18.5 Advanced kinematics of planar motion

In this section we consider three types of problems where the kinematics involves solution of differential equations. In most cases this means computer solution is involved for this type of problem. Here are the three problem types:

- **I. Closed kinematic chains.** The main simple example is a 4-bar linkage with one bar grounded. This system has one degree of freedom, but it is difficult to directly calculate the positions, velocities and accelerations of all points in terms of one variable. Instead, the constraint that the linkage is closed is sometimes most-easily expressed as a differential equation.
- **II. Rolling contact with not-round objects.** For non-round rollers and cams solving for configuration, velocity and acceleration can sometimes be best done with integration. A side benefit from studying this topic is the observation that all non-translational motions are equivalent to rolling of some kind.
- **III. Contact with ideal wheels and skates, looking down.** Cars, tricycles, trailers, grocery carts and sleighs have wheels and have dynamics that is sometimes well characterized by planar analysis, where the plane is the horizontal plane. In this view the simple model of a wheel is as something that prevents sideways motion but allows motion in the direction of travel (like for some of the trike and car problems in 1-D constrained motion). Such problems are called *non-holonomic* (see box 18.3 on page 3).

Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to the next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than do they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

Example: Four bar linkage: configuration variables
It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ of any one of its joints.

However, given a configuration, the constraint on the rates and accelerations is relatively easy to express, always yielding linear equations.

Example: Four bar linkage: configuration rates
If you write the relative velocities of the ends of the bars in terms of configuration rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ and then write the chain closure equation you get a linear equation in the rates. Likewise if you write the closure condition in terms of acceleration. The coefficients in these equations are likely to be complex functions of the configuration, so integrating these equations requires numerics. But the constraint is linear.

Thus, as shown in the last sample of Sect. 10.4, one way to calculate the evolving configurations of a closed chain is to integrate the velocity relations numerically.

Rolling of not round objects

When two objects roll on each other they maintain contact and do not slip relative to each other. That is to say rolling of one rigid curve $\mathcal{B}$ on another $\mathcal{A}$ means:

- The instantaneous relative motion of $\mathcal{B}$ with respect to $\mathcal{A}$ is a rotation about the contact point at the common tangent C, and
- The sequence of points C moves the same distance on both curves.

For simplicity let's take $\mathcal{A}$ to be a curve fixed in space on which rigid curve $\mathcal{B}$ rolls. Take a reference point of interest fixed on body $\mathcal{B}$ to be O'. So,

$$\vec{\boldsymbol{v}}_{\mathrm{O'}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{O'/C}} \qquad \text{where } \vec{\boldsymbol{\omega}} = \dot{\theta}_{\mathcal{B}} \hat{\boldsymbol{k}} \tag{18.66}$$

and $\theta_{\mathcal{B}}$ is, say, the rotation of a $\hat{\boldsymbol{\imath}}'$ axis fixed in $\mathcal{B}$ relative to a $\hat{\boldsymbol{\imath}}$ axis fixed in $\mathcal{A}$. If we use the rotation $\theta_{\mathcal{B}}$ of body $\mathcal{B}$ as our configuration variable, we now know how to find the velocity of all points in terms of their positions and the rotation rate. Thus we can find the rate of change of the configuration. To proceed as time progresses we also need to know how the position of point C evolves. Not the material point C on either body, but the location of mutual contact.

If we assume that both curves are parameterized by arc-length going counter-clock wise, if we take curvature as positive if directed towards the interior of each curve's body, then the condition of maintaining contact requires that

$$\vec{\boldsymbol{v}}_{\mathrm{C}} = \dot{s}\hat{\boldsymbol{e}}_t \qquad \text{where } \hat{\boldsymbol{e}}_t \text{ is the tangent to fixed curve } \mathcal{A}.$$

and s is the advance along curve $\mathcal{A}$. To maintain tangency, the angles must be maintained so

$$\dot{s} = \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\dot{\theta}.$$

18.3 Skates, wheels and non-holonomic constraints

Of the words in this book "non-holonomic" is probably the most obscure. This is because the subject of mechanics was mostly stolen from engineers by physicists about 100 years ago. And physicists, the authors of most introductions to mechanics, had no use for non-holonomic mechanics as it wasn't useful for the development of quantum mechanics. So many people are unaware of the word, the subject or its utility.

