \hat{j} - \omega_{AB} 1 \cos 60^\circ \hat{i} - \omega_{c/B} [1 \sin 60^\circ \hat{j} + 1 \cos 60^\circ \hat{i}]$$

$$\Rightarrow 0 = -\omega_{AB} 1 \cos 60^\circ - \omega_{c/B} 1 \cos 60^\circ \Rightarrow \omega_{c/B} = -\omega_{AB} \quad \therefore \underline{\omega}_{BC} = -2 \frac{\text{rev}}{\text{s}} \hat{k}$$

Acceleration of D = ?

$$\underline{a}_D = \underline{a}_c = \underline{a}_B + \underline{a}_{c/B}$$

$$\Rightarrow \underline{a}_c = \underline{a}_c \hat{j} = -\omega_{AB}^2 [1 \sin 60^\circ \hat{i} + 1 \cos 60^\circ \hat{j}] + \underline{\dot{\omega}}_{BC} \hat{k} \times [-1 \sin 60^\circ \hat{i} + 1 \cos 60^\circ \hat{j}] -$$

$$\Rightarrow \left\{ \underline{a}_c \right\} = \left[-\omega_{AB}^2 1 \sin 60^\circ + \omega_{BC}^2 1 \sin 60^\circ - \dot{\omega}_{BC} 1 \cos 60^\circ \right] \hat{i} + \left[-\omega_{AB}^2 1 \cos 60^\circ - \omega_{BC}^2 1 \cos 60^\circ - \dot{\omega}_{BC} 1 \sin 60^\circ \right] \hat{j}$$

$$\left\{ \right\} \cdot \hat{i} \Rightarrow \dot{\omega}_{BC} = 0 \quad \text{and} \quad \left\{ \right\} \cdot \hat{j} \Rightarrow \underline{a}_c = -(\omega_{AB}^2 + \omega_{BC}^2) 1 \cos 60^\circ$$

$$\therefore \underline{a}_c = \underline{a}_D = -2 \cdot \omega_{AB}^2 1 \cos 60^\circ \hat{j} \quad (\text{or}) \quad \underline{a}_D = -1 \omega_{AB}^2 \hat{j}$$

Given that $\omega_{AB} = 2 \text{ rev/s} = 2 \cdot 2\pi \text{ rad/s}$

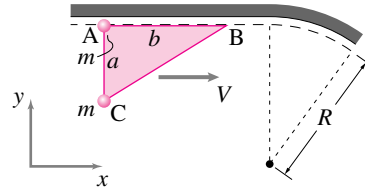
$$\therefore \text{Force on P at the instant shown, } \underline{F} = m \underline{a}_D = 2(1 \text{ lbm})(0.5 \text{ ft})(4\pi^2 \frac{\text{rad}^2}{\text{s}^2}) \hat{j}$$

$$\therefore \underline{F} = -16\pi^2 \hat{j} \text{ poundals}$$

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19.2.13 The large masses m at C and A were supported by the light triangular plate ABC , whose corners follow the guide. B enters the curved guide with velocity V . Neglecting gravity, find the vertical reactions (in the y direction) at A and B . (Hint: for the rigid planar body ABC , find $\ddot{y}_A$ and $\ddot{y}_B$ in terms of $\ddot{a}_y^{cm}$ and

