18.4 Kinematics of 2-D mechanisms

An *ideal mechanism* or *linkage* is a collection of rigid objects constrained to move relative to the ground and each other by hinges but which still has some possible motion(s). People also use the word mechanism or linkage more loosely to include any collection of machine parts connected by any means.

The analysis of the kinematics of mechanisms is an important part of machine design. Mechanisms *synthesis*, coming up with a mechanism design which has desired motions, is obviously key in creative design, and nowadays in computer aided design.

Finally, the determination of the dynamics of a mechanism, how it will move and with what forces, is completely dependent on understanding the kinematics of the mechanism. The whole subject of mechanism kinematic analysis, although in some sense a subset of dynamics, is actually a huge and infinitely complex subject in itself[1] and also a useful subject in itself. Often kinematics is the central interest in machine design, and mechanics (force and acceleration) analysis is only carried out if something that shouldn't do so, breaks or shakes. This section presents some of the basic ideas in kinematic analysis. The overall question in mechanism kinematic analysis is this:

> Given a collection of parts and a description of how they are connected, in what ways can they move?

Without getting into the details of the motions yet, the first question to answer is simpler than finding the motions, but just counting them: In how many ways can the mechanism move?

[1] **Very advanced aside.** The general kinematics of planar linkages is as complicated as the classification of closed surfaces in any number of dimensions: spheres, donuts, spheres with two holes, etc. For every surface there is a corresponding 2D mechanism. In impenetrable math-speak: Any orientable manifold is a connected component of the configuration space of some planar linkage. This idea was proposed by Bill Thurston in the 1970's and later proved by Kapovich and Millson. We report this just so you don't feel bad that you don't know now, and never will know, all there is to know about planar mechanisms.

Degrees of freedom (DOF)

The number of degrees of freedom (DOF) n_{DOF} of a mechanism is the number of different ways it can move. More precisely

> The number of degrees of freedom n_{DOF} of a mechanism is the minimum number of configuration variables needed to describe all possible configurations of the mechanism.

The minimum number of configuration variables n_{DOF} is a property of the mechanism. The choice of what these variables are, however is not unique to a given mechanism.

Example: A particle in a plane has 2 degrees of freedom.
The set of 'configurations' of a particle in a plane is the set of positions of the particle. This is fully described by its x and y coordinates. Thus $n_{\text{DOF}} = 2$. But the configuration is also determined by the particle's polar coordinates R and θ. And there are an infinite number of other pairs of numbers that could

be used to describe the configurations (*e.g.*, the x' and y' coordinates, the w and z coordinates with $w = e^x$ and $z = e^y$, etc). The minimum number of configuration variables, 2, is unique, but the choice of variables is not.

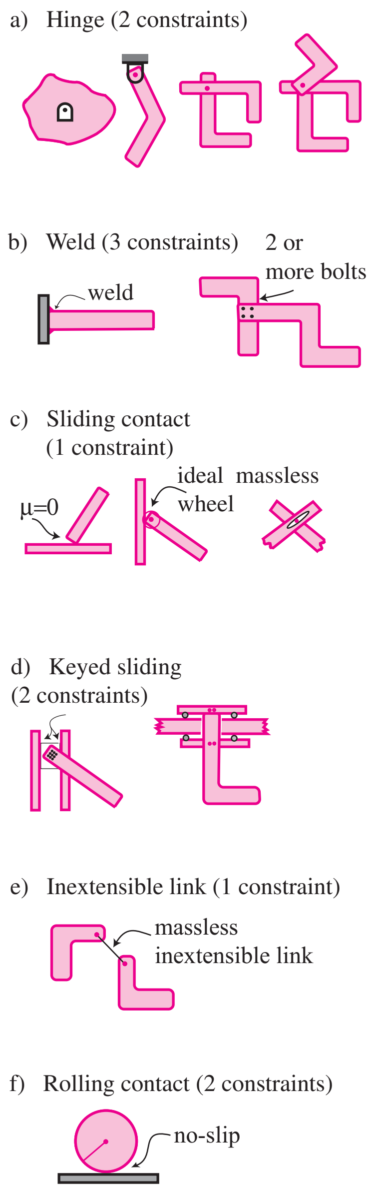

Figure 18.50: The number of degrees of freedom that are eliminated by various constraints. See text for explanation.

