19.1 Mechanics of a constrained particle

The kinematics of time-varying base vectors help us deal with some more difficult particle motion mechanics problems. For one point mass it is easy to write balance of linear momentum. It is:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}.$$

The mass of the particle m times its vector acceleration $\vec{\boldsymbol{a}}$ is equal to the total force on the particle $\vec{\boldsymbol{F}}$. No problem.

Now, however, we can write this equation in five somewhat distinct ways.

1. In general abstract vector form: $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.
2. In cartesian coordinates: $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}} = m[\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}]$.
3. In polar coordinates:
$$F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$
4. In path coordinates: $F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n = m[\dot{v}\hat{\boldsymbol{e}}_t + (v^2/\rho)\hat{\boldsymbol{e}}_n]$

All of these equations are always right. Additionally, for a given particle moving under the action of a given force there are many more correct equations that can be found by shifting the origin and orientation of the coordinate systems. For example for a moving frame $\mathcal{B}$ with origin $0'$, rotation rate $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ and angular acceleration $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$:

5. $\vec{\boldsymbol{F}} = m\left\{\vec{\boldsymbol{a}}_{0'} \quad - \omega_{\mathcal{B}}^2\vec{\boldsymbol{r}} \quad + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}} \quad + \vec{\boldsymbol{a}}_{/\mathcal{B}} \quad + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{/\mathcal{B}}\right\}$

where, to simplify the notation, all motions are relative to $\mathcal{F}$ and positions relative to 0 unless explicitly indicated by a $/\mathcal{B}$ or $/0'$. This is quite a collection of kinematic tools. In general we want to choose the best tools for the job. But to get a sense let's first look at a simple problem using each of these kinematic approaches, some of which are rather inappropriate.

A particle that moves with no net force

In the special case that a particle has no force on it we know intuitively, or from the verbal statement of Newton's First Law, that the particle travels in a straight line at constant speed. As a first example, let's try to find this result using the vector equations of motion five different ways: in the general abstract form, in cartesian coordinates, in polar coordinates, in path coordinates, and relative to a moving frame (see fig. 19.1).

General abstract form. The equation of linear momentum balance is $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ or, if there is no force, $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$, which means that $d\vec{\boldsymbol{v}}/dt = \vec{\boldsymbol{0}}$. So $\vec{\boldsymbol{v}}$ is a constant. We can call this constant $\vec{\boldsymbol{v}}_0$. So after some time the particle is where it was at $t = 0$, say, $\vec{\boldsymbol{r}}_0$, plus its velocity $\vec{\boldsymbol{v}}_0$ times time. That is:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{v}}_0 t. \tag{19.1}$$

This vector relation is a parametric equation for a straight line. The particle moves in a straight line, as expected.

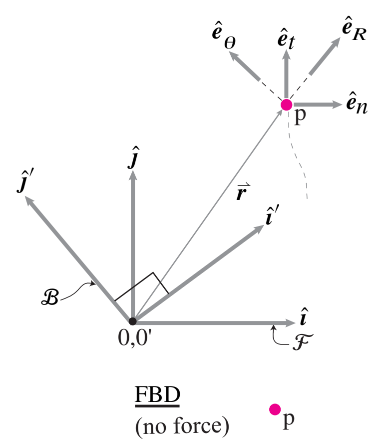


Figure 19.1: A particle P moves. One can track its motion using the general vector form $\vec{\boldsymbol{r}}$, Cartesian coordinates in the fixed frame $\mathcal{F} = 0\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}$, Polar coordinates using $\hat{\boldsymbol{e}}_R\&\hat{\boldsymbol{e}}_\theta$, path coordinates $\hat{\boldsymbol{e}}_t\&\hat{\boldsymbol{e}}_n$ and cartesian coordinates in a rotating frame $\mathcal{B} = 0'\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'$.

Cartesian coordinates. If instead we break the linear momentum balance equation into cartesian coordinates we get

$$F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}} = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}).$$

Because the net force is zero and the net mass is not negligible,

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \text{ and} \qquad \ddot{z} = 0.$$

These equations imply that $\dot{x}$, $\dot{y}$, and $\dot{z}$ are all constants, let's call them v_{x0}, v_{y0}, v_{z0}. So x, y, and z are given by

$$x = x_0 + v_{x0}t, \quad y = y_0 + v_{y0}t, \quad \& \; z = z_0 + v_{z0}t.$$

We can put these components into their place in vector form to get:

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} = (x_0 + v_{x0}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{y0}t)\hat{\boldsymbol{\jmath}} + (z_0 + v_{z0}t)\hat{\boldsymbol{k}}. \quad (19.2)$$

Note that there are six free constants in this equation representing the initial position and velocity. Equation 19.2 is a cartesian representation of equation 19.1; it describes a straight line being traversed at constant rate.