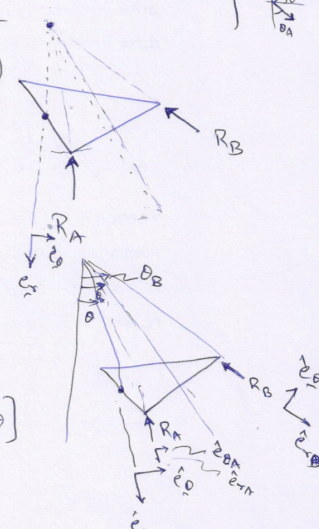
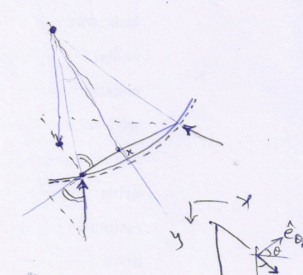
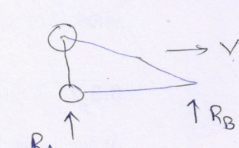
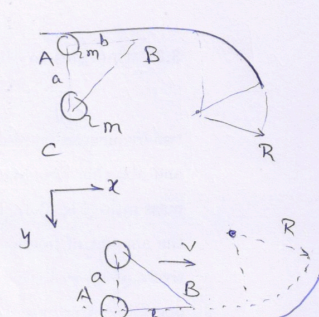
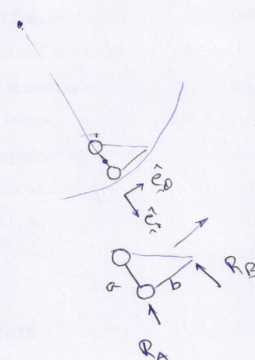
$\ddot{\theta}$, assuming $\dot{\theta} = 0$ initially.)



Problem 19.13

18-2.13

Find Reaction at ~~B~~ A & B .



$$\begin{aligned} \underline{a}_{cm} &= \underline{\dot{\omega}} \times R \underline{\hat{e}}_r + \underline{\omega} \times (\underline{\omega} \times R \underline{\hat{e}}_r) \\ &= \ddot{\theta} R \underline{\hat{e}}_\theta - \dot{\theta}^2 R \underline{\hat{e}}_r \end{aligned}$$

$$\sum \underline{F} = m \underline{a}_{cm}$$

$$\left\{ R_A (-\underline{\hat{e}}_r) + R_B (-\underline{\hat{e}}_r) = m (\ddot{\theta} R \underline{\hat{e}}_\theta - \dot{\theta}^2 R \underline{\hat{e}}_r) \right\}$$

$$\left\{ \cdot \underline{\hat{e}}_\theta \Rightarrow R_A (-\underline{\hat{e}}_r \cdot \underline{\hat{e}}_\theta) + R_B (-\underline{\hat{e}}_r \cdot \underline{\hat{e}}_\theta) = m \ddot{\theta} R (\underline{\hat{e}}_\theta \cdot \underline{\hat{e}}_\theta) \right.$$

$$\underline{\hat{e}}_{rA} = \cos \theta_A \underline{\hat{j}} + \sin \theta_A \underline{\hat{i}}$$

$$\underline{\hat{e}}_{rB} = \cos \theta_B \underline{\hat{j}} + \sin \theta_B \underline{\hat{i}}$$

$$\underline{\hat{e}}_\theta = \cos \theta \underline{\hat{j}} - \sin \theta \underline{\hat{i}}$$

$$\Rightarrow \left\{ \cdot \underline{\hat{e}}_\theta = R_A \left[-\left[\sin \theta_A \cos \theta - \sin \theta \cos \theta_A \right] \right] + R_B \left[-\sin \theta_B \cos \theta + \cos \theta_B \sin \theta \right] \right.$$

$$= m R \ddot{\theta}$$

$$\Rightarrow R_A [\sin(\theta - \theta_A)] + R_B [\sin(\theta - \theta_B)] = m R \ddot{\theta}$$

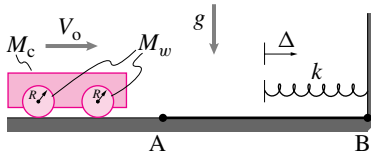
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19.2.14 An idealized model for a car comprises a rigid chassis of mass M_c and four identical rigid disks (wheels) of mass M_w and radius R , as shown in the figure. Initially in motion with speed v_0 , the car is momentarily brought to rest by compressing an initially uncompressed spring of stiffness k . Assume no frictional losses.

- Assuming no wheel slip, determine the compression Δ of the spring required to stop the car.
- While the car is in contact with the spring but still moving forward, in which direction is the *tangential* force on any one of the wheels (due to contact with the ground)

pointing? Why? (Illustrate with a sketch).

- Repeat part (a) assuming now that the ground is perfectly frictionless from points A to B shown in the figure.