For planar mechanisms one can often determine the number of degrees of freedom by the following formula

$$3\cdot\underbrace{(\text{\# of rigid objects})}_{n_{\text{bod}}}+2\cdot\underbrace{(\text{\# of particles})}_{n_{\text{part}}}-\underbrace{(\text{\# of constraints})}_{n_{\text{con}}} = n_{\text{DOF}} \qquad (18.36)$$

The formula starts with the number of ways one rigid object can move (2 translations and a rotation makes 3) and one particle can move (just 2 translations) and then subtracts the restrictions to the motion. In eqn. (18.36) the number n_{con} of constraints is counted as the number of degrees of freedom restricted by the connections.

Examples of connections and their effect on n_{DOF}

See fig. 18.50 for some standard idealized connections and their number of constraints (assuming they are already constrained to a plane).

a) **2** for a **pin joint**: a pin joint restricts relative motion in 2 directions but still allows relative rotation. If three objects are connected at one pin then it counts as two pin joints and thus $2 \times 2 = 4$ reductions in the number of degrees of freedom. There are 6 reductions for 4 objects connected at one pin, etc.

b) **3** for a **welded connection**: a weld restricts relative translation in two directions as well as relative rotation $(2 + 1 = 3)$. So two parts that are welded together have $2 \cdot 3 - 3 = 3$ degrees of freedom. That is, any collection of rigid objects welded together is the same as one rigid object. The word ‘weld’ is meant to include any collection of bolts, glue, string, rivets or bailing wire that prevents any relative motion.

c) **1** for a **sliding contact**: the sliding contact restricts relative translation normal to the contact surfaces and allows translation tangent to the surfaces. Relative rotation is also allowed.

d) **2** for a **keyed sliding contact**: allows relative translation in one direction but disallows translation in one direction as well as rotation.

e) **1** for a **massless link hinged at its ends to two objects**: this keeps the distance between two points fixed which is one restriction (alternatively the bar adds 3 degrees of freedom and each hinge subtracts 2 $(+3 - 2 \times 2 = -1$ degree of freedom).

f) **2** for a **rolling contact**: relative slip is not allowed nor is interpenetration.

Be warned

> If some of the constraints are redundant then a system can have more degrees of freedom than eqn. (18.36) indicates. But a mechanism never has fewer degrees of freedom than eqn. (18.36) indicates.

Some simple mechanisms

Figure 18.51 shows some examples of simple mechanisms and the number of degrees of freedom. In each case we look at eqn. (18.36): $3 \cdot n_{\text{bod}} + 2 \cdot n_{\text{part}} - n_{\text{con}} = n_{\text{DOF}}$.

a) An **object connected to ground by a hinge** has **1** degree of freedom; the set of all possible configurations can be described by the angle of the object: $n_{\text{bod}} = 1, n_{\text{con}} = 2 \quad \Rightarrow \quad n_{\text{DOF}} = 1$.

b) An **unconstrained object** has **3** degrees of freedom; the set of all configurations can be described by the x and y coordinates of a reference point and by the rotation θ: $n_{\text{bod}} = 1, n_{\text{con}} = 0 \quad \Rightarrow \quad n_{\text{DOF}} = 3$.

c) A **bead on a wire** has **1** degree of freedom; its configuration is fully determined by the distance the bead has advanced along the wire relative to a reference mark: $n_{\text{part}} = 1, n_{\text{con}} = 1 \quad \Rightarrow \quad n_{\text{DOF}} = 1$.

d) A **statically determinate truss** has **0** degrees of freedom; a statically determinate truss has no way to deform: $n_{\text{bod}} = n_{\text{bars}}, n_{\text{part}} = 0, n_{\text{con}} = 2 \cdot n_{\text{pins}} + n_{\text{ground const}} \quad \Rightarrow n_{\text{DOF}} = 0$. Note that the number of pin restrictions in our count here is more than twice the number of joints in the study of trusses in statics. In statics, we focussed on joints as restricted by bars. Here we look at bars as restricted by joints. A given joint counts 2, 4 or 6 pin restrictions depending on whether it connects 2, 3 or 4 bars. Thus the 11 bar truss shown has 7 joints (by truss-analysis counting) but, by the counting here it has $2 \times 2 + 2 \times 4 + 3 \times 6 = 30$ pin restrictions plus 3 ground restrictions for 33 restrictions in all. Without restrictions the 11 bars had 33 degrees of freedom. So, because 33-33 = 0, and none of the restrictions is redundant, the truss has zero degrees of freedom.

e) A **rolling wheel** has **1** degree of freedom; its configuration is fully determined either by the net angle θ it has rolled or by the x coordinate of its center: $n_{\text{bod}} = 1, n_{\text{con}} = 2 \quad \Rightarrow \quad n_{\text{DOF}} = 1$.

f) A **double pendulum** or two-link robot arm has **2** degrees of freedom; its configuration is determined by the net rotations of its two links (or by the rotation of the first link and the relative rotation of the second link): $n_{\text{bod}} = 2, n_{\text{con}} = 4 \quad \Rightarrow \quad n_{\text{DOF}} = 2$.

g) A **cart with two rolling wheels** has **1** degree of freedom; given the position of the cart, say x, the rotation of the wheels is determined. Similarly for a **bicycle**. For this model of a bicycle we are

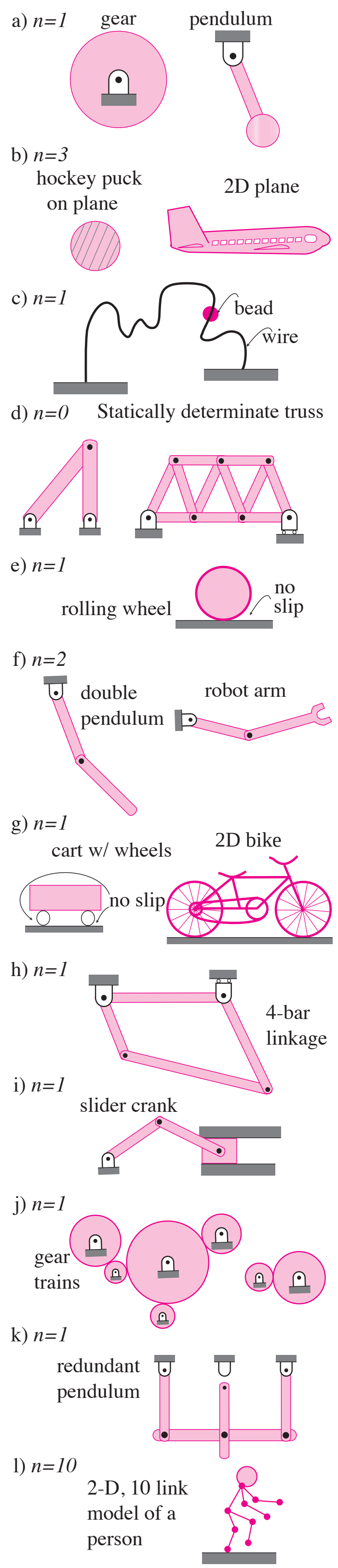

Figure 18.51: Some simple mechanisms and their number of degrees of freedom, n (called n_{DOF} in the text).

neglecting the steering degree of freedom and also the freedoms of the crank and pedals. For a vehicle with 2 wheels in 2D we have: $n_{\text{bod}} = 3, n_{\text{con}} = 4 \times 2 = 8$ (2 hinges and 2 rolling contacts) $\Rightarrow \quad n_{\text{DOF}} = 9 - 8 = 1$.

h) A **“four” bar linkage** has **1** degree of freedom; the angle of any one of the bars determines the angles of the others: $n_{\text{bod}} = 4, n_{\text{con}} = 2{\times}4{+}2{+}1 = 11$ (there are 4 pin joints between the bars, one pin joint to ground and one roller connection to the ground) $\Rightarrow \quad n_{\text{DOF}} = 1$.

i) A **slider crank** has **1** degree of freedom; the rotation of the crank determines the configuration of the system $n_{\text{bod}} = 3, n_{\text{con}} = 3\cdot 2 + 2$ (there are 3 pins and one keyed connection) $\Rightarrow \quad n_{\text{DOF}} = 1$.

j) An ideal **gear train** (with all gears pinned to ground) has **1** degree of freedom; the amount of rotation of any one gear determines the rotation of all of the gears: In this case the counting formula is *wrong*. Say there are 2 gears, then $n_{\text{bod}} = 2, n_{\text{con}} = 3 \cdot 2 = 6$ (two pins and one rolling contact) $\Rightarrow \quad n_{\text{DOF}} = 0 \neq 1$. The rolling constraint prevents interpenetration, but this was already prevented by the hinges at the center of the gears. The constraints are redundant and the system has more degrees of freedom than eqn. (18.36) indicates.

k) A **redundant swing** with one horizontal bar suspended by 3 parallel struts has **1** degree of freedom; the angle of one upright link determines the full configuration of the mechanism. The counting formula is again wrong: $n_{\text{bod}} = 4, n_{\text{con}} = 6\cdot 2 = 12 \quad \Rightarrow \quad n_{\text{DOF}} = 0$ underestimates the number of degrees of freedom because the constraints are redundant.

l) A **2-D 10-link model of a person** with one foot on the ground has **10** degrees of freedom; the angles of the 10 links (4 arm links, 4 leg links, a body and a head) determine the full configuration of the mechanism. There are no redundant constraints so the counting formula also works using $n_{\text{bod}} = 10, n_{\text{hinges}} = 10$:

$$n_{\text{DOF}} = 3 * n_{\text{bod}} - n_{\text{con}} \tag{18.37}$$
$$= 3 * n_{\text{bod}} - 2 \cdot n_{\text{hinges}} = 30 - 2 \cdot 10 = 10 \tag{18.38}$$

What are the 10 hinges? One at the ground, 1 at each knee and elbow, 2 at the hip and 3 at the shoulders (one hinge for each object linked to the trunk).

Configuration variables

Once we know the number of degrees of freedom n_{DOF} of a system it is often useful to settle on one set of n_{DOF} configuration variables. In this book n_{DOF} will be 1, 2 or at most 3. Thus we pick 1,2 or 3 variables.

Example: Straight line motion
Chapter 14 on straight line motion was mostly about one-degree-of-freedom systems ($n_{\text{DOF}} = 1$). These systems could all be characterized by the single configuration variable x, the displacement along the line of a reference point on the

object relative to a reference point on the ground. All the positions,velocities and accelerations of all points in the system could be found in terms of x, $\dot{x}$ and $\ddot{x}$ (in fact all points had $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}}$).

Example: Circular motion about a fixed axis
In chapters 15 and 16 we were almost entirely focussed on systems with one degree of freedom well characterized by the one configuration variable, the rotation angle θ. For such motions positions, velocities, and accelerations of all points were determined by the initial positions of the points $\theta, \dot{\theta}$ and $\ddot{\theta}$ by equations which you know well by now.

For more general motions we almost always take inspiration from the two examples above. We use the translation of a conspicuous point, or we use the rotation of a conspicuous object for a configuration variable. And more of the same if the system has more than 1 degree of freedom. The natural choice of configuration variables for some simple mechanisms is given in the text discussing fig. 18.51.

Often our main kinematic task is to express the full configuration of the system as well as all the velocities and accelerations of all its parts in terms of the positions of the parts, the configuration variables, and their first and second time derivatives.

Adding relative angular velocities

One last simple kinematic fact is needed before we can plug and chug with the theory we have so far and apply it to general kinematic mechanisms. It concerns the addition of rotations and rotation rates. The following example basically tells the whole story

Example: Double pendulum and the addition of rotation rates
The commonly used configuration variables for the double pendulum shown in fig. 18.52 are θ_1 and θ_2. To actually know the configuration of the system obviously we need to know ϕ which is given by

$$\phi = \theta_1 + \theta_2 \quad \text{so} \quad \dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 \quad \text{and so} \quad \ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2.$$

[Aside: One reason for choosing θ_2 instead of ϕ as a configuration variable is that if one was measuring or controlling the second link, say as a robotic arm, the angle θ_2 can be measured more easily than ϕ. Also, it turns out (in hindsight) that the differential equations of motion are slightly simpler using θ_2 instead of ϕ.]

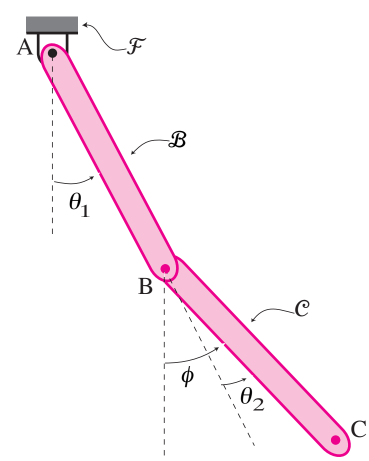

Figure 18.52: A double pendulum's configuration variables are most often taken to be θ_1 and θ_2. But one then needs to know that $\phi = \theta_1 + \theta_2$ and that $\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2$ and $\ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2$.

Looking at the bars as being glued to reference frames ($\mathcal{F}$ for the fixed frame, $\mathcal{B}$ for bar AB, and $\mathcal{C}$ for bar BC), the above example shows that

$$\vec{\omega}_{\mathcal{C}/\mathcal{F}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}} \tag{18.39}$$

which is often written with the simple notation

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

Which can only be given strict meaning by the more elaborate eqn. (18.39) above it.

Example: Double pendulum (see previous example)
Take $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} = \dot{\theta}_1\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} = \dot{\theta}_2\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} = \dot{\phi}\hat{\boldsymbol{k}}$ and eqn. (18.39) is self evident from the addition of angles.

Similarly we have

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{F}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} \qquad (18.40)$$

which is often written with the simple notation

$$\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{\alpha}}_1 + \vec{\boldsymbol{\alpha}}_2.$$

For those going on to study 3-D mechanics, one should note that, unlike eqn. (18.39), eqn. (18.40) does *not* hold in 3-D.

Kinematics of mechanisms

One approach to mechanisms is to do what one can with high-school geometry and trigonometry, the laws of sines (see page 110), and so on.

Example: Rod on step using geometry and trigonometry
One end of a rod slides on the ground. The other end slides on a corner at A (see fig. 18.53). Given that $\vec{\boldsymbol{v}}_{\mathrm{B}} = v_B\hat{\boldsymbol{\imath}}$ we can find $\dot{\phi}$ as follows:

$$\frac{h}{\ell_{OB}} = \tan\phi \Rightarrow \left\{ \frac{\ell_{OB}}{h} = \frac{\cos\phi}{\sin\phi} \right\}$$

$$\frac{d}{dt}\left\{ \quad \right\} \Rightarrow \frac{\dot{\ell}_{OB}}{h} = \frac{-\dot{\phi}}{\sin^2\phi} \Rightarrow \dot{\phi} = -\frac{v_B\sin^2\phi}{h}$$

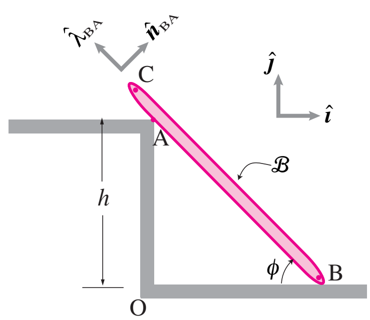

Figure 18.53: A rigid rod slides so that it always touches the ground and the corner at A. One would like to know the relation between $\dot{\phi}$ and $\vec{\boldsymbol{v}}_{\mathrm{B}}$.

As the above calculation shows, this problem doesn't need the heavy machinery of our moving-frame vector methods. But it provides an instructive example.

Example: Rod on step using moving-frame methods (see previous example)
We look at point A and note that we can think of it as a fixed point in the fixed frame $\mathcal{F}$ and also as a point that is moving relative to the translating and rotating frame $\mathcal{B}$. We evaluate its velocity both ways.

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{A}} &= \vec{\boldsymbol{v}}_{\mathrm{A}} \\
\vec{\boldsymbol{0}} &= \vec{\boldsymbol{v}}_{\mathrm{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{A/B}} + \vec{\boldsymbol{v}}_{\mathrm{A}/\mathcal{B}} && (\text{eqn. } (18.28)) \\
\{\vec{\boldsymbol{0}} &= v_B\hat{\boldsymbol{\imath}} - \dot{\phi}\hat{\boldsymbol{k}} \times (\ell_{BA}\hat{\boldsymbol{\lambda}}_{BA}) + v_{A