SAMPLE 19.1 A bead on a straight wire. A straight wire is hung between points A and B in the xy plane as shown in the figure. A bead slides down the wire from point A. Write the geometric constraint equation for the bead's motion and derive the conditions on velocity and acceleration components of the bead due to the constraint.

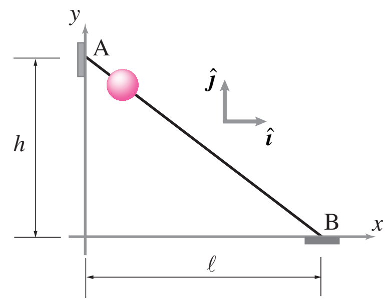

Figure 19.7

Solution The constraint on the bead's motion is that its path must be along the wire, *i.e.*, a straight line between points A and B. Thus the geometric constraint on the motion is expressed by the equation of the path which is

$$y = h - \frac{h}{\ell}x.$$

Since the bead is constrained to move on this path, its velocity and acceleration vectors are also constrained to be directed along AB. This imposes conditions on their x and y components that are easily derived by differentiating the geometric constraint equation with respect to t. Thus,

$$\begin{aligned} \dot{y} &= -\frac{h}{\ell}\dot{x}, \\ \ddot{y} &= -\frac{h}{\ell}\ddot{x}. \end{aligned}$$

$$\boxed{y = h - \frac{h}{\ell}x, \quad \dot{y} = -\frac{h}{\ell}\dot{x}, \quad \ddot{y} = -\frac{h}{\ell}\ddot{x}}$$

SAMPLE 19.2 A particle sliding on a parabolic path. A particle slides on a parabolic trough given by $y = ax^2$ where a is a constant. Write the geometric constraints of motion (on the path, velocity, and acceleration) of the particle. Write the velocity and acceleration of the particle at a generic location (x, y) on its path.

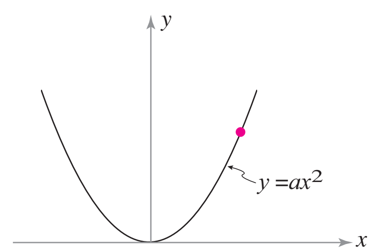

Figure 19.8

Solution The geometric constraint on the path of the particle is already given, $y = ax^2$. Differentiating the path constraint with respect to time, we get the constraint on velocity and acceleration components.

$$\begin{aligned} \dot{y} &= 2ax\dot{x}, \\ \ddot{y} &= 2ax\ddot{x} + 2a\dot{x}^2. \end{aligned}$$

Now, at a point (x, y), we can write the velocity and acceleration of the particle as

$$\begin{aligned} \vec{\boldsymbol{v}} &= \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}, \\ \vec{\boldsymbol{a}} &= \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} = \ddot{x}\hat{\boldsymbol{\imath}} + (2ax\ddot{x} + 2a\dot{x}^2)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + (2ax\ddot{x} + 2a\dot{x}^2)\hat{\boldsymbol{\jmath}}}$$

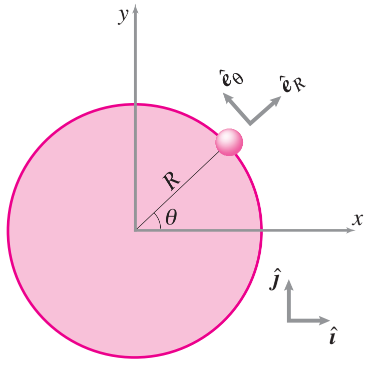

Figure 19.9

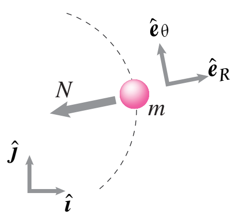

Figure 19.10

SAMPLE 19.3 Circular motion of a particle. A particle is constrained to move on a frictionless circular path of radius R_0 with constant angular speed $\dot{\theta}$. There is no gravity. Find the equation of motion of the particle in the x-direction and show that this motion is simple harmonic.

Solution This is a simple problem that you have solved before, probably a few times. Here, we do this problem again just to show how it works out with the constraint machinery in evidence. The geometric constraint on the path of the particle is $R = R_0$ (in polar coordinates). This constraint gives us $\dot{R} = 0$ and $\ddot{R} = 0$. Then the acceleration of the particle (in polar coordinates), $\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$ reduces to $\vec{a} = -R\dot{\theta}^2\hat{e}_R$, (of course).

The free-body diagram of the particle shows that there is only one force acting on the particle, the normal reaction N of the path acting in the $\hat{e}_R$ direction. Therefore, the linear momentum balance gives,

$$N\hat{e}_R = m\vec{a} = m(-R\dot{\theta}^2\hat{e}_R) \quad \Rightarrow \quad N = -mR\dot{\theta}^2.$$

But, to write the equation of motion in the x-direction, we need to write the linear momentum balance in the x-direction. We can write $\sum \vec{F} = m\vec{a}$ using mixed basis vectors as $N\hat{e}_R = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$. Dotting this equation with $\hat{\imath}$, we get

$$\ddot{x} = \frac{N}{m}(\hat{e}_R \cdot \hat{\imath}) = \frac{-mR\dot{\theta}^2}{m}\cos\theta = -\dot{\theta}^2(R\cos\theta) = -\dot{\theta}^2 x$$

or, $\ddot{x} + \dot{\theta}^2 x = 0$, which is the equation of simple harmonic motion in x. You can easily show that the motion in the y-direction is also simple harmonic ($\ddot{y} + \dot{\theta}^2 y = 0$).

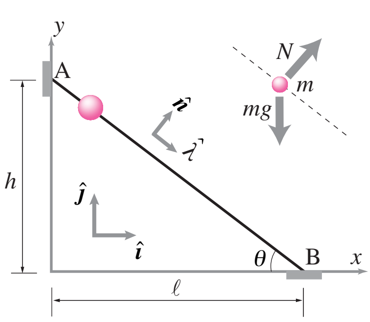

Figure 19.11

SAMPLE 19.4 A bead slides on a straight wire. Consider the problem of the bead sliding on a straight, inclined, frictionless wire of Sample 19.1 again. Find the position of the bead $x(t)$ and $y(t)$ assuming it slides under gravity starting from rest at A.