Polar/cylindrical coordinates. When there is no force, in polar coordinates we have:

$$\underbrace{F_R}_{0}\,\hat{\boldsymbol{e}}_R + \underbrace{F_\theta}_{0}\,\hat{\boldsymbol{e}}_\theta + \underbrace{F_z}_{0}\,\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$

This vector equation leads to the following three scalar differential equations, the first two of which are coupled non-linear equations (neither can be solved without the other).

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \\ \ddot{z} &= 0 \end{aligned}$$

A tedious calculation will show that these equations are solved by the following functions of time:

$$\begin{aligned} R &= \sqrt{d^2 + [v_0(t - t_0)]^2} \qquad (19.3) \\ \theta &= \theta_0 + \tan^{-1}[v_0(t - t_0)/d] \\ z &= z_0 + v_{z0}t, \end{aligned}$$

where θ_0, d, t_0, v_0, z_0, and v_{z0} are constants. Note that, though eqn. (19.4) looks different than eqn. (19.2), there are still 6 free constants. From the physical interpretation you know that eqn. (19.4) must be the parametric equation of a straight line. And, indeed, you can verify that picking arbitrary constants and using a computer to make a polar plot of eqn. (19.4) does in fact show a straight line. From eqn. (19.4) it seems that polar coordinates' main function is to obfuscate rather than clarify. For the simple case that a particle moves with no force at all, we have to solve non-linear differential equations whereas using cartesian coordinates we get linear equations which are easy to solve and where the solution is easy to interpret.

But, if we add a central force, a force like earth's gravity acting on an orbiting satellite (the force on the satellite is directed towards the center of the earth), the equations become almost intolerable in cartesian coordinates. But, in polar coordinates, the solution is almost as easy (which is not all that easy for most of us) as the solution 19.4. So the classic analytic solutions of celestial mechanics are usually expressed in terms of polar coordinates.

Path coordinates. When there is no force, $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is expressed in path coordinates as

$$\underbrace{F_t}_{0}\,\hat{\boldsymbol{e}}_t + \underbrace{F_n}_{0}\,\hat{\boldsymbol{e}}_n = m(\dot{v}\hat{\boldsymbol{e}}_t + \underbrace{(v^2/\rho)\hat{\boldsymbol{e}}_n}_{v^2\vec{\boldsymbol{\kappa}}}).$$

That is,

$$\dot{v} = 0 \quad \text{and} \quad v^2/\rho = 0.$$

So the speed v must be constant and the radius of curvature ρ of the path infinite. That is, the particle moves at constant speed in a straight line.

Relative to a rotating reference frame Let's look at the equations using a frame $\mathcal{B}$ that shares an origin with $\mathcal{F}$ but is rotating at a constant rate $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}}$ relative to $\mathcal{F}$. Thus

$$\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \vec{\mathbf{0}} \quad \text{and} \quad \vec{\boldsymbol{a}}_{0'/0} = \vec{\mathbf{0}}$$

and we have that $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is written as

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\vec{\mathbf{0}} &= m\Big\{\underbrace{\vec{\boldsymbol{a}}_{0'}}_{\vec{\mathbf{0}}} \quad -\omega_{\mathcal{B}}^2\vec{\boldsymbol{r}} \quad + \underbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{B}}}_{\vec{\mathbf{0}}}\times\vec{\boldsymbol{r}} \quad + \vec{\boldsymbol{a}}_{/\mathcal{B}} \quad + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}}\times\vec{\boldsymbol{v}}_{/\mathcal{B}}\Big\} \\
&= -\omega^2\vec{\boldsymbol{r}} + \vec{\boldsymbol{a}}_{/\mathcal{B}} + 2\omega\hat{\boldsymbol{k}}\times\vec{\boldsymbol{v}}_{/\mathcal{B}}.
\end{aligned} \tag{19.4}$$