In two dimensions the word non-holonomic in effect means the mechanics of objects constrained by ideal skates or massless ideal wheels. Often these non-holonomic constraints are described as "non integrable". Literally, the word non-holonomic means "not whole". But in what sense is a rolling ideal wheel "non-integrable" or less "whole" than anything else?

A constrained rigid body. Consider a rigid body that is free to slide on a plane. It has three degrees of freedom described by $x_{O'}, y_{O'}$ and θ, all measured relative to a fixed reference frame $O\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}$. Point C on the body has relative position $\vec{\boldsymbol{r}}_{C/O'} = x'_C\hat{\boldsymbol{\imath}}' + y'_C\hat{\boldsymbol{\jmath}}'$ where x'_C and y'_C are constants. Now let's constrain the body at point C one of these two different ways:

- **a)** Pin the body to the ground with an ideal hinge at point C. This keeps point C from moving but allows the body to rotate (holonomic).
- **b)** Put an ideal wheel or skate under the body at C that prevents sliding sideways to the skate but allows point C to move parallel to the skate and also allows rotation about the skate (non-holonomic).

Pin Constraint. In the first case, for the pin, we could describe the constraint with the phrase 'point C on the body can have no velocity' and then write and calculate:

$$\begin{aligned}
\vec{\mathbf{0}} &= \vec{\boldsymbol{v}}_C \\
\vec{\mathbf{0}} &= \vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{C/O'} \qquad (18.62)\\
\vec{\mathbf{0}} &= \dot{x}_{O'}\hat{\boldsymbol{\imath}} + \dot{y}_{O'}\hat{\boldsymbol{\jmath}} + \dot{\theta}\hat{\boldsymbol{k}}\times(x'_C\hat{\boldsymbol{\imath}}' + y'_C\hat{\boldsymbol{\jmath}}') \\
\{\,\vec{\mathbf{0}} &= \dot{x}_{O'}\hat{\boldsymbol{\imath}} + \dot{y}_{O'}\hat{\boldsymbol{\jmath}} + \left(\dot{\theta}x'_C\hat{\boldsymbol{\jmath}}' - \dot{\theta}y'_C\hat{\boldsymbol{\imath}}'\right)\,\}
\end{aligned}$$

$$\{\}\cdot\hat{\boldsymbol{\imath}} \quad\Rightarrow\quad \dot{x}_{O'} - \dot{\theta}\sin\theta x'_C - \dot{\theta}\cos\theta y'_C = 0$$

$$\{\}\cdot\hat{\boldsymbol{\jmath}} \quad\Rightarrow\quad \dot{y}_{O'} + \dot{\theta}\cos\theta x'_C - \dot{\theta}\sin\theta y'_C = 0.$$

$$\Rightarrow\quad \frac{d}{dt}\left(x_{O'} + \cos\theta x'_C - \sin\theta y'_C\right) = 0$$

$$\Rightarrow\quad \frac{d}{dt}\left(y_{O'} + \sin\theta x'_C + \cos\theta y'_C\right) = 0$$

The last two equations are two differential equations in the three variables $x_{O'}, y_{O'}$ and θ. They are "integrable" in the sense that they are equivalent to

$$\begin{aligned}
& x_{O'} + \cos\theta x'_C - \sin\theta y'_C &= C_1 \\
\text{and}\quad & y_{O'} + \sin\theta x'_C + \cos\theta y'_C &= C_2 \qquad (18.63)
\end{aligned}$$

where C_1 and C_2 are integration constants that need to be set by the starting configuration. Solving for $x_{O'}$ and $y_{O'}$ in terms of θ:

$$\begin{aligned}
x_{O'} &= C_1 - \cos\theta x'_C + \sin\theta y'_C \\
y_{O'} &= C_2 - \sin\theta x'_C - \cos\theta y'_C.
\end{aligned}$$

Here we have derived the obvious, that a pinned body has one independent configuration variable θ, but we did so starting with a vector expression of constraint in terms of velocities (eqn. (18.62)). Then we wrote the constraint as two scalar constraints on the derivatives of configuration variables and then "integrated" them to write constraints on the configuration variables, finally eliminating two of the configuration variables.