Problem 19.14

Answer:

$$(a) \Delta = \sqrt{\frac{(M_c + 6M_w)v_0^2}{k}}$$

(b) Tangential forces point toward the wall!

$$(c) \Delta = \sqrt{\frac{(M_c + 4M_w)v_0^2}{k}}$$

19.2.14

a) Compression required to stop the car.

No slip and A to B is not frictionless

Using energy balance: $E_i = E_f$

Initial energy, $E_i = K.E$

$$= \frac{1}{2} M_c v_0^2 + 4 \left(\frac{1}{2} I \omega^2 + \frac{1}{2} M_w v_0^2 \right)$$

$$= \frac{1}{2} M_c v_0^2 + 2 \left(\frac{1}{2} M_w R^2 \right) \left(\frac{v_0}{R} \right)^2 + 2 M_w v_0^2$$

$$= \frac{1}{2} M_c v_0^2 + 3 M_w v_0^2$$

Final energy, $E_f = P.E$

$$= \frac{1}{2} k \Delta^2$$

$\therefore E_f = E_i \Rightarrow \frac{1}{2} k \Delta^2 = \frac{1}{2} M_c v_0^2 + 3 M_w v_0^2$

$$\Rightarrow \Delta = \frac{v_0}{\sqrt{k}} \sqrt{M_c + 6 M_w} = \sqrt{\frac{(M_c + 6 M_w) v_0^2}{k}}$$

b) While the car is in contact with the spring but still moving forward, the tangential force at the point of contact of wheels with the ground remain directed towards the wall. Even though the $\uparrow$ force on the wheel is in the opposite direction to its motion, the direction of the tangential force at the point of contact with the ground is in the direction of wheels' motion / heading.

c) If AB is frictionless, $E_i = \frac{1}{2} M_c v_0^2 + 4 \left(\frac{1}{2} M_w v_0^2 \right)$

$$E_f = E_i \Rightarrow \frac{1}{2} k \Delta^2 = \frac{1}{2} (M_c + 4 M_w) v_0^2$$

$\therefore$ Deflection of the spring required to bring the car to rest is, $\Delta = \sqrt{\frac{(M_c + 4 M_w) v_0^2}{k}}$

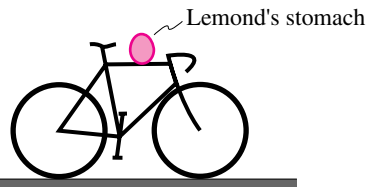
Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

19.2.15 Assume Greg Lemond's riding "tuck" was so good that you can neglect air resistance when you think about him and his bike. Further, you can regard his and the bike's combined mass (all 70 kg) as concentrated at a point in his stomach somewhere. Greg's left foot has just fallen off the pedal so he is only pedaling with his right foot, which at the moment in question is at its lowest point in the motion. You note that, relative to the ground the right foot is only going $3/4$ as fast as the bike (since it is going backwards relative to the bike), though you can't make out all the radii of his frictionless gears and rigid round wheels. Greg, ever in touch with his body, tells you he is pushing back on the pedal with a force of 70 N. You would like to know Greg's acceleration.

- a) In your first misconceived experiment you set up a 70 kg bicy-

cle in your laboratory and balance it with strings that cause no fore or aft forces. You tie a string to the vertically down right pedal and pull back with a force of 70 N. What acceleration do you measure?

- b) What is Greg's actual acceleration? [Hint: Greg's massless leg is pushing forward on his body with a force of 70 N] You can neglect the mass of the wheels and other transmission parts (chain, crank, etc).



Problem 19.15

16.39 (A)

Assume the wheels are massless and roll without slip.

If you pull the right pedal with $\vec{F}_P = -70 \text{ N } \hat{i}$, the FBD is shown on the left.

Since the wheels are massless, there is no friction on the front and a forward friction force on the back wheel.

The friction force $F_A \hat{i}$ on the rear wheel is static friction which does no work to the system. So we can use energy method.