Solution To find the position of the bead, we need to write the equation of motion and solve it. This is a single DOF system and, therefore, one scalar equation of motion should suffice.

The free-body diagram of the bead is shown in fig. 19.11. Using basis vectors $(\hat{\lambda}, \hat{n})$ and $(\hat{\imath}, \hat{\jmath})$ we write the LMB for the bead as

$$-mg\hat{\jmath} + N\hat{n} = m\vec{a} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}).$$

We can easily eliminate the constraint force N from this equation by dotting this equation with $\hat{\lambda}$, which gives

$$\begin{aligned} -\cancel{m}g(\hat{\jmath} \cdot \hat{\lambda}) &= \cancel{m}[\ddot{x}(\hat{\imath} \cdot \hat{\lambda}) + \ddot{y}(\hat{\jmath} \cdot \hat{\lambda})] \\ \Rightarrow \quad g\sin\theta &= \ddot{x}\cos\theta - \ddot{y}\sin\theta. \end{aligned}$$

But, from the geometric constraint $y = h - (h/\ell)x$, we have $\ddot{y} = -(h/\ell)\ddot{x} = -(\tan\theta)\ddot{x}$. Therefore,

$$g\sin\theta = \ddot{x}\cos\theta + \ddot{x}\tan\theta\sin\theta \quad \Rightarrow \quad \ddot{x} = g\sin\theta\cos\theta.$$

Since $g\sin\theta\cos\theta$ is constant, we integrate the equation of motion easily to find $x(t) = \frac{1}{2}g\sin\theta\cos\theta t^2$ since $x(0) = 0, \dot{x}(0) = 0$. And, since $y = h - x\tan\theta$, we have $y(t) = h - \frac{1}{2}g\sin^2\theta t^2$.

$$x(t) = \tfrac{1}{2}ght^2\sin\theta\cos\theta, \quad y(t) = h - \tfrac{1}{2}gt^2\sin^2\theta$$

SAMPLE 19.5 A bead sliding down a parabolic trough. Consider the problem of Sample 19.2 again. Find the equation of motion of the bead.

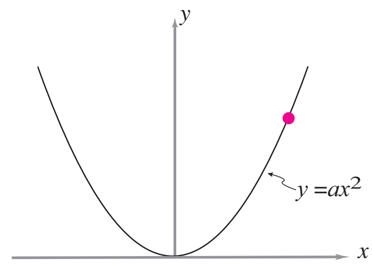

Figure 19.12

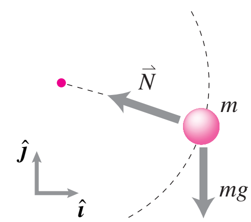

Figure 19.13

Solution This is, again, a one DOF system. Therefore, we will get a single scalar equation of motion. The free-body diagram shown in fig. 19.13 shows two forces acting on the bead. The constraint force N acts normal to the path. Let $\hat{e}_t$ and $\hat{e}_n$ be unit vectors tangential and normal to the path, respectively. Then the linear momentum balance gives

$$-mg\hat{\jmath} + N\hat{e}_n = m\vec{a} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}).$$

To eliminate the unknown constraint force N from this equation, we can take a dot product of this equation with $\hat{e}_t$. However, we must first find $\hat{e}_t$. Now $\hat{e}_t$ is the unit tangent vector. So, we can find it by finding a tangent vector to the path (remember gradient of a function ∇f?) and then dividing it by the length of the vector. That is doable but a little complicated. All we need here is the dot product with a vector *normal* to $\hat{e}_n$. Why not use the velocity vector $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$? The velocity vector is always tangential to the path. Furthermore, we know that from the geometric constraint ($y = ax^2$), $\dot{y} = 2ax\dot{x}$ and $\ddot{y} = 2ax\ddot{x} + 2a\dot{x}^2$. Therefore, $\vec{v} = \dot{x}\hat{\imath} + 2ax\dot{x}\hat{\jmath}$. Now, dotting the LMB equation with $\vec{v}$ we get,

$$\begin{aligned}
-\cancel{m}g\overbrace{(\hat{\jmath}\cdot\vec{v})}^{2ax\dot{x}} &= \cancel{m}\left[\,\ddot{x}\overbrace{(\hat{\imath}\cdot\vec{v})}^{\dot{x}} + \ddot{y}\overbrace{(\hat{\jmath}\cdot\vec{v})}^{2ax\dot{x}}\,\right] \\
-2gax\cancel{\dot{x}} &= \ddot{x}\cancel{\dot{x}} + 2ax\ddot{y}\cancel{\dot{x}} \\
&= \ddot{x} + 2ax\underbrace{(2ax\ddot{x} + 2a\dot{x}^2)}_{\ddot{y}} \\
&= \ddot{x}(1 + 4a^2x^2) + 4a^2x\dot{x}^2.
\end{aligned}$$

Rearranging the terms above, we get the required equation of motion:

$$\ddot{x} + \frac{4a^2x}{1+4a^2x^2}\dot{x}^2 + \frac{2gax}{1+4a^2x^2} = 0.$$

As you can see, this is a nonlinear ODE. Analytical solution of this equation is rather difficult. We can, however, always solve it numerically. Note that a solution of this equation only gives you $x(t)$, *i.e.,* the x coordinate of the position of the bead. But, you can always find the y coordinate since $y = ax^2$.

$$\boxed{\ddot{x} + \frac{4a^2x}{1+4a^2x^2}\dot{x}^2 + \frac{2gax}{1+4a^2x^2} = 0}$$

Comment: Note that if we consider x and $\dot{x}$ to be very small so that we can ignore the $\dot{x}^2$ terms completely, and take $1 + 4a^2x^2 \approx 1$, then the equation of motion becomes

$$\ddot{x} + (2ga)\,x = 0$$

which is the equation of simple harmonic motion with frequency $\sqrt{2ga}$. Thus, if we consider a shallow parabola, and release the bead close to the origin, it executes simple harmonic motion, much like a simple pendulum. This is an intuitively realizable motion.

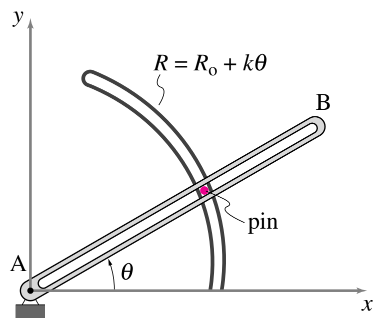

Figure 19.14: A pin is constrained to move in a groove and a slotted arm.