Skate constraint. Now consider the same body constrained by a skate or ideal wheel at C instead of a pin. The skate is aligned with the $\hat{\boldsymbol{\imath}}'$ so point C can only move in the $\hat{\boldsymbol{\imath}}'$ direction. The body is still free to rotate about the point C (to steer). Thus,

$$\begin{aligned}
0 &= \vec{\boldsymbol{v}}_C\cdot\hat{\boldsymbol{\jmath}}' \\
0 &= \left(\vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{C/O'}\right)\cdot\hat{\boldsymbol{\jmath}}' \qquad (18.64)\\
0 &= \left(\dot{x}_{O'}\hat{\boldsymbol{\imath}} + \dot{y}_{O'}\hat{\boldsymbol{\jmath}} + \dot{\theta}\hat{\boldsymbol{k}}\times(x'_C\hat{\boldsymbol{\imath}}' + y'_C\hat{\boldsymbol{\jmath}}')\right)\cdot\hat{\boldsymbol{\jmath}}' \\
0 &= -\dot{x}_{O'}\sin\theta + \dot{y}_{O'}\cos\theta + \dot{\theta}x'_C \\
\Rightarrow\quad 0 &= \frac{d}{dt}F(x_{O'}, y_{O'}, \theta)\,?
\end{aligned}$$

As for the hinge where we found 2 constant functions, we might want to find the function $F(x_{O'}, y_{O'}, \theta)$ that satisfies the differential equation above, namely

$$\frac{d}{dt}F(x_{O'}, y_{O'}, \theta) = -\dot{x}_{O'}\sin\theta + \dot{y}_{O'}\cos\theta + \dot{\theta}x'_C. \qquad (18.65)$$

Another math nightmare. How do we find this F? You can't find one. This is the crux of the matter. Neither your calculus professor nor Ramanujan could find one either. No computer can find one, or even a numerical approximation of a solution. There is no function $F(x_{O'}, y_{O'}, \theta)$ that solves eqn. (18.65). The solution fundamentally does not exist. That is why we say the skate/wheel constraint eqn. (18.64) is "non-integrable".

Parallel parking We can use physical reasoning to show that no function F can solve eqn. (18.65). If such an F did exist it would mean that only the set of configurations with position $x_{O'}, y_{O'}$ and angles θ consistent with $F = constant$ would be allowed by the skate constraint (assuming F depends nontrivially on at least one of the variables). This means there would be some angles and positions that the body couldn't get to. Remember, we are not doing mechanics, just kinematics. So we can see what configurations are geometrically allowed while still respecting the constraint. The simple observation that motivates the answer is this:

> Even though the skate constrains $\vec{\boldsymbol{v}}_C$ to not have a sideways component, point C can get to a point that is straight sideways.

How? Like a car can move sideways into a parking space without skidding sideways; by parallel parking. More generally, the body can get to any position and any orientation by the following moves. First rotate the body so the skate aims to its new goal. Then slide the skate to its new goal. And finally rotate the body to its new desired orientation.

Thus, the skate constraint does not disallow *any* configurations! Yet the constraint does disallow some velocities (the skate can't go sideways). In this way, the skate constraint is not "whole". It constrains velocities without constraining configurations.

Counting degrees of freedom. How many degrees of freedom does a body with a skate constraint have? There are two different answers. By counting possible configurations there are three degrees of freedom (it takes three variables to describe all possible configurations). But at any configuration the velocity can be described by 2 numbers ($\dot{\theta}$ and $v_{i'}$). Whenever the number of configuration degrees of freedom is greater than the number of velocity degrees of freedom (for example, 3>2) there are non-holonomic constraints.

One might like more examples. But besides artificial mathematical ones, there are none. The only smooth non-holonomic constraint in 2D mechanics is the ideal skate or wheel.

Altogether this gives

$$\vec{\boldsymbol{v}}_{\mathrm{C}} = \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\dot{\theta}\hat{\boldsymbol{e}}_t \quad \text{(not the velocity of any material point).}$$

To find the acceleration of material point O' on $\mathcal{B}$ we differentiate eqn. (18.66) with respect to time:

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{O'}} &= \frac{d}{dt}\vec{\boldsymbol{v}}_{\mathrm{O'}} \\
&= \frac{d}{dt}\left(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{O'/C}}\right) \\
&= \frac{d}{dt}\left(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{r}}_{\mathrm{O'}} - \vec{\boldsymbol{r}}_{\mathrm{C}})\right) \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{O'/C}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\mathrm{O'}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{\mathrm{C}} \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{O'/C}} - \omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{\mathrm{O'/C}} - \vec{\boldsymbol{\omega}} \times \left(\frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\dot{\theta}\hat{\boldsymbol{e}}_t\right) \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{O'/C}} - \omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{\mathrm{O'/C}} + \left(\frac{\dot{\theta}_{\mathcal{B}}^2}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\hat{\boldsymbol{e}}_n\right)
\end{aligned}$$

where $\hat{\boldsymbol{e}}_n$ is normal to the curves and directed towards the interior of $\mathcal{B}$. Thus the acceleration of all points on $\mathcal{B}$ is the same as if the body were pinned at C *plus* an acceleration due to rolling. This rolling acceleration is small if either of the bodies is sharp (has very large κ) and large if the bodies are nearly conformal.

Body curve and space curve

As one rigid body moves arbitrarily on a plane with some non-zero rotation rate we can find a point at relative position $\vec{\boldsymbol{r}}_{\mathrm{C/O'}}$ where $\vec{\boldsymbol{r}}_{\mathrm{C}} = \vec{\boldsymbol{0}}$. That is a place where the velocity due to rotation about O' exactly cancels the velocity *of* O'.

$$\vec{\boldsymbol{0}} = \vec{\boldsymbol{v}}_{\mathrm{O'}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{C/O'}}$$

Crossing both sides with $\hat{\boldsymbol{k}}$ and using that $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ we get

$$\vec{\boldsymbol{r}}_{\mathrm{C/O'}} = \frac{\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{\mathrm{O'}}}{\omega_{\mathcal{B}}}$$

as the point "on" the body that has no velocity. This point does not literally have to be on the body, rather it is fixed to the reference frame defined by the body.

As motion progresses a sequence of such points C is traced on the ground. Similarly a sequence of points is traced on the body. These two sequences are called the space curve and the body curve (or "polohodie" and "herpolhodie" in older books). The motion of body $\mathcal{B}$ is thus a rolling of the body curve on the space curve.

As a machine designer this means you can generate any desired motion by rolling of appropriate shapes. Move the object in the desired manner, draw the space curve and body curve, make parts with those shapes, and the desired motion occurs by a rolling of those shapes.

Ideal wheels and skates, looking down

If we look down on an ideal skate or wheel at point C on a rigid body and assume that the skate is oriented with the positive $\hat{\boldsymbol{\imath}}'$ axis at point C on the body then we know that

$$\vec{\boldsymbol{v}}_{\mathrm{C}} = v_C \hat{\boldsymbol{\imath}}'$$

and hence the velocity of any point G on the body of interest is

$$\begin{aligned}\vec{\boldsymbol{v}}_{\mathrm{G}} &= \vec{\boldsymbol{v}}_{\mathrm{C}} + \vec{\boldsymbol{v}}_{\mathrm{G/C}} \\ &= v_c \hat{\boldsymbol{\imath}}' + \dot{\theta}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}}.\end{aligned}$$

The acceleration is found by differentiating this expression as

$$\begin{aligned}\vec{\boldsymbol{a}}_{\mathrm{G}} = \frac{d}{dt}\vec{\boldsymbol{v}}_{\mathrm{G}} &= \vec{\boldsymbol{a}}_{\mathrm{C}} + \vec{\boldsymbol{a}}_{\mathrm{G/C}} \\ &= \frac{d}{dt}\left(v_c \hat{\boldsymbol{\imath}}' + \dot{\theta}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}}\right) \\ &= \dot{v}_C \hat{\boldsymbol{\imath}}' + v_C \dot{\hat{\boldsymbol{\imath}}} + \ddot{\theta}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}} - \dot{\theta}^2 \vec{\boldsymbol{r}}_{\mathrm{G/C}} \\ &= \underbrace{\dot{v}_C \hat{\boldsymbol{\imath}}' + v_C \dot{\theta}\hat{\boldsymbol{\jmath}}'}_{\vec{\boldsymbol{a}}_{\mathrm{C}}} + + \underbrace{\ddot{\theta}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}} - \dot{\theta}^2 \vec{\boldsymbol{r}}_{\mathrm{G/C}}}_{\vec{\boldsymbol{a}}_{\mathrm{G/C}}}.\end{aligned}$$