Now, using the rotating base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, we have that $F_{x'}\hat{\boldsymbol{\imath}}' + F_{y'}\hat{\boldsymbol{\jmath}}' = 0\hat{\boldsymbol{\imath}}' + 0\hat{\boldsymbol{\jmath}}'$ and

$$\vec{\boldsymbol{r}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}', \quad \vec{\boldsymbol{v}}_{\mathcal{B}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}', \quad \text{and} \quad \vec{\boldsymbol{a}}_{\mathcal{B}} = \ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}'$$

so eqn. (19.4) can be rewritten as

$$\vec{\mathbf{0}} \;=\; -\omega^2(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') + (\ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}') + 2\omega\hat{\boldsymbol{k}}\times(\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}')$$

which in turn can be broken into components and written as:

$$\begin{aligned} \ddot{x}' &= \omega^2 x' + 2\omega \dot{y}' \\ \ddot{y}' &= \omega^2 y' - 2\omega \dot{x}' \end{aligned} \tag{19.5}$$

which makes up a pair of second order linear differential equations. With some work, someone good at ODEs can solve this with pencil and paper. But most of us would use a computer for such a system. If, in some consistent units, we had

$$y'(0) = 0, \dot{y}'(0) = 0, x'(0) = 0, \dot{x}'(0) = 1, \omega = 1$$

then the solution turns out to be

$$\begin{aligned} x' &= \quad t\cos(\omega t) \\ y' &= -t\sin(\omega t) \end{aligned}$$

as you can check by substituting into eqn. (19.6). That is, a particle which goes in a straight line away from the origin goes, in spirals as seen in the rotating frame①.

①And the straight line motion could have been more complicated yet, as seen in the rotating frame, with variable rate rotation and acceleration.This is why we are lucky the earth is not spinning fast or at highly variable rates. Otherwise the motion we would see, us humans here on the rotating earth, would be particles with no force on them spiraling around all over the place. And that's what Isaac Newton would have seen. And he would have written "A particle in motion tends to go in crazy spirals, and a particle that is initially at rest also goes bonkers" and the equations we use through out this book would have been much harder to discover.

Constrained motion

A particle in a plane has 2 degrees of freedom. There is basically only one kind of constraint — to a path. When constrained to a path the particle has one remaining degree of freedom so its configuration can be described with one variable. For a given problem you must think about

- What force(s) constrain the motion to the path?
- What do you want to use for a configuration variable?

You use these ideas to

- draw an appropriate free-body diagram, and then
- calculate the velocity and acceleration in terms of the configuration variable and its derivatives.

After these key first steps you plug into equations of motion and solve for what you are interested in. Of course the needed math could be difficult or impossible, but the work is somewhat routine from a mechanics point of view.

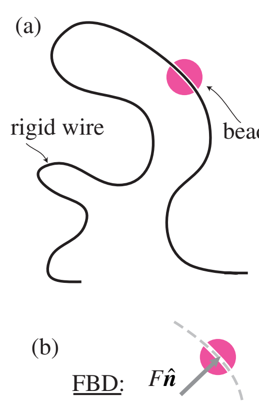


Figure 19.2: A point-mass bead slides on a rigid immobile frictionless wire. The free-body diagram shows that the only force on the bead is in the direction normal to the wire.

Example: Bead on frictionless wire

A bead slides on a frictionless wire with a crazy but smooth shape. No forces are applied to the bead besides the constraint force (see fig. 19.2).

Fig. 19.2b shows a free-body diagram where $\hat{\boldsymbol{n}}$ is the normal to the wire at the point of interest. It doesn't matter if you use for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_n$, or $\hat{\boldsymbol{n}} =$ a vector always, say, to the left, just so long as you know what you mean by $\hat{\boldsymbol{n}}$. The free-body diagram shows that you know that the constraint force is normal to the path (the frictionless wire) but that you don't know how big it is (F is an unknown scalar).

For some purposes, especially general problems like this where no specific path is given, the most appropriate configuration variable is s, the arc length along the path. If the path is given we assume we know the position at any given arc length by the functions

$$x(s) \quad \text{and} \quad y(s).$$

So

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}(s), \vec{\boldsymbol{v}} = \dot{s}\hat{\boldsymbol{e}}_t, \quad \text{and} \quad \vec{\boldsymbol{a}} = (\dot{s}^2/\rho)\hat{\boldsymbol{e}}_n + \ddot{s}\hat{\boldsymbol{e}}_t.$$

Now we can write linear momentum balance

$$\begin{aligned} \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \qquad (19.6)\\ \{F\hat{\boldsymbol{n}} &= m\left((\dot{s}^2/\rho)\hat{\boldsymbol{e}}_n + \ddot{s}\hat{\boldsymbol{e}}_t\right)\} \\ \{\}\cdot\hat{\boldsymbol{e}}_t \quad \Rightarrow \quad \dot{v} &= 0 \quad \text{and} \\ \{\}\cdot\hat{\boldsymbol{e}}_n \quad \Rightarrow \quad F &= mv^2/\rho. \end{aligned}$$

Eqn. 19.6 tells us that the bead moves at constant speed, no matter what the shape of the wire. It also tells us that the more curved the wire, the bigger the constraint force needed to keep the bead on the wire.