SAMPLE 19.6 Constrained motion of a pin. During a small interval of its motion, a pin of 100 grams is constrained to move in a groove described by the equation $R = R_0 + k\theta$ where $R_0 = 0.3\,\text{m}$ and $k = 0.05\,\text{m}$. The pin is driven by a slotted arm AB and is free to slide along the arm in the slot. The arm rotates at a constant speed $\omega = 6\,\text{rad/s}$. Find the magnitude of the force on the pin at $\theta = 60^\circ$.

Solution Let $\vec{F}$ denote the net force on the pin. Then from the linear momentum balance

$$\vec{F} = m\vec{a}$$

where $\vec{a}$ is the acceleration of the pin. Therefore, to find the force at $\theta = 60^\circ$ we need to find the acceleration at that position.

From the given figure, we assume that the pin is in the groove at $\theta = 60^\circ$. Since the equation of the groove (and hence the path of the pin) is given in polar coordinates, it seems natural to use polar coordinate formula for the acceleration. For planar motion, the acceleration is

$$a = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

We are given that $\dot{\theta} \equiv \omega = 6\,\text{rad/s}$ and the radial position of the pin $R = R_0 + k\,\theta$. Therefore, (1)

(1) Note that R is a function of θ and θ is a function of time, therefore R is a function of time. Although we are interested in finding $\dot{R}$ and $\ddot{R}$ at $\theta = 60^\circ$, we cannot first substitute $\theta = 60^\circ$ in the expression for R and then take its time derivatives (which will be zero).

$$\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{dt} = 0 \quad (\text{ since } \dot{\theta} = \text{constant}) \\
\dot{R} &= \frac{d}{dt}(R_0 + k\theta) = k\dot{\theta} \quad \text{and} \\
\ddot{R} &= k\ddot{\theta} = 0.
\end{aligned}$$

Substituting these expressions in the acceleration formula and then substituting the numerical values at $\theta = 60^\circ$, (remember, θ must be in radians!), we get

$$\begin{aligned}
\vec{a} &= -\overbrace{(R_0 + k\theta)\dot{\theta}^2}^{R\dot{\theta}^2}\,\hat{e}_R + \overbrace{2k\dot{\theta}^2}^{2\dot{R}\dot{\theta}}\,\hat{e}_\theta \\
&= -(0.3\,\text{m} + 0.05\,\text{m}\cdot\frac{\pi}{3})\cdot(6\,\text{rad/s})^2\,\hat{e}_R + 2\cdot 0.05\,\text{m}\cdot(6\,\text{rad/s})^2\,\hat{e}_\theta \\
&= -13.63\,\text{m/s}^2\hat{e}_R + 3.60\,\text{m/s}^2\hat{e}_\theta.
\end{aligned}$$

Therefore the net force on the pin is

$$\begin{aligned}
\vec{F} &= m\vec{a} \\
&= 0.1\,\text{kg}\cdot(-13.63\hat{e}_R + 3.60\hat{e}_\theta)\,\text{m/s}^2 \\
&= (-1.36\hat{e}_R + 0.36\hat{e}_\theta)\,\text{N}
\end{aligned}$$

and the magnitude of the net force is

$$F = |\vec{F}| = \sqrt{(1.36\,\text{N})^2 + (0.36\,\text{N})^2} = 1.41\,\text{N}.$$

$F = 1.41\,\text{N}$

SAMPLE 19.7 **A puck sliding on a frictional rotating table.** A horizontal turntable rotates with constant angular speed $\omega = 100\,\text{rpm}$. A puck of mass $m = 0.1\,\text{kg}$ is gently placed on the rotating turntable. The puck begins to slide. The coefficient of friction between the puck and the turntable is 0.25. Find the equation of motion of the puck using

1. A fixed reference frame with cartesian coordinates
2. A rotating reference frame with cartesian coordinates

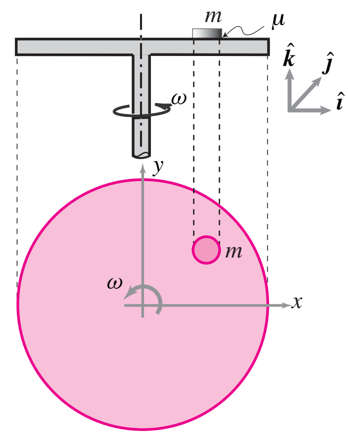

Figure 19.15

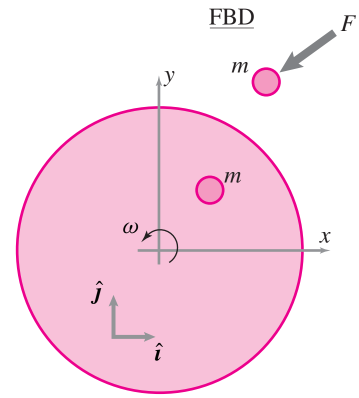

Figure 19.16: A partial free-body diagram of the puck. For linear momentum balance we need to consider only the forces acting in the plane of motion.

Solution

1. **Equation of motion using a fixed reference frame:** The puck has two DOF on the turntable. So, we will need two configuration variables, say x and y, and we will have to find equation of motion for each variable.

   Let us use a fixed cartesian coordinate system with the origin at the center of the turntable. Let $\vec{r}_P = x\hat{\imath} + y\hat{\jmath}$ be the position of the puck at some instant t, so that its velocity is $\vec{v}_P = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ and acceleration is $\vec{a}_P = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$.