It is interesting to note that the Coriolis-like term $v_C\dot{\theta}\hat{\boldsymbol{\jmath}}'$ does not have the usual factor of 2 one encounters in holonomic problems. To find the trajectory of the point C, say, one needs to integrate the velocity like this:

$$\begin{aligned}\dot{x} &= \vec{\boldsymbol{v}}_{\mathrm{C}} \cdot \hat{\boldsymbol{\imath}} \\ &= v_C \hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\imath}} \\ &= v_C \cos\theta \\ \dot{y} &= \vec{\boldsymbol{v}}_{\mathrm{C}} \cdot \hat{\boldsymbol{\jmath}} \\ &= v_C \hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}} \\ &= v_C \sin\theta.\end{aligned}$$

SAMPLE 18.15 Kinematics of a four bar linkage. A four bar linkage ABCD is shown in the figure (fourth bar is the ground AD) at some instant t_0. The driving bar AB rotates with angular velocity $\vec{\omega}_{AB} = \dot{\theta}(t)\hat{\boldsymbol{k}}$. Find the angular velocities of rods BC and CD as a function of $\dot{\theta}$. How can you solve for the positions of the bars at any t if the initial configuration is as shown in the figure?

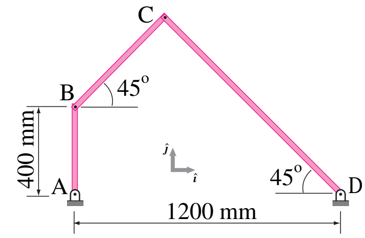


Figure 18.64

Solution Let the angles that rods AB, BC, and CD make with the horizontal (x-axis) be θ, β, and ϕ, respectively. Then, we can write $\vec{\omega}_{BC} = \dot{\beta}\hat{\boldsymbol{k}}$ and $\vec{\omega}_{CD} = \dot{\phi}\hat{\boldsymbol{k}}$. We have to find $\dot{\beta}$ and $\dot{\phi}$.

Note that the motion of point B is a simple circular motion about point A with given angular velocity $\vec{\omega}_{AB}$. Thus, the velocity of point B is known. Now, we can find the velocity of point C two ways: (i) by considering rod CB: $\vec{v}_C = \vec{v}_B + \vec{v}_{C/B} = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$, and (ii) by considering rod CD: $\vec{v}_C = \vec{\omega}_{CD} \times \vec{r}_{C/D}$. Either way the velocity must be the same. Thus, we have a 2-D vector equation with two unknowns $\dot{\beta}$ and $\dot{\phi}$. We can get two independent scalar equations from the vector equation and thus we can solve for the desired unknowns.

Let us use the rotating base vectors $(\hat{\boldsymbol{\lambda}}_1, \hat{\boldsymbol{n}}_1), (\hat{\boldsymbol{\lambda}}_2, \hat{\boldsymbol{n}}_2)$, and $(\hat{\boldsymbol{\lambda}}_3, \hat{\boldsymbol{n}}_3)$ with rods AB, BC, and CD, respectively. Note that these base vectors are basically the $(\hat{\boldsymbol{e}}_R, \hat{\boldsymbol{e}}_\theta)$ pairs; we use $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}})$ just for the sake of easy subscripting. Now,

$$\begin{aligned} \vec{v}_B &= \vec{\omega}_{AB} \times \vec{r}_{B/A} = \ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 \\ \vec{v}_C &= \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = \ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 + \ell_2 \dot{\beta} \hat{\boldsymbol{n}}_2 \qquad (18.67) \\ \text{also,} \quad \vec{v}_C &= \vec{\omega}_{CD} \times \vec{r}_{C/D} = \ell_3 \dot{\phi} \hat{\boldsymbol{n}}_3 \qquad (18.68) \end{aligned}$$

Thus, from eqn. (18.67) and (18.68), we have,

$$\ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 + \ell_2 \dot{\beta} \hat{\boldsymbol{n}}_2 = \ell_3 \dot{\phi} \hat{\boldsymbol{n}}_3 \qquad (18.69)$$