Because this is a 1 DOF system, any one equation of motion should give us the result. Instead of linear momentum balance we could have used power balance to get the same result like this:

$$\begin{aligned} P &= \dot{E}_{\text{K}} \\ \Rightarrow \quad \vec{\boldsymbol{F}}_{\text{tot}}\cdot\vec{\boldsymbol{v}} &= \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\} \\ \Rightarrow \quad F\hat{\boldsymbol{e}}_n\cdot v\hat{\boldsymbol{e}}_t &= mv\dot{v} \\ \Rightarrow \quad 0 &= \dot{v}. \end{aligned}$$

This is natural enough. The only force on the particle is perpendicular to its motion, so does no work. So the particle must have constant kinetic energy and its speed must be constant.

The example above doesn't seem that applicable; how often does one see beads on frictionless wires? It is a bit more useful than it appears. For example, if a car is coasting on an open road and tire resistance and sideslip can be ignored, the reasoning above shows that the car is neither speeded nor slowed by turning. Similarly, an idealization of an airplane wing is as something which only causes force perpendicular to motion. So in quick maneuvers where the gravity force is relatively small, the plane maintains its speed.

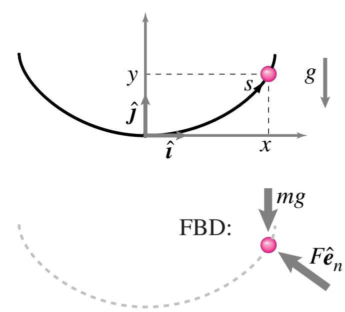


Figure 19.3: A bead slides on a frictionless wire on the curve implicitly defined by $y = cs^2/2$, where s is arclength along the curve measured from the origin.

Example: The hard brachistochrone problem.
Here is a puzzle proposed by Johann Bernoulli in June 1696 2: given that a roller coaster has to coast from rest at one place to another place that is no higher, what shape should the track be to make the trip as quick as possible.

The solution. Finding the solution, or even verifying it, is a problem in the calculus of variations, *i.e.,* too advanced for this book. The solution turns out to be the brachistochrone curve that obeys the following relationship between arclength s from the origin and vertical position y (drawn accurately in fig. 19.3):

$$y = \frac{1}{2}cs^2 \qquad (\text{for}\,|s| \leq 1/c). \qquad (19.7)$$

Starting at the origin this curve is close to $y = cx^2/2$ but gets a bit higher (bigger y) because for a given value of x, s is greater than x. The curve terminates at vertical tangents at $s = \pm 1/c$ where $y = 1/2c$. (see fig. 19.3). To solve the

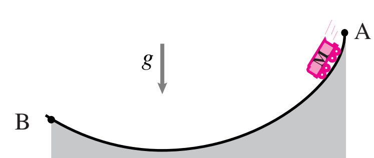


Figure 19.4: The roller coaster that gets from A to B the fastest is the one with a track in the shape of the brachistochrone $y = cs^2/2$.

puzzle this curve is scaled (by choosing a value of c) and displaced so that it has a vertical tangent at A and also so the curve goes through B. The idea that the hard math seems to be expressing, is that the particle should first build up as much speed as it can (by going straight down) and then head off in the right direction (see fig. 19.4).

On the other hand, here is an easier problem that is a virtual setup for the techniques now at hand.

Example: The easy brachistochrone problem

How long does it take for a particle to slide back and forth on a frictionless wire with $y = cs^2/2$ as driven by gravity? (see fig. 19.3)

Let's use s as our configuration variable. The power balance equation is:

$$\begin{aligned}
& & P &= \dot{E}_K \\
(F\hat{\boldsymbol{e}}_n)\cdot(v\hat{\boldsymbol{e}}_t) = 0 \quad &\Rightarrow & -mg\hat{\boldsymbol{\jmath}}\cdot(\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}) &= \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\} \\
&\Rightarrow & -mg\dot{y} &= m\frac{d}{dt}\frac{\dot{s}^2}{2} \\
&\Rightarrow & -mgcs\dot{s} &= m\dot{s}\ddot{s} \\
\text{Assuming } \dot{s} \neq 0 \quad &\Rightarrow & \ddot{s} + gcs = 0. &
\end{aligned}$$

This, remarkably, is the simple harmonic oscillator equation with general solution

$$s = A\cos(\sqrt{gc}\,t) + B\sin(\sqrt{gc}\,t).$$

Thus the period of oscillation (T such that $\sqrt{gc}T = 2\pi$) is

$$T = \frac{2\pi}{\sqrt{gc}}$$

which is independent of the amplitude of oscillation ③.