   The free-body diagram of the puck should show three forces — the force of gravity (in $-\hat{k}$ direction), the normal reaction of the turntable (in $\hat{k}$ direction) and the friction force $\vec{F}$. Since, there is no motion in the vertical direction, we know that $N = mg$ and that $F = |\vec{F}| = \mu N = \mu mg$. But, what is the direction of the friction force? Well, we know that it acts in the opposite direction of the *relative slip*, so that

$$\vec{F} = -\mu N \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|}.$$

   So, we need to find $\vec{v}_{\text{rel}}$. Now $\vec{v}_{\text{rel}}$ is the velocity of the puck relative to the turntable, or more precisely, relative to the point on the turntable just underneath the puck. Let us denote that point by P'. Clearly, P' goes in circles with constant speed, so that its velocity is

$$\vec{v}_{P'} = \vec{\omega} \times \vec{r}_{P'} = \dot{\theta}\hat{k} \times (x\hat{\imath} + y\hat{\jmath}) = \dot{\theta}x\hat{\jmath} - \dot{\theta}y\hat{\imath}.$$

   Therefore, the relative velocity, $\vec{v}_{\text{rel}}$ is

$$\vec{v}_{\text{rel}} = \vec{v}_P - \vec{v}_{P'} = (\dot{x} + \dot{\theta}y)\hat{\imath} + (\dot{y} - \dot{\theta}x)\hat{\jmath}$$

   Now the linear momentum balance for the puck in the xy plane gives

$$\begin{aligned}
-\mu \cancel{m} g \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|} &= \cancel{m}(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\
\Rightarrow \quad \ddot{x} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|}(\vec{v}_{\text{rel}} \cdot \hat{\imath}) = -\frac{\mu g(\dot{x} + \dot{\theta}y)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}} \\
\text{and} \quad \ddot{y} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|}(\vec{v}_{\text{rel}} \cdot \hat{\jmath}) = -\frac{\mu g(\dot{y} - \dot{\theta}x)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}}.
\end{aligned}$$

   These are coupled nonlinear ODEs that represent the equations of motion of the puck.

$$\boxed{\ddot{x} = -\frac{\mu g(\dot{x}+\dot{\theta}y)}{\sqrt{(\dot{x}+\dot{\theta}y)^2+(\dot{y}-\dot{\theta}x)^2}}, \quad \ddot{y} = -\frac{\mu g(\dot{y}-\dot{\theta}x)}{\sqrt{(\dot{x}+\dot{\theta}y)^2+(\dot{y}-\dot{\theta}x)^2}}}$$

   Note that these equations are valid only as long as there is relative slip between the puck and the turntable. If the puck stops sliding due to friction, it simply goes in circles with the turntable and, therefore, its equations of motion then are $\ddot{x} = -\dot{\theta}^2 x$, $\ddot{y} = -\dot{\theta}^2 y$.

2. **Equation of motion using a rotating reference frame:** Now we derive the equations of motion using a rotating reference frame, $\mathcal{B}$, with (x', y') coordinate axes, fixed to the rotating turntable. Let the position of the puck in the rotating frame be $\vec{\boldsymbol{r}}_{P/O'} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'$. Note that the velocity of the puck in the rotating reference frame is $\vec{\boldsymbol{v}}_{P/\mathcal{B}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'$, and acceleration is $\vec{\boldsymbol{a}}_{P/\mathcal{B}} = \ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}\hat{\boldsymbol{\jmath}}'$.

Now, from the linear momentum balance ($\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) for the puck, we get

$$-\mu \cancel{m} g \frac{\vec{\boldsymbol{v}}_{\text{rel}}}{|\vec{\boldsymbol{v}}_{\text{rel}}|} = \cancel{m}(\vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{P/\mathcal{B}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}})$$

where we have used the three term acceleration formula for $\vec{\boldsymbol{a}}_P$. Here,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= -\dot{\theta}^2(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') \\
\vec{\boldsymbol{a}}_{P/\mathcal{B}} &= \ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}' \\
2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}} &= -2\dot{\theta}\dot{y}'\hat{\boldsymbol{\imath}}' + 2\dot{\theta}\dot{x}'\hat{\boldsymbol{\jmath}}'
\end{aligned}$$

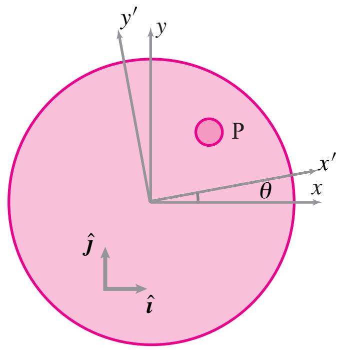

Figure 19.17: Axes (x', y') represent the rotating frame $\mathcal{B}$ fixed to the rotating turntable; ie (x', y') rotate with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{k}}$.

Note that the point P', coincident with P and fixed on the turntable, is stationary with respect to the rotating frame. Therefore, the relative velocity of P as observed in the rotating frame is $\vec{\boldsymbol{v}}_{\text{rel}} = \vec{\boldsymbol{v}}_{P/\mathcal{B}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'$. Substituting these terms in the LMB equation above, we have

$$-\mu g \frac{\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} = (-\dot{\theta}^2 x' + \ddot{x}' - 2\dot{\theta}\dot{y}')\hat{\boldsymbol{\imath}}' + (-\dot{\theta}^2 y' + \ddot{y}' + 2\dot{\theta}\dot{x}')\hat{\boldsymbol{\jmath}}'$$

Dotting this equation with $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, respectively, we get

$$\begin{aligned}
\ddot{x}' &= \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} + 2\dot{\theta}\,\dot{y}' \\
\ddot{y}' &= \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} - 2\dot{\theta}\,\dot{x}'
\end{aligned}$$

These are the required equations of motion for the puck in the rotating frame.

$$\ddot{x}' = \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} + 2\dot{\theta}\,\dot{y}', \quad \ddot{y}' = \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} - 2\dot{\theta}\,\dot{x}'$$

Once we find a solution $x'(t)$ and $y'(t)$ of these equations, we can find the solution in the fixed frame by transforming (x', y') to (x, y) through

$$\left\{ \begin{array}{c} x(t) \\ y(t) \end{array} \right\} = \left[\begin{array}{cc} \cos(\dot{\theta}t) & -\sin(\dot{\theta}t) \\ \sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{array} \right] \left\{ \begin{array}{c} x'(t) \\ y'(t) \end{array} \right\}$$

Also, note that when the solution of the equations of motion in the rotating reference frame brings the puck to halt, the puck stops with respect to the rotating turntable. To an observer in the fixed frame, the puck will be going in circles with a constant $\dot{\theta}$.