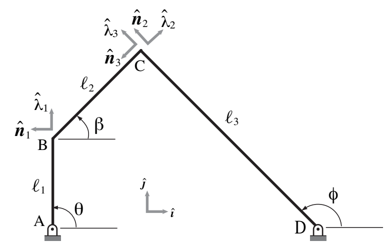


Figure 18.65

Dotting eqn. (18.69) with $\hat{\boldsymbol{\lambda}}_2$ (to eliminate $\dot{\beta}$ term), we get

$$\begin{aligned} \ell_1 \dot{\theta} (\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2) &= \ell_3 \dot{\phi} (\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2) \\ \Rightarrow \quad \dot{\phi} &= \frac{\ell_1}{\ell_3} \frac{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2)}{(\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2)} \dot{\theta}. \qquad (18.70) \end{aligned}$$

Similarly, dotting eqn. (18.69) with $\hat{\boldsymbol{\lambda}}_3$ (to eliminate $\dot{\phi}$ term), we get

$$\dot{\beta} = -\frac{\ell_1}{\ell_2} \frac{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_3)}{(\hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{\lambda}}_3)} \dot{\theta}. \qquad (18.71)$$

We are practically done at this point with the kinematics — we have found $\dot{\beta}$ and $\dot{\phi}$ as functions of $\dot{\theta}$. The various dot products are just geometry and vector algebra. To write them explicitly, we note that $\hat{\boldsymbol{\lambda}}_1 = \cos\theta \hat{\boldsymbol{\imath}} + \sin\theta \hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{n}}_1 = -\sin\theta \hat{\boldsymbol{\imath}} + \cos\theta \hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{\lambda}}_2 = \cos\beta \hat{\boldsymbol{\imath}} + \sin\beta \hat{\boldsymbol{\jmath}}$, etc. Thus,

$$\begin{aligned} \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2 &= -\sin\theta\cos\beta + \cos\theta\sin\beta = \sin(\beta - \theta) \\ \hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2 &= \sin(\beta - \phi) \\ \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_3 &= \sin(\phi - \theta), \quad \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{\lambda}}_3 = \sin(\phi - \beta). \end{aligned}$$

Substituting the appropriate expressions in eqn. (18.71) and (18.70), we get

$$\dot{\beta} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi - \theta)}{\sin(\phi - \beta)} \dot{\theta}, \qquad \dot{\phi} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta - \theta)}{\sin(\beta - \phi)} \dot{\theta}.$$

$$\boxed{\vec{\omega}_{BC} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi-\theta)}{\sin(\phi-\beta)} \dot{\theta}\, \hat{\boldsymbol{k}}, \qquad \vec{\omega}_{CD} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta-\theta)}{\sin(\beta-\phi)} \dot{\theta}\, \hat{\boldsymbol{k}}}$$

Note that the expressions for $\dot{\beta}$ and $\dot{\phi}$ are coupled, nonlinear, first order ordinary differential equations. To be able to find $\dot{\theta}, \beta(t)$ and $\phi(t)$, we need to integrate $\dot{\theta}, \dot{\beta}$, and $\dot{\phi}$. Here, we set up these differential equations for numerical integration. Although, we can use any given $\dot{\theta}(t)$ (*e.g.,* αt or $\dot{\theta}_0 \sin(\Omega t)$ or whatever), for definiteness in our numerical integration, let us take a constant $\dot{\theta}$, that is, let $\dot{\theta} = C = 10\,\text{rad/s}$ (say). So, our equations are,

$$\dot{\theta} = C, \qquad \dot{\beta} = \frac{\ell_1}{\ell_2}\frac{\sin(\phi-\theta)}{\sin(\beta-\phi)}C, \qquad \dot{\phi} = \frac{\ell_1}{\ell_3}\frac{\sin(\beta-\theta)}{\sin(\beta-\phi)}C,$$

and the initial conditions are $\theta(0) = \pi/2, \quad \beta(0) = \pi/4, \quad \phi(0) = 3\pi/4$.

Here is a pseudocode that we use to integrate these equations numerically for a period of $2\pi/C = \pi/5$ seconds (one complete revolution of AB).

```
ODEs = {thetadot = C,
        betadot  = (l1/l2)*sin(phi-theta)/sin(beta-phi)*C,
        phidot   = (l1/l3)*sin(beta-theta)/sin(beta-phi)*C}
IC   = {theta(0) = pi/2, beta(0) = pi/4, phi(0) = 3*pi/4}
Set  C=10, l1=.4, l2=.4*sqrt(2), l3=.8*sqrt(2)
Solve ODEs with IC for t=0 to t=pi/5
```

After we get the angles, we can compute the xy coordinates of points B and C at each instant and plot the mechanism at those instants. Plots thus obtained from our numerical solution are shown in fig. 18.66 where the configuration of the mechanism is shown at 9 equally spaced times between $t = 0$ to $t = \pi/5$.