The key to this quick solution was using a configuration variable that made the expression for the velocity simple, and using an equation of motion that didn't involve the unknown reaction force F which we also didn't care about. We could have got the same equation of motion by writing $\vec{F} = m\vec{a}$ and eliminated $F\hat{\boldsymbol{e}}_n$ by dotting both sides with a convenient vector orthogonal to $F\hat{\boldsymbol{e}}_n$, say $\vec{v}$.

The brachistochrone is a famous curve that has various interesting properties (*e.g.,* Box 19.1 on page 7).

.

Example: A collar on two rotating rods

Consider a pair of collars hinged together as a point mass m at P. Each slides frictionlessly on a rod about whose rotation everything is known (see fig. 19.5. What is the force of rod 1 on the mass? For this 2 degree of freedom system let's use configuration variables θ_1 and θ_2, and two sets of rotating base vectors: $\hat{\boldsymbol{\lambda}}_1\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{\lambda}}_2\hat{\boldsymbol{n}}_2$. These rotating base vectors can be written in terms of the θs, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the standard manner. Assume we know ℓ_1 and ℓ_2 in this configuration. First find $\dot{\ell}_1$ and $\dot{\ell}_2$ by thinking of the velocity of the collar two different ways:

$$\begin{aligned}
\vec{v} \quad &= \quad \vec{v} \qquad\qquad (19.8)\\
\{\,\dot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \dot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 \quad &= \quad \dot{\ell}_2\hat{\boldsymbol{\lambda}}_2 + \dot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\,\} \\
\{\}\cdot\hat{\boldsymbol{n}}_2 \quad &\Rightarrow \quad \dot{\ell}_1 = \frac{\dot{\theta}_2\ell_2 - \dot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2}{\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{n}}_2} \\
\{\}\cdot\hat{\boldsymbol{n}}_1 \quad &\Rightarrow \quad \dot{\ell}_2 = \frac{\dot{\theta}_1\ell_1 - \dot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1}{\hat{\boldsymbol{\lambda}}_2\cdot\hat{\boldsymbol{n}}_1}.
\end{aligned}$$

② **The original brachistochrone (least time) puzzle:**

"I, Johann Bernoulli, greet the most clever mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem whose possible solution will bestow fame and remain as lasting monument. Following the example set by Pascal, Fermat, etc., I hope to earn the gratitude of the entire scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall then publicly declare him worthy of praise.

"*Let two points A and B be given in a vertical plane. Find the curve that a point M, moving on a path AMB must follow such that, starting from A, it reaches B in the shortest time under its own gravity.*"

Newton. Besides Bernoulli's brother Jakob, one of the people to solve the puzzle was 55 year old Isaac Newton. He attempted to keep his solution, said to have been worked out in one evening while he also invented the calculus of variations, anonymous. But Bernoulli supposedly saw through this deception, commenting "I recognize the lion by his paw print", which was presumably not a comment about Newton's handwriting.

③ Actually, the amplitude can't be arbitrarily large. The solution to the defining eqn. (19.7) only makes sense for $|s| < 1/c$. For $|s| > 1/c$ there is no curve satisfying eqn. (19.7).

Having found $\dot{\ell}_1$ and $\dot{\ell}_2$ we can find the velocity $\vec{\boldsymbol{v}}$ by evaluating either side of eqn. (19.8). Now we apply identical reasoning with the acceleration. The result looks messy, but the approach is straightforward:

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= \vec{\boldsymbol{a}} \qquad (19.9)\\
\{\, \ddot{\ell}_1 \hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1 \ell_1 \hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1 \dot{\theta}_1 \hat{\boldsymbol{n}}_1 &= \ddot{\ell}_2 \hat{\boldsymbol{\lambda}}_2 + \ddot{\theta}_2 \ell_2 \hat{\boldsymbol{n}}_2 + 2\dot{\ell}_2 \dot{\theta}_2 \hat{\boldsymbol{n}}_2 \,\}\\
\{\} \cdot \hat{\boldsymbol{n}}_2 \quad \Rightarrow \quad \ddot{\ell}_1 &= \frac{\ddot{\theta}_2 \ell_2 + 2\dot{\ell}_2 \dot{\theta}_2 - \ddot{\theta}_1 \ell_1 \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2 - 2\dot{\ell}_1 \dot{\theta}_1 \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2}{\hat{\boldsymbol{\lambda}}_1 \cdot \hat{\boldsymbol{n}}_2}\\
\{\} \cdot \hat{\boldsymbol{n}}_1 \quad \Rightarrow \quad \ddot{\ell}_2 &= \frac{\ddot{\theta}_1 \ell_1 + 2\dot{\ell}_1 \dot{\theta}_1 - \ddot{\theta}_2 \ell_2 \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{n}}_1 - 2\dot{\ell}_2 \dot{\theta}_2 \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{n}}_1}{\hat{\boldsymbol{\lambda}}_2 \cdot \hat{\boldsymbol{n}}_1}
\end{aligned}
$$

We use the results from eqn. (19.8) for $\dot{\ell}_1$ and $\dot{\ell}_2$ to evaluate the right hand sides of the expressions for $\ddot{\ell}_1$ and $\ddot{\ell}_2$ in eqn. (19.9). So now either the left hand side or the right hand side of the second of Eqns. 19.9 can be used to evaluate the acceleration $\vec{\boldsymbol{a}}$, all the terms in both expressions have been found.