SAMPLE 19.8 A collar sliding on a frictional rod. A collar of mass $m = 0.5\,\mathrm{lbm}$ slides on a massless rigid rod OA of length $\ell = 8\,\mathrm{ft}$. The rod rotates counterclockwise with a constant angular speed $\dot{\theta} = 5\,\mathrm{rad/s}$. The coefficient of friction between the rod and the collar is $\mu = 0.3$. At time $t = 0\,\mathrm{s}$, the bar is horizontal, the collar is at $R_0 = 1\,\mathrm{ft}$ and has radial speed $\dot{R}_0 = \mu\dot{\theta}R_0$ towards the pivot O. Ignore gravity.

1. How does the position of the collar change with time (*i.e.,* what is the equation of motion of the collar)?
2. Plot the path of the collar starting from $t = 0\,\mathrm{s}$ till the collar shoots off the end of the bar.
3. How long does it take for the collar to leave the bar?

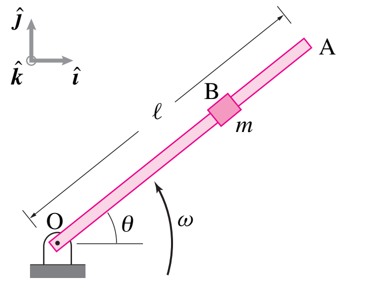

Figure 19.18: A collar slides on a frictional bar and finally shoots off the end. The bar rotates with constant angular speed.

Solution

1. A Free-Body Diagram of the collar at a general position (R, θ) is shown in Fig. 19.19. The geometry of the position vector and basis vectors is shown in Fig. 19.20. In vector notation, the forces on the collar are $\vec{\boldsymbol{N}} = N\hat{\boldsymbol{e}}_\theta$ acting normal to the rod and the force of friction $\vec{\boldsymbol{F}}_s = -\mu N\hat{\boldsymbol{e}}_R$ acting along the rod. Thus linear momentum balance for the collar is:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -\mu N\hat{\boldsymbol{e}}_R + N\hat{\boldsymbol{e}}_\theta &= m[(\ddot{R} - R\dot{\theta}^2)\,\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_{0})\,\hat{\boldsymbol{e}}_\theta \end{aligned} \qquad (19.11)$$

Note that $\ddot{\theta} = 0$ because the rod is rotating at a constant rate. Dotting both sides of eqn. (19.11) with $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ we get

$$\begin{aligned} [\text{Eqn. (19.11)}]\cdot\hat{\boldsymbol{e}}_R \quad \Rightarrow \quad -\mu N &= m(\ddot{R} - R\dot{\theta}^2) \\ \text{or} \quad \ddot{R} - R\dot{\theta}^2 &= -\frac{\mu N}{m} \\ [\text{Eqn. (19.11)}]\cdot\hat{\boldsymbol{e}}_\theta \quad \Rightarrow \quad N &= 2m\dot{R}\dot{\theta}. \end{aligned}$$

Eliminating N from the last two equations we get

$$\ddot{R} + 2\mu\dot{\theta}\,\dot{R} - \dot{\theta}^2\,R = 0.$$

Since $\dot{\theta} = \omega$ is constant, the above equation is of the form

$$\ddot{R} + C\,\dot{R} - \omega^2\,R = 0 \qquad (19.12)$$

where $C = 2\mu\omega$ and $\omega = \dot{\theta}$.

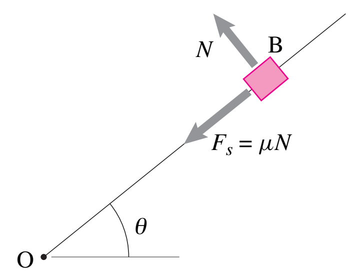

Figure 19.19: Free-Body Diagram of the collar. The only forces on the collar are from the bar: the normal force N and the friction force $F_s = \mu N$.

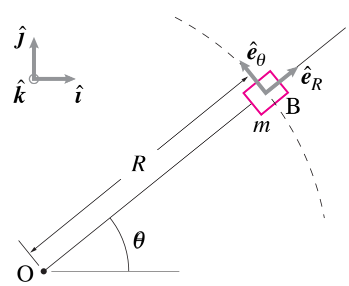

Figure 19.20: Geometry of the collar position at an arbitrary time during its slide on the rod.

Note that we could derive eqn. (19.12) from eqn. (19.11) in a single step by taking a dot product of eqn. (19.11) with $\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta$. Why? Look at the left hand side of eqn. (19.11). The net reaction force is $N(-\mu\hat{\boldsymbol{e}}_R + \hat{\boldsymbol{e}}_\theta)$ acting in the direction $(-\mu\hat{\boldsymbol{e}}_R + \hat{\boldsymbol{e}}_\theta)$. If we dot the reaction force with a vector normal to its direction, we get rid of the reaction force. A vector normal to $(-\mu\hat{\boldsymbol{e}}_R + \hat{\boldsymbol{e}}_\theta)$ is $\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta$.

$$\begin{aligned} [\text{Eqn. (19.11)}]\cdot(\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta) \quad \Rightarrow \quad 0 &= m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + 2\dot{R}\dot{\theta}\hat{\boldsymbol{e}}_\theta]\cdot(\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta) \\ \Rightarrow \quad 0 &= \ddot{R} + 2\mu\dot{\theta}\,\dot{R} - \dot{\theta}^2\,R \end{aligned}$$

which is the same equation as eqn. (19.12).

Solution of equation (19.12): The characteristic equation②associated with

②If we assume a solution of the form $R(t) = e^{\lambda t}$ with some unknown λ and plug back into the given differential equation, we get an algebraic equation in λ which is called the characteristic equation. For more information, see box 11.3 on page 563, eqn. (11.9).