Let the velocity of the bike be $\vec{V}_{\text{bike}} = V_b \hat{i}$

The velocity of the pedal is then $\vec{V}_{\text{pedal}} = \frac{3}{4} V_b \hat{i}$

$\therefore$ Power of $\vec{F}_P$

$$P_P = \vec{F}_P \cdot \vec{V}_{\text{pedal}} = -\frac{3}{4} V_b \times 70 = -52.5 V_b$$

(V_b in terms of "m/s")

Since $\vec{F}_P$ is the only force that does work to this system,

$$P_P = \dot{E}_K \quad \text{where } \dot{E}_K \text{ is rate of change of the kinetic energy of the system}$$

In this case

$$E_K = \frac{1}{2} m V_b^2 \Rightarrow \dot{E}_K = m V_b \dot{V}_b$$

$$\Rightarrow -52.5 V_b = 70 V_b \dot{V}_b \Rightarrow \dot{V}_b = -0.75 \text{ m/s}^2$$

$\therefore$ The acceleration of the bike is $\boxed{\vec{a}_{\text{bike}} = \dot{V}_b \hat{i} = -0.75 \text{ m/s}^2 \hat{i}}$

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(B) If Greg is riding on the bike, he pushes the pedal with $\vec{F}_p = -70 \text{ N } \hat{i}$ and he also receive the reaction force from the pedal of $70 \text{ N } \hat{i}$.

Velocity of the pedal is $\vec{V}_{\text{pedal}} = \frac{3}{4} V_b \hat{i}$

velocity of gr Greg is $\vec{V}_{\text{Greg}} = V_b \hat{i}$

Similary the total power to the system (Greg and bike) is

$$P = \vec{F}_p \cdot \vec{V}_{\text{pedal}} - \vec{F}_p \cdot \vec{V}_{\text{Greg}}$$

$$= -70 \text{ N } \hat{i} \cdot \left(\frac{3}{4} V_b \hat{i} - V_b \hat{i} \right) = 17.5 V_b \hat{i}$$

(V_b in terms of m/s)

$$\therefore P = \dot{E}_k = m V_b \dot{V}_b$$

$$\Rightarrow 17.5 V_b = 70 V_b \dot{V}_b \Rightarrow \dot{V}_b = 0.25 \text{ m/s}^2$$

$\therefore$ Greg's accelerative is

$$\boxed{\begin{aligned} \vec{a}_{\text{Greg}} &= \dot{\vec{V}}_{\text{Greg}} = \dot{V}_b \hat{i} \\ &= 0.25 \text{ m/s}^2 \hat{i} \end{aligned}}$$

This result means Greg ~~Lemond~~ accelerates forward by pushing the pedal backwards.

19.2.16 Which way does the bike accelerate? A bicycle with all frictionless bearings is standing still on level ground. A horizontal force F is applied on one of the pedals as shown. There is no slip between the wheels and the ground. The bicycle is gently balanced from falling over sideways. It is heavy enough so that both wheels stay on the ground. Does the bicycle accelerate forward, backward, or not at all? Make any reasonable assumptions about the dimensions and mass.

Justify your answer as clearly as you can, clearly enough to convince a person similar to yourself but who has not seen the experiment performed.



Problem 19.16

BIKE PEDAL-FORCE SOLUTION: WHICH WAY DOES THE BIKE GO?

A Ruina, April 20, 2009

Question. You stand next to a normal bike on flat ground. You delicately balance the bike with the steering straight. Your friend stands next to the bike and pushes backwards (to the left on the figure) on the bottom pedal. Which way does the bike accelerate?

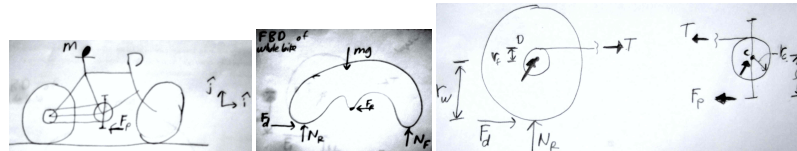
Assumptions. You apply negligible force. Your friend applies $-F_p \hat{i}$ to the pedal only. The rear wheel rolls. The only negligible mass is of the bike as a whole, m . All the bearings have negligible friction. The gearing of the bike is normal. The tension in the bottom part of the chain is negligible (not needed, but simplifies calculation). The key dimensions are

ℓ = the crank length. The crank is the piece from the middle of the bottom bracket to the pedal.

r_c = the radius of the chain wheel. The chain wheel is the gear that is in the front of the bike. It chain wheel is rigidly attached to the crank and rotates with the crank.

r_f = radius of the free-wheel sprocket. This is also called the cluster. It is the gear at the back of the bike. When there is tension on the upper chain, and the rear ratchet is thus engaged, the rear sprocket rotates with the rear wheel.

r_w = radius of the rear wheel.