To find the forces we use linear momentum balance and the free-body dia-

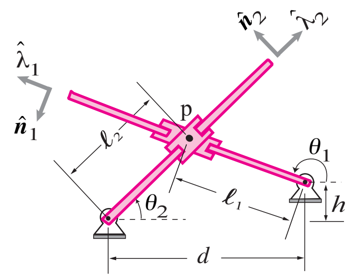
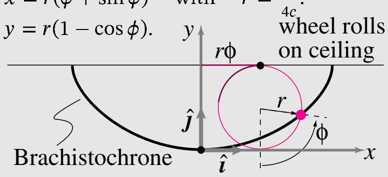


Figure 19.5: A point-mass collar slides simultaneously on 2 rods.

19.1 Some brachistochrone curiosities

The brachistochrone is a cycloid. There is no straightforward way to draw the curve $y = cs^2/2$ because the formula doesn't tell you the x coordinates of the points. You could find them by integrating $dx = \sqrt{ds^2 - dy^2}$ numerically or with calculus tricks. But it turns out (see below) that the curve with $y = cs^2/2$ is described by the parametric equations

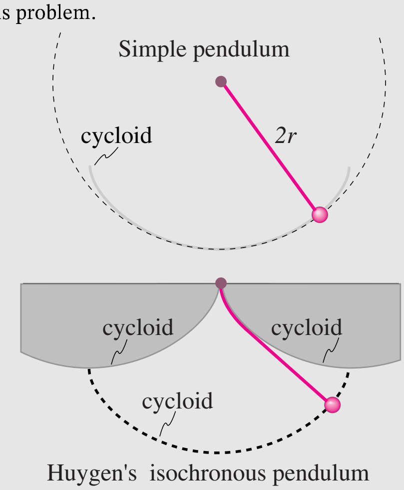
$$
\begin{aligned}
x &= r(\phi + \sin\phi) \quad \text{with} \quad r = \tfrac{1}{4c}.\\
y &= r(1 - \cos\phi).
\end{aligned}
$$



This is the path of a particle on the perimeter of a wheel that rolls against a horizontal ceiling a distance $2r$ above the origin, as you can verify by adding up distances in the picture above (see page 937). We will show below that the upside down cycloid and the curve $y = cs^2/2$ are one and the same.

Note that the osculating circle of this cycloid at its lowest point has radius $4r = 1/c$, just the length of a simple pendulum that, for small oscillations, has the same frequency of oscillation as the bead on the brachistochrone. A point mass swinging on a string is like a bead on a frictionless circular wire and this, in turn, is close to the motion of a bead on a brachistochrone wire for small oscillations.

Galileo (1564-1642). Well before Bernoulli's challenge, Galileo was interested in things rolling and sliding on ramps. He knew that the shortest distance between two points is a straight line, and had noted that a ball rolling down an appropriately curved ramp gets to its destination faster than a ball traveling the shortest route. A ball going on a straight ramp just doesn't pick up much speed, and when it finally has its greatest speed the trip is over. Imagine sliding straight sideways; it takes forever on a straight-line route. Better, he must have reasoned, to get the ball rolling fast at the start and then go fast for most of its journey, possibly slowing at the end. Galileo thought the best shape was the bottom of a circle (or fraction thereof), which isn't far off either in shape or concept, but isn't quite right. Galileo was apparently obsessed with cycloids for other reasons but didn't see their connection to this problem.