Eqn. (19.12) is

$$\begin{aligned}\lambda^2 + C\lambda - \omega^2 &= 0\\ \Rightarrow \quad \lambda &= \frac{-C \pm \sqrt{C^2+4\omega^2}}{2}\\ &= \omega(-\mu \pm \sqrt{\mu^2+1}).\end{aligned}$$

Therefore, the solution of Eqn. (19.12) is

$$\begin{aligned}R(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t}\\ &= Ae^{(-\mu+\sqrt{\mu^2+1})\omega t} + Be^{(-\mu-\sqrt{\mu^2+1})\omega t}.\end{aligned}$$

Substituting the given initial conditions: $R(0) = R^0 = 1\,\text{ft}$ and $\dot{R}(0) = -\mu\dot{\theta}R^0$, we get3

3From the general solution and the given initial conditions, we have:

$$\begin{aligned}R_0 &= A + B\\ -\mu\omega R_0 &= A\lambda_1 + B\lambda_2\end{aligned}$$

Solving these two equations simultaneously, we get $A = \frac{\lambda_2+\mu\omega}{\lambda_2-\lambda_1}R_0$, and $B = -\frac{\lambda_1+\mu\omega}{\lambda_2-\lambda_1}R_0$. Substituting the values of λ_1 and λ_2, we get $A = B = \frac{R_0}{2} = \frac{1\,\text{ft}}{2}$.

$$R(t) = \frac{1\,\text{ft}}{2}\left[e^{(-\mu+\sqrt{\mu^2+1})\omega t} + e^{(-\mu-\sqrt{\mu^2+1})\omega t}\right]. \tag{19.13}$$

$$\boxed{R(t) = \frac{1\,\text{ft}}{2}\left[e^{(-\mu+\sqrt{\mu^2+1})\omega t} + e^{(-\mu-\sqrt{\mu^2+1})\omega t}\right].}$$

2. To draw the path of the collar we need both R and θ. Because $\dot{\theta} = 5\,\text{rad/s} =$ constant,

$$\theta = \dot{\theta}\,t = (5\,\text{rad/s})\,t.$$

Now we can take various values of t from 0 s to, say, 1 s, and calculate values of θ and R. Plotting all these values of R and θ, however, does not give us the correct path after the collar reaches the end of the bar, at R = length of the bar = 8 ft. So we need to find the time t_f when $R(t_f) = 8\,\text{ft}$. Equation (19.13) is a nonlinear algebraic equation for t which we can solve iteratively on a computer, or with some patience, even on a calculator. One way to find t_f would be to simply plot $R(t)$ and find the intersection with $R = 8$ (see fig. 19.21) and read the corresponding value of t. Following either of these methods we find that $t_f \approx 0.74518$ s here. Now, we can plot the path of the collar by computing R and θ from $t = 0$ to $t = t_f$ and making a polar plot on a computer as follows (pseudocode).

```
tf = 0.74518                    % final value of t
t = 0:tf/100:tf;                % take 101 points in [0 tf]
R0 = 1; w = 5; mu = .3;         % initialize variables
f1 = -mu + sqrt(mu^2 +1);       % first partial exponent
f2 = -mu - sqrt(mu^2 +1);       % second partial exponent
R = 0.5*R0*(exp(f1*w*t) + exp(f2*w*t));  % calculate R
theta = w*t;                    % calculate theta
polarplot(theta,r)
```

The plot produced thus is shown in Fig. 19.22.

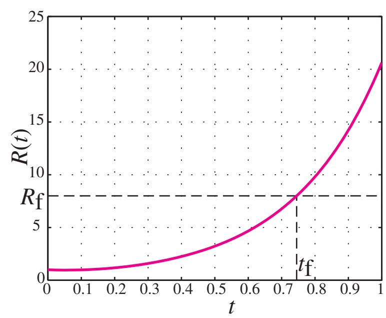

Figure 19.21: Finding the final time t_f from the graph of $R(t)$ such that $R(t_f) = R_f = 8\,\text{ft}$.

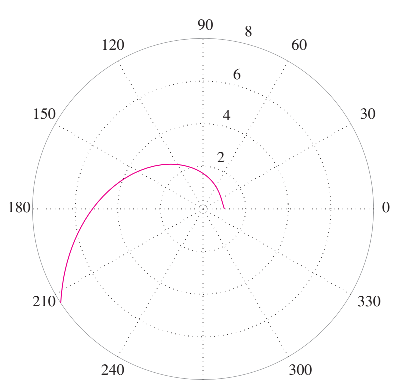

Figure 19.22: Plot of the path of the collar until it leaves the rod.

3. The time t_f computed above was

$$t_f = 0.7452\,\text{s}.$$

By plugging this value in the expression for $R(t)$ (Eqn. (19.13) we get, indeed,

$$R = 8\,\text{ft}.$$

$$\boxed{t_f = 0.7452\,\text{s}}$$

Note, although we have not checked it explicitly, the increase in kinetic energy of the particle comes from the work of the force from the rod on the particle:

$$\Delta E_K = \int \vec{F} \cdot \vec{v}\,dt$$

SAMPLE 19.9 **A collar sliding on a rotating rod in 3-D.** A massless and frictionless rod AB is rotating about the vertical axis through point A. The rod is bent at an angle ϕ from the vertical and is rotating with a constant angular speed ω about the vertical axis. A small collar of mass m slides on the rod. Assume that at time $t = 0$ the collar is released from a rest position with respect to the rod at a distance R_0 from the axis of rotation. There is no gravity.

1. Find the equation of motion for the collar (a differential equation for the position of the collar).
2. How does the distance of the collar from the vertical axis change with time?
3. For $\phi = \pi/2$, show that the solution obtained in (ii) above is the same as that obtained in Sample 19.8 for $\mu = 0$.

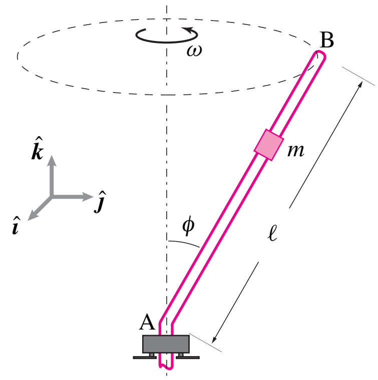

Figure 19.23: A collar of mass m slides on a massless and frictionless bar, bent at an angle ϕ with the vertical axis. The bar is rotating at a constant angular speed ω about the z-axis.