Correct Answers.

- (1) The bike accelerates to the left. Try it. It always goes backwards. The next answer is different and is also correct.
- (2) The answer depends on the bike. Despite (1) above, the answer actually depends on what bike you try it on. For any bike you have ever seen or can buy, in any gear, answer (1) is correct. However, if the gearing is sufficiently low (how low? read solution below) then the bike will go forwards.

Reasoning.

A. The bike starts with no kinetic energy. It ends with some kinetic energy. Therefore some force had to do some positive work. The only force that has non-zero dot product with the velocity of the point it is touching is $-F_p \hat{i}$. Therefore *the pedal has to go backwards*. Does this mean the bike goes forwards or backwards? When a bike goes forwards the bottom pedal goes backwards relative to the bike. But relative to the ground it goes forwards. Otherwise people wouldn't ride bikes. The whole idea of a bike is that you go faster with a given leg motion than you do when you walk or run. So the bottom pedal goes the same direction as the bike. Since the pedal goes backwards, so does the bike.

B. Same as above, but let's calculate the pedal velocity and see if it really is in the same direction as the bike. Let's define ω_w and ω_c as the counterclockwise angular velocities of the rear wheel and crank,

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

2

respectively. Assume bike and pedal have absolute velocity $v_b \hat{i}$ and $v_p \hat{i}$, then

$$\begin{aligned}
 (1) \quad v_p &= v_b + v_{p/b} \\
 (2) \quad &= v_b + \omega_c \ell \\
 (3) \quad &= v_b + \omega_w \frac{r_f}{r_c} \ell \\
 (4) \quad &= v_b + (-v_b/r_w) \frac{r_f}{r_c} \ell \\
 (5) \quad &= v_b \left(1 - \frac{r_f \ell}{r_c r_w} \right).
 \end{aligned}$$

So the pedal goes forward when the bike goes forward so long as $\frac{r_f \ell}{r_c r_w} < 1$. Thus the bike goes back when the pedal goes back with the same inequality. Such bikes all go back when you push back on the pedal. On the other hand, The bike would go forwards when you push back on the pedal if the bike has a ‘very low gear’. You could get such low gears from having some combination of a big rear sprocket radius r_f , a big crank length ℓ , a small front chain ring radius r_c and a small rear wheel radius r_w .

C. Here’s a different approach. Given F_p moment balance about C for the crank and chain wheel tells us the chain tension

$$T = \frac{\ell}{r_c} F_p.$$

Moment balance about the rear axle, for the rear wheel and freewheel gear tell us that the drive force F_d is

$$F_d = \frac{r_f}{r_w} T = \frac{r_f}{r_w} \underbrace{\left(\frac{\ell}{r_c} F_p \right)}_T.$$

Now look at linear momentum for the whole bike:

$$\begin{aligned}
 (6) \quad -F_p + F_d &= ma_b \\
 (7) \quad -F_p + \underbrace{\frac{r_f}{r_w} \frac{\ell}{r_c} F_p}_{F_d} &= ma_b \\
 (8) \quad \Rightarrow a_b &= -\frac{F_p}{m} \left(1 - \frac{r_f \ell}{r_w r_c} \right).
 \end{aligned}$$

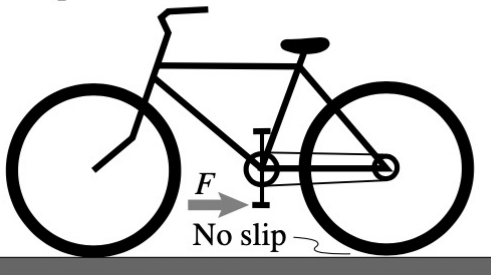
So, now by different reasoning, if $\frac{r_f \ell}{r_c r_w} < 1$ then a push back of F_p causes a negative acceleration a_b of the bike. That is, if $\frac{r_f \ell}{r_c r_w} < 1$ (which it is for all real bikes) then the bike accelerates backwards when you push backwards on the pedal.