A constant period pendulum. For clock time keeping, a pendulum is better than a bead on a wire because the friction of sliding is avoided. Unfortunately, a simple pendulum has a period which is longer if the amplitude is bigger. Not much longer, 18%

(continued...)

gram

$$\begin{aligned} \vec{F}_{\text{tot}} &= m\vec{a} \qquad (19.10)\\ \{F_1\hat{n}_1 + F_2\hat{n}_2 &= m\{\ddot{\ell}_1\hat{\lambda}_1 + \ddot{\theta}_1\ell_1\hat{n}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{n}_1\}\} \end{aligned}$$

$$\{\}\cdot\hat{\lambda}_2 \quad \Rightarrow \quad F_2 = \frac{\{\ddot{\ell}_1\hat{\lambda}_1 + \ddot{\theta}_1\ell_1\hat{n}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{n}_1\}\cdot\hat{\lambda}_2}{\hat{n}_1\cdot\hat{\lambda}_2}$$

$$\{\}\cdot\hat{\lambda}_1 \quad \Rightarrow \quad F_1 = \frac{\{\ddot{\ell}_1\hat{\lambda}_1 + \ddot{\theta}_1\ell_1\hat{n}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{n}_1\}\cdot\hat{\lambda}_1}{\hat{n}_2\cdot\hat{\lambda}_1}$$

When actually evaluating the expressions above one can write the base vectors in terms of $\hat{\imath}$ and $\hat{\jmath}$ or use geometry.

Often when working out a problem it is best to not substitute numbers until the end of a problem. This example shows the opposite. If we left the expressions for $\dot{\ell}_1$ and $\dot{\ell}_2$ with letters and substituted that into the expressions for $\ddot{\ell}_1$ and $\ddot{\ell}_2$ and left those expressions intact while substituting for the acceleration $\vec{a}$ we would have large expressions for the force components F_1 and F_2. On the other hand, by using numbers as the calculation progresses the formulas do not grow so much in complexity.

19.1 Some brachistochrone curiosities (continued)

if the swinging is $\pm 90^\circ$ and only 1.7% longer if the amplitude is $\pm 30^\circ$, but enough to annoy clock designers. A bead sliding frictionlessly on the path $y = s^2/2$ has the nice property that the period does not depend on the amplitude. But any real bead sliding on any real wire has substantial friction. So, at first blush the brachistochrone curve, despite its nice constant-period property, cannot be used to keep time.

But Huygens, one of the smart old timers, looked for a curve that, when a string wraps around it, makes the end follow the brachistochrone curve. To this day you can see fancy old clocks with this wrapping device, a solid piece with two cuspoidal shapes at the hinge of the swinging ("isochronous" or "tautochrone") pendulum which wraps around it.

Geometry. The two key features discussed above, that the curve $y = cs^2/2$ is a cycloid, and that a cycloid can be generated by wrapping a string around another cycloid, can be found from the geometric construction below. Two cycloids are shown, one from wheel 1 rolling under line L_1 and another from wheel 2 rolling under line L_2 a distance $2r$ below. Both wheels have radius r. Imagine that the cycloids A_1M_1B and A_2M_2B are drawn by wheels always arranged with vertically aligned rolling contact points C_1 and C_2 and with points M_1 and M_2 initially aligned vertically a distance $4r$ apart at A_1 and A_2.

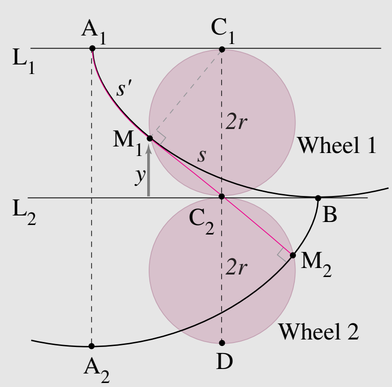
The two cycloids are thus the same shape but are displaced with one being $2r$ below and πr sideways from the other.

Because both wheels have rolled the same distance (A_1C_1) they have rotated the same amount and M_2 is as far forward of C_1C_2D as M_1 is behind. Similarly M_1 is as far above L_2 as M_2 is below. So the line M_1M_2 is bisected by the point C_2.

Because C_1C_2 is the diameter of a circle with M_1 on the perimeter, angle $C_1M_1C_2$ is a right angle. Because material point C_1 on the wheel has zero velocity the velocity of M_1 (and thus the tangent to the curve) is orthogonal to C_1M_1. Thus the line M_1M_2 is tangent to the upper cycloid.

The rolling of wheel 2 instantaneously rotating about C_2 makes the tangent to the lower cycloid orthogonal to M_1M_2, the condition for the motion of M_2 to be from the wrapping of an inextensible line around the curve A_1M_1B. This shows that cycloid A_2M_2B is generated by the wrapping of a line anchored at A_1 about the upper cycloid. And this is Huygen's wrapping mechanism for making a pendulum bob follow a cycloid. Because of this wrapping generation, the arc-length $s' + s$ of A_1M_1B must be $4r$ and the arc length s of M_1B is M_1M_2 so the length M_1C_2 is $s/2$. By the similarity of the two right triangles that share the length $s/2$ of M_1C_2:

$$\frac{y}{s/2} = \frac{s/2}{2r} \quad \Rightarrow \quad y = \frac{1}{4r}\frac{s^2}{2} = c\frac{s^2}{2}$$



which shows that the upper cycloid is the curve $y = cs^2/2$ if $c = 1/4r$, where s is measured from B. This was the equation used to show the constant period nature of the sliding motion of a bead on a frictionless cycloidal curve using power balance.

