SAMPLE 18.15 **Kinematics of a four bar linkage.** A four bar linkage ABCD is shown in the figure (fourth bar is the ground AD) at some instant t_0. The driving bar AB rotates with angular velocity $\vec{\omega}_{AB} = \dot{\theta}(t)\hat{\boldsymbol{k}}$. Find the angular velocities of rods BC and CD as a function of $\dot{\theta}$. How can you solve for the positions of the bars at any t if the initial configuration is as shown in the figure?

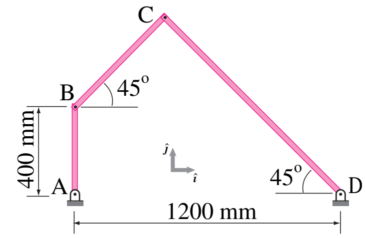

Figure 18.64

Solution Let the angles that rods AB, BC, and CD make with the horizontal (x-axis) be θ, β, and ϕ, respectively. Then, we can write $\vec{\omega}_{BC} = \dot{\beta}\hat{\boldsymbol{k}}$ and $\vec{\omega}_{CD} = \dot{\phi}\hat{\boldsymbol{k}}$. We have to find $\dot{\beta}$ and $\dot{\phi}$.

Note that the motion of point B is a simple circular motion about point A with given angular velocity $\vec{\omega}_{AB}$. Thus, the velocity of point B is known. Now, we can find the velocity of point C two ways: (i) by considering rod CB: $\vec{v}_C = \vec{v}_B + \vec{v}_{C/B} = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$, and (ii) by considering rod CD: $\vec{v}_C = \vec{\omega}_{CD} \times \vec{r}_{C/D}$. Either way the velocity must be the same. Thus, we have a 2-D vector equation with two unknowns $\dot{\beta}$ and $\dot{\phi}$. We can get two independent scalar equations from the vector equation and thus we can solve for the desired unknowns.

Let us use the rotating base vectors $(\hat{\boldsymbol{\lambda}}_1, \hat{\boldsymbol{n}}_1), (\hat{\boldsymbol{\lambda}}_2, \hat{\boldsymbol{n}}_2)$, and $(\hat{\boldsymbol{\lambda}}_3, \hat{\boldsymbol{n}}_3)$ with rods AB, BC, and CD, respectively. Note that these base vectors are basically the $(\hat{\boldsymbol{e}}_R, \hat{\boldsymbol{e}}_\theta)$ pairs; we use $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}})$ just for the sake of easy subscripting. Now,

$$
\begin{aligned}
\vec{v}_B &= \vec{\omega}_{AB} \times \vec{r}_{B/A} = \ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 \\
\vec{v}_C &= \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = \ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 + \ell_2 \dot{\beta} \hat{\boldsymbol{n}}_2 \qquad (18.67) \\
\text{also,} \quad \vec{v}_C &= \vec{\omega}_{CD} \times \vec{r}_{C/D} = \ell_3 \dot{\phi} \hat{\boldsymbol{n}}_3 \qquad (18.68)
\end{aligned}
$$

Thus, from eqn. (18.67) and (18.68), we have,

$$
\ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 + \ell_2 \dot{\beta} \hat{\boldsymbol{n}}_2 = \ell_3 \dot{\phi} \hat{\boldsymbol{n}}_3 \qquad (18.69)
$$

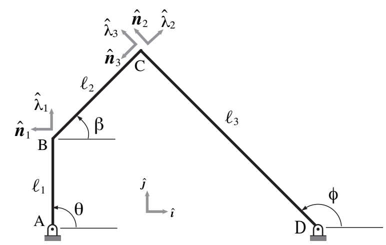

Figure 18.65

Dotting eqn. (18.69) with $\hat{\boldsymbol{\lambda}}_2$ (to eliminate $\dot{\beta}$ term), we get

$$
\begin{aligned}
\ell_1 \dot{\theta} (\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2) &= \ell_3 \dot{\phi} (\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2) \\
\Rightarrow \quad \dot{\phi} &= \frac{\ell_1}{\ell_3} \frac{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2)}{(\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2)} \dot{\theta}. \qquad (18.70)
\end{aligned}
$$

Similarly, dotting eqn. (18.69) with $\hat{\boldsymbol{\lambda}}_3$ (to eliminate $\dot{\phi}$ term), we get

$$
\dot{\beta} = -\frac{\ell_1}{\ell_2} \frac{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_3)}{(\hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{\lambda}}_3)} \dot{\theta}. \qquad (18.71)
$$

We are practically done at this point with the kinematics — we have found $\dot{\beta}$ and $\dot{\phi}$ as functions of $\dot{\theta}$. The various dot products are just geometry and vector algebra. To write them explicitly, we note that $\hat{\boldsymbol{\lambda}}_1 = \cos\theta \hat{\boldsymbol{\imath}} + \sin\theta \hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{n}}_1 = -\sin\theta \hat{\boldsymbol{\imath}} + \cos\theta \hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{\lambda}}_2 = \cos\beta \hat{\boldsymbol{\imath}} + \sin\beta \hat{\boldsymbol{\jmath}}$, etc. Thus,

$$
\begin{aligned}
\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2 &= -\sin\theta\cos\beta + \cos\theta\sin\beta = \sin(\beta - \theta) \\
\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2 &= \sin(\beta - \phi) \\
\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_3 &= \sin(\phi - \theta), \quad \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{\lambda}}_3 = \sin(\phi - \beta).
\end{aligned}
$$

Substituting the appropriate expressions in eqn. (18.71) and (18.70), we get

$$
\dot{\beta} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi - \theta)}{\sin(\phi - \beta)} \dot{\theta}, \qquad \dot{\phi} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta - \theta)}{\sin(\beta - \phi)} \dot{\theta}.
$$

$$
\boxed{\vec{\omega}_{BC} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi-\theta)}{\sin(\phi-\beta)} \dot{\theta}\, \hat{\boldsymbol{k}}, \qquad \vec{\omega}_{CD} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta-\theta)}{\sin(\beta-\phi)} \dot{\theta}\, \hat{\boldsymbol{k}}}
$$

Note that the expressions for $\dot{\beta}$ and $\dot{\phi}$ are coupled, nonlinear, first order ordinary differential equations. To be able to find $\dot{\theta}, \beta(t)$ and $\phi(t)$, we need to integrate $\dot{\theta}, \dot{\beta}$, and $\dot{\phi}$. Here, we set up these differential equations for numerical integration. Although, we can use any given $\dot{\theta}(t)$ (*e.g.,* αt or $\dot{\theta}_0 \sin(\Omega t)$ or whatever), for definiteness in our numerical integration, let us take a constant $\dot{\theta}$, that is, let $\dot{\theta} = C = 10\,\mathrm{rad/s}$ (say). So, our equations are,

$$\dot{\theta} = C, \qquad \dot{\beta} = \frac{\ell_1}{\ell_2}\frac{\sin(\phi-\theta)}{\sin(\beta-\phi)}C, \qquad \dot{\phi} = \frac{\ell_1}{\ell_3}\frac{\sin(\beta-\theta)}{\sin(\beta-\phi)}C,$$

and the initial conditions are $\theta(0) = \pi/2, \quad \beta(0) = \pi/4, \quad \phi(0) = 3\pi/4$.

Here is a pseudocode that we use to integrate these equations numerically for a period of $2\pi/C = \pi/5$ seconds (one complete revolution of AB).

```
ODEs = {thetadot = C,
        betadot  = (l1/l2)*sin(phi-theta)/sin(beta-phi)*C,
        phidot   = (l1/l3)*sin(beta-theta)/sin(beta-phi)*C}
IC   = {theta(0) = pi/2, beta(0) = pi/4, phi(0) = 3*pi/4}
Set  C=10, l1=.4, l2=.4*sqrt(2), l3=.8*sqrt(2)
Solve ODEs with IC for t=0 to t=pi/5
```

After we get the angles, we can compute the xy coordinates of points B and C at each instant and plot the mechanism at those instants. Plots thus obtained from our numerical solution are shown in fig. 18.66 where the configuration of the mechanism is shown at 9 equally spaced times between $t = 0$ to $t = \pi/5$.

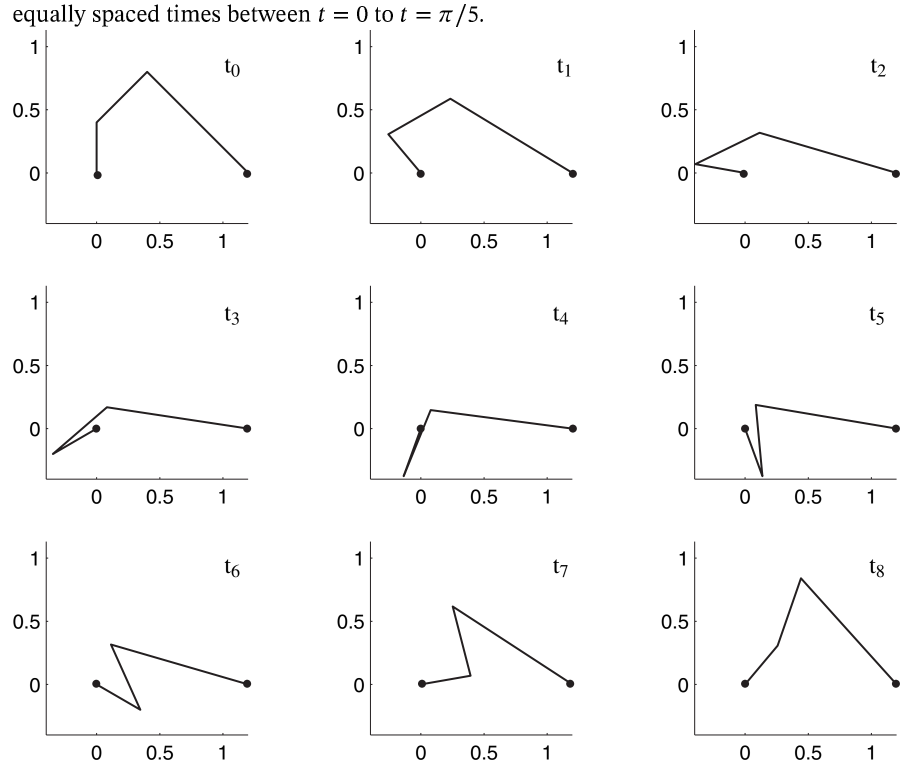

Figure 18.66: Several configurations of the mechanism at equal intervals of time during one complete revolution of the driving link. After t_8 the mechanism returns to the initial configuration.