MAE 2030 HW14 Problem 76 Solution

19.2.16

Dopei Yu

19.2.16 Which way does the bike accelerate? A bicycle with all frictionless bearings is standing still on level ground. A horizontal force F is applied on one of the pedals as shown. There is no slip between the wheels and the ground. The bicycle is gently balanced from falling over sideways. It is heavy enough so that both wheels stay on the ground. Does the bicycle accelerate forward, backward, or not at all? Make any reasonable assumptions about the dimensions and mass. Justify your answer as clearly as you can, clearly enough to convince a person similar to yourself but who has not seen the experiment performed.



Filename: pfigure-s94h14p2

Problem 19.2.16

Given: horizontal force F applied at pedal
both wheels on ground
no slip
balanced

Find: direction of acceleration of bicycle

Solution: multiple FBDs

Assumptions: • pedal gears are massless

• bicycle wheels are massless

• only upper half of chain is in tension (bottom half slack)

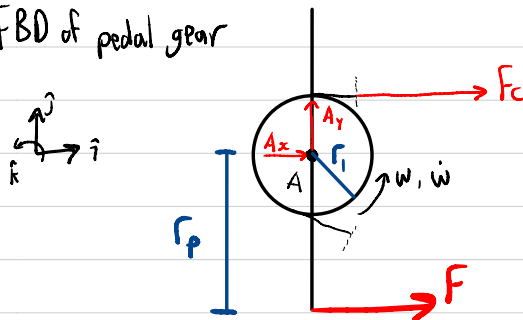
• wheels and pedals are uniform

• front and back pedal gears are same radius

• bicycle is rigid (all pts. have same $\vec{a}$)

• bicycle has mass m

FBD of pedal gear



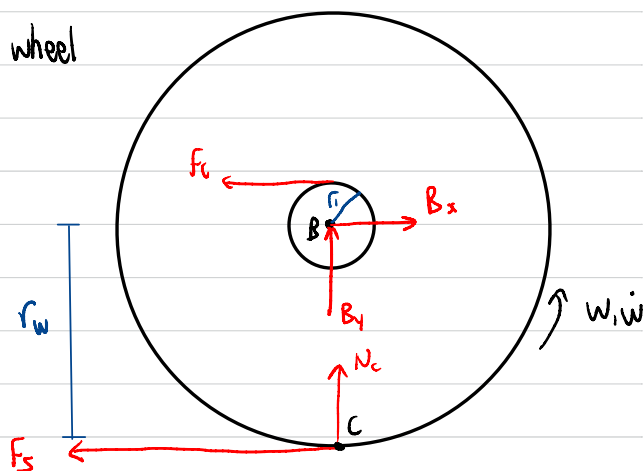
Assuming massless pedal:

$$\text{AMB}_{1A}: F r_p - F_c r_1 = 0$$

$$\textcircled{1} F_c = \frac{r_p}{r_1} F \quad \text{and since } r_p > r_1, F_c > F$$

* Assume only top half of chain is in tension (bottom half is slack) *

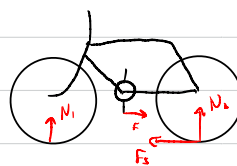
FBD of rear wheel



$$\text{AMB}_{/B} : F_c r_i - F_s r_w = 0 \quad \text{plug in ①}$$

$$F_s = \frac{r_i}{r_w} F_c = \frac{r_i}{r_w} \frac{r_e}{r_i} F = \frac{r_e}{r_w} F \quad \text{②}$$

LMB of entire bike in x-direction:



$$\sum F_x = F - F_s = m a_G$$

$$F - \frac{r_e}{r_w} F = m a_G \rightarrow a_G = \left(1 - \frac{r_e}{r_w}\right) \frac{F}{m}$$

since $r_e < r_w$, $1 - \frac{r_e}{r_w} > 0$, and thus a_G is positive.

Also, since both wheels are in contact with ground, $\vec{a}_G = a_G \hat{i}$.

Since a_G is positive, the bicycle accelerates to the right.