As is the case with most mechanism-mechanics problems, the hard work in getting the dynamics equations is in the kinematics. Generally there are no great short-cuts. There are alternative methods. In this case the location of the base points and the two angles determine the base and two angles of a triangle. This triangle can be solved for the location of the point P. Once that position is known in terms of θ_1 and θ_2 the velocity and acceleration can be found by differentiation.

As a robot manipulator, this design has the advantage that no motors need to be displaced. It has the disadvantage of requiring good sliding joints.

An alternative solution of the kinematics of this problem would be to use trigonometry to find the position of point P in terms of the angles θ_1 and θ_2. Then the acceleration of point P is found by taking two time derivatives. The result is approximately equal in the complexity of its appearance to the results used above. That method requires more cleverness at the start (solving an angle-side-angle triangle) and then just brute force differentiation using the chain rule and the product rule.

Example: Inverted pendulum with a vibrating base

Assume that the base $0'$ of an inverted point-mass pendulum of length ℓ is vibrated according to (see fig. 19.6)

$$\vec{\boldsymbol{r}}_{0'} = d\sin\omega t\hat{\boldsymbol{\imath}}.$$

The point P thus has acceleration

$$\begin{aligned}\vec{\boldsymbol{a}}_{\mathrm{P}} &= \vec{\boldsymbol{a}}_{0'} + \vec{\boldsymbol{a}}_{\mathrm{P}/0'}\\ &= -d\omega^2\sin\omega t\hat{\boldsymbol{\imath}} + \ddot{\theta}\ell\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2\ell\hat{\boldsymbol{e}}_R\end{aligned}$$

Now apply linear momentum balance as

$$\begin{aligned}\vec{\boldsymbol{F}}_{\mathrm{tot}} &= m\vec{\boldsymbol{a}}_{\mathrm{P}}\\ \{-mg\hat{\boldsymbol{\imath}} + -T\hat{\boldsymbol{e}}_R &= m\{-d\omega^2\sin\omega t\hat{\boldsymbol{\imath}} + \ddot{\theta}\ell\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2\ell\hat{\boldsymbol{e}}_R\}\}\\ \{\}\cdot\hat{\boldsymbol{e}}_\theta \quad\Rightarrow\quad -g\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta &= -d\omega^2\sin\omega t\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta + \ddot{\theta}\ell\\ \Rightarrow\quad g\sin\theta &= d\omega^2\sin\omega t\sin\theta + \ddot{\theta}\ell\end{aligned}$$

which you write as

$$0 = \ddot{\theta} + (d\omega^2\sin\omega t - g)\sin\theta/\ell \quad \text{or} \quad \ddot{\theta} = (g - d\omega^2\sin\omega t)\sin\theta/\ell$$

depending on whether you are analytically or numerically inclined. This is a second order non-linear ordinary differential equation. If $\omega = 0$ or $d = 0$ then this is the classic inverted pendulum equation and has solutions that show that the pendulum doesn't stay near upright. But, you can find by analytic cleverness or numerical integration that for some values of d and ω that the pendulum does not fall down! Just shaking the base keeps the pendulum up ($\omega^2 d > g$ for all cases where this is possible). This isn't just academic nonsense, the device can be built and the balancing demonstrated.

One alternative to using linear momentum balance in the equations above would be to use angular momentum balance about the point 0'. The resulting vector equation

$$\ell\hat{\boldsymbol{e}}_R \times (-mg\hat{\boldsymbol{\imath}}) = \vec{\boldsymbol{r}}_{\mathrm{P}/0'} \times (m\vec{\boldsymbol{a}})$$

yields the same second order scalar ODE.

The vibrating mechanism shown is a "Scotch yoke". An eccentric disk is mounted to the shaft of a constant angular velocity motor. The rectangular slot moves up and down sinusoidally as the disk wobbles.

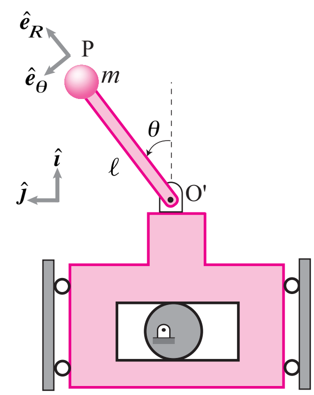


Figure 19.6: The base of a pendulum is vertically vibrated.