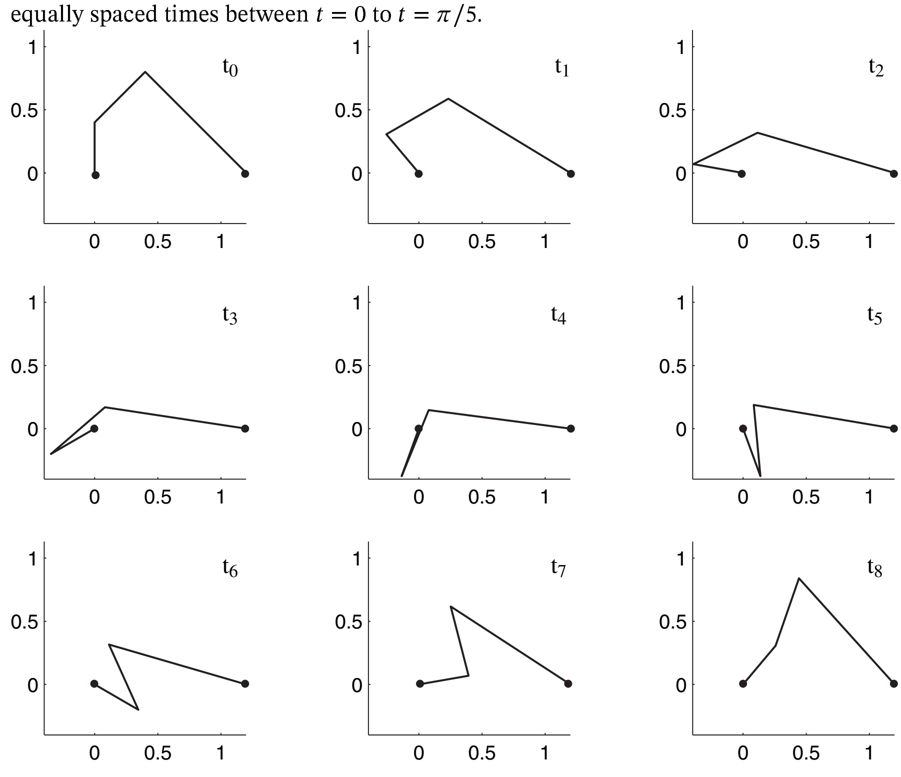


Figure 18.66: Several configurations of the mechanism at equal intervals of time during one complete revolution of the driving link. After t_8 the mechanism returns to the initial configuration.

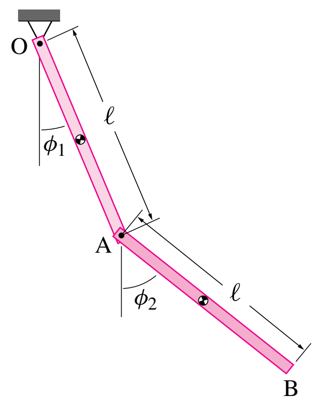
18.5 Advanced kinematics of planar motion

18.5.1 Double pendulum. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. At the instant shown, ϕ_1, $\dot{\phi}_1$, ϕ_2, and $\dot{\phi}_2$ are known. For the instant shown, answer the following questions in terms of ℓ, ϕ_1, $\dot{\phi}_1$, ϕ_2, and $\dot{\phi}_2$.

a) What is the absolute velocity of point A?

b) What is the velocity of point B relative to point A?

c) What is the absolute velocity of point B?



Problem 18.5.1

18.5.2 Double pendulum, Again. For the double pendulum in problem 18.5.1, $\ddot{\phi}_1$ and $\ddot{\phi}_2$ are also known at the instant shown. For the instant shown, answer the following questions in terms of ℓ, ϕ_1, $\dot{\phi}_1$, $\ddot{\phi}_1$, ϕ_2, $\dot{\phi}_2$, and $\ddot{\phi}_2$.

a) What is the absolute acceleration of point A?

b) What is the acceleration of point B relative to point A?

c) What is the absolute acceleration of point B?