Solution Since the bar is bent and it rotates about the vertical axis, it sweeps a conical surface about the axis of rotation. As the collar slides on the rod, it traces a path on this surface. Let (R, θ, z) be the cylindrical coordinates of the collar at any general time t (Fig. 19.24(a)). Since the rod is frictionless and there is no gravity, the only force acting on the collar is the normal reaction from the rod. This force is shown in the free-body diagram of the collar in Fig. 19.24(b). Note that there are a lot of possible directions for a normal to the rod at the collar. In fact, any vector in the plane perpendicular to the rod at the collar is normal to the rod. So, at this point let us write

$$\vec{\boldsymbol{N}} = N\hat{\boldsymbol{n}}$$

where $\hat{\boldsymbol{n}}$ is a unit normal to the rod. Thus, if $\hat{\boldsymbol{\lambda}}$ is a unit vector along the rod, then

$$\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{n}} = 0 \qquad (19.14)$$

where

$$\hat{\boldsymbol{\lambda}} = \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \sin\phi\,\hat{\boldsymbol{e}}_R + \cos\phi\,\hat{\boldsymbol{k}}.$$

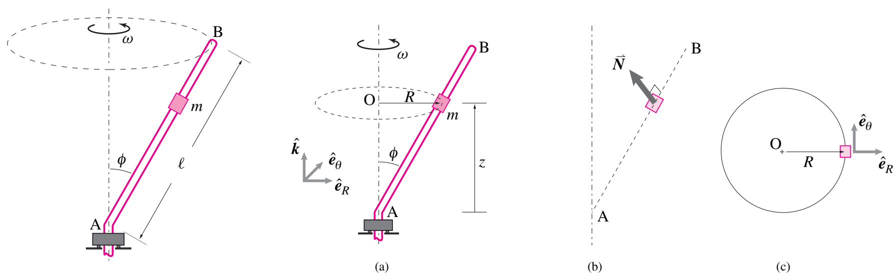

Figure 19.24: (a) Cylindrical coordinates R and z of the collar (θ is not shown) and the orientation of the cylindrical basis vectors. (b) Free-body diagram of the collar (there is no gravity). (c) Instantaneous R-$theta$ plane of the collar.

1. **Equation of Motion:** We now write the linear momentum balance for the collar:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

where

$$\begin{aligned}\sum \vec{F} &= N\hat{\boldsymbol{n}} \\ \vec{\boldsymbol{a}} &= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_{0})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}\end{aligned}$$

Dotting both sides of the equation $\sum \vec{F} = m\vec{\boldsymbol{a}}$ by $\hat{\boldsymbol{\lambda}}$ we get

$$\begin{aligned} N\underbrace{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}}}_{0} &= m[(\ddot{R} - R\dot{\theta}^2)\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{\lambda}}}_{\sin\phi} + 2\dot{R}\dot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}}}_{0} + \ddot{z}\underbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\lambda}}}_{\cos\phi}] \\ \Rightarrow \qquad & (\ddot{R} - R\dot{\theta}^2)\sin\phi + \ddot{z}\cos\phi = 0. \end{aligned}$$

This expression is *an* equation of motion of the collar. However, it is only a single equation in terms of derivatives of two variables R and z. Now, from geometry

$$R = z\tan\phi \qquad \Rightarrow \qquad \ddot{z} = \ddot{R}/\tan\phi.$$

Substituting this relationship in the equation of motion we get

$$\ddot{R}(\sin\phi + \frac{\cos\phi}{\tan\phi}) - R\dot{\theta}^2 = 0.$$

Noting that $\dot{\theta} = \omega =$ a constant, we may write the above equation, with some trigonometric simplifications, as

$$\ddot{R} + (\omega\ \sin\phi)^2 R = 0 \qquad (19.15)$$

which is an equation of motion of the collar in terms of its distance R from the vertical axis.

2. **Solution of the equation of motion:** The solution of Eqn. (19.15) is given by ④

④ You may find the solution by solving the corresponding *characteristic equation* (see the solution of Eqn. (19.15) in Sample 19.8, for example), or by looking up the table of "The Simplest ODEs and Their Solutions" on page 22 of the text, or by guessing a solution yourself if you have some experience with ODEs.

$$R(t) = C_1 e^{(\omega\ \sin\phi)t} + C_2 e^{-(\omega\ \sin\phi)t}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions. Substituting the given initial conditions: $R(0) = R_0$ and $\dot{R}(0) = 0$ we get

$$C_1 = C_2 = \frac{R_0}{2}.$$

Therefore, the solution may be written as

$$R(t) = \frac{R_0}{2}\left[e^{(\omega\ \sin\phi)t} + e^{-(\omega\ \sin\phi)t}\right]. \qquad (19.16)$$

3. **Special case, $\phi = \pi/2$:** Substituting $\phi = \pi/2$ in Eqn. (19.16) we get

$$R(t) = \frac{R_0}{2}[e^{\omega t} + e^{-\omega t}]$$

which is the same solution as obtained in Eqn. (19.16) in Sample 19.8 for $\mu = 0$ and $R_0 = 1\,\text{ft}$.

SAMPLE 19.10 **Osculating circle in 3-D.** A small bead is driven down a a wire-frame bent in the shape of a conical helix by a tiny motor imbedded in the bead. The combined mass of the bead and the motor is $m = 200$ gm. The shape of the helix is given: $R = R_0\theta,\ z = 2R_0\theta$ and where $R_0 = 0.3$ m. At the instant when $\theta = 2$ radians, the angular speed and the angular acceleration of the bead are $\dot\theta = 1$ rad/s and $\ddot\theta = 2\ \text{rad/s}^2$. Find

1. the *net* normal force on the bead and
2. the radius of the osculating circle at the instant given.

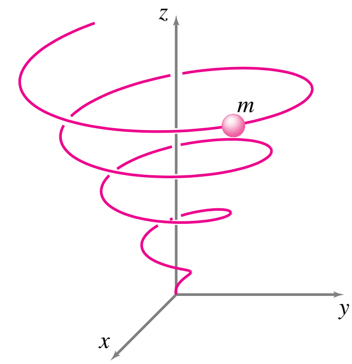

Figure 19.25: A motor driven bead slides down a helical wire-frame described by the equations $R = R_0\theta,\ z = 2R_0\theta$.

Solution

1. We can find the net normal force on the bead from the linear momentum balance of the bead (see the free-body diagram of the bead):

$$\begin{aligned}
\sum \vec{\boldsymbol F} &= m\vec{\boldsymbol a}\\
N\hat{\boldsymbol e}_n + T\hat{\boldsymbol e}_t - mg\hat{\boldsymbol k} &= m(a_t\hat{\boldsymbol e}_t + a_n\hat{\boldsymbol e}_n)\\
\Rightarrow\quad F_n &\equiv \sum \vec{\boldsymbol F}\cdot\hat{\boldsymbol e}_n = ma_n.
\end{aligned}$$

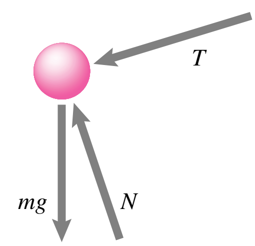

Figure 19.26: Free-body diagram of the bead. Note that the *net* normal force is $F_n = \sum \vec{\boldsymbol F}\cdot\hat{\boldsymbol e}_n$ where $\sum \vec{\boldsymbol F} = N\hat{\boldsymbol e}_n + T\hat{\boldsymbol e}_t - mg\hat{\boldsymbol k}$.

Thus we need to find the normal acceleration of the bead. We can write the position of the bead as

$$\begin{aligned}
\vec{\boldsymbol r} &= \underbrace{R}_{R_0\theta}\hat{\boldsymbol e}_R + \underbrace{z}_{2R_0\theta}\hat{\boldsymbol k}\\
\Rightarrow\quad \vec{\boldsymbol v} &\equiv \dot R\hat{\boldsymbol e}_R + R\dot\theta\hat{\boldsymbol e}_\theta + \dot z\hat{\boldsymbol k}\\
&= R_0\dot\theta\hat{\boldsymbol e}_R + R_0\theta\dot\theta\hat{\boldsymbol e}_\theta + 2R_0\dot\theta\hat{\boldsymbol k}\\
&= R_0\dot\theta(\hat{\boldsymbol e}_R + \theta\hat{\boldsymbol e}_\theta + 2\hat{\boldsymbol k}),\\
\text{and}\quad \vec{\boldsymbol a} &\equiv (\ddot R - R\dot\theta^2)\hat{\boldsymbol e}_R + (2\dot R\dot\theta + R\ddot\theta)\hat{\boldsymbol e}_\theta + \ddot z\hat{\boldsymbol k}\\
&= (R_0\ddot\theta - R_0\theta\dot\theta^2)\hat{\boldsymbol e}_R + (2R_0\dot\theta^2 + R_0\theta\ddot\theta)\hat{\boldsymbol e}_\theta + 2R_0\ddot\theta\hat{\boldsymbol k}\\
&= R_0[(\ddot\theta - \theta\dot\theta^2)\hat{\boldsymbol e}_R + (\dot\theta^2 + \theta\ddot\theta)\hat{\boldsymbol e}_\theta + 2\ddot\theta\hat{\boldsymbol k}]
\end{aligned}$$

Substituting the given numerical values for $\theta, \dot\theta, R_0$ and $\ddot\theta$ in the above expressions for $\vec{\boldsymbol v}$ and $\vec{\boldsymbol a}$ we get the velocity and the acceleration of the bead at the instant of interest:

$$\begin{aligned}
\vec{\boldsymbol v} &= 0.3\,\text{m/s}(\hat{\boldsymbol e}_R + 2\hat{\boldsymbol e}_\theta + 2\hat{\boldsymbol k})\\
\vec{\boldsymbol a} &= 1.2\,\text{m/s}^2(2\hat{\boldsymbol e}_\theta + \hat{\boldsymbol k}).
\end{aligned}$$

In path coordinates,

$$\vec{\boldsymbol a} = \vec{\boldsymbol a}_t + \vec{\boldsymbol a}_n = a_t\hat{\boldsymbol e}_t + a_n\hat{\boldsymbol e}_n$$

where

$$\begin{aligned}
\hat{\boldsymbol e}_t &= \frac{\vec{\boldsymbol v}}{|\vec{\boldsymbol v}|} &\quad (19.17)\\
&= \frac{0.3\,\text{m/s}(\hat{\boldsymbol e}_R + 2\hat{\boldsymbol e}_\theta + 2\hat{\boldsymbol k})}{0.3\sqrt{1+4+4}\ \text{m/s}} &\quad (19.18)\\
&= \frac{1}{3}(\hat{\boldsymbol e}_R + 2\hat{\boldsymbol e}_\theta + 2\hat{\boldsymbol k}). &\quad (19.19)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{a}}_t &= (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t \\
&= (\frac{4.8 + 2.4}{3}\ \mathrm{m/s^2})\hat{\boldsymbol{e}}_t \\
&= 0.8\,\mathrm{m/s^2}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta + 2\hat{\boldsymbol{k}}) \\
\vec{\boldsymbol{a}}_n &= \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_t \\
&= [2.4\hat{\boldsymbol{e}}_\theta + 1.2\hat{\boldsymbol{k}} - 0.8\,\mathrm{m/s^2}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta + 2\hat{\boldsymbol{k}})] \\
&= (-0.8\hat{\boldsymbol{e}}_R + 0.8\hat{\boldsymbol{e}}_\theta - 0.4\hat{\boldsymbol{k}})\,\mathrm{m/s^2} \\
a_n &= |\vec{\boldsymbol{a}}_n| = 1.2\,\mathrm{m/s^2} \\
\hat{\boldsymbol{e}}_n &= \frac{\vec{\boldsymbol{a}}_n}{a_n} \\
&= -\frac{2}{3}\hat{\boldsymbol{e}}_R + \frac{2}{3}\hat{\boldsymbol{e}}_\theta - \frac{1}{3}\hat{\boldsymbol{k}}.
\end{aligned}$$

Hence, the net normal force on the bead

$$\begin{aligned}
\vec{\boldsymbol{F}}_n &= m a_n \hat{\boldsymbol{e}}_n \\
&= 0.2\,\mathrm{kg} \cdot 1.2\,\mathrm{m/s^2} \cdot \frac{1}{3}(-2\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta - \hat{\boldsymbol{k}}) \\
&= (-0.16\hat{\boldsymbol{e}}_R + 0.16\hat{\boldsymbol{e}}_\theta - 0.08\hat{\boldsymbol{k}})\,\mathrm{N}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{F}}_n = 0.08\,\mathrm{N}(-2\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{k}})}$$

2. For calculating the radius ρ of the osculating circle, we note that

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow \quad \rho &= \frac{v^2}{a_n} = \frac{(0.9\,\mathrm{m/s})^2}{1.2\,\mathrm{m/s^2}} = 0.675\,\mathrm{m}.
\end{aligned}$$

$$\boxed{\rho = 0.675\,\mathrm{m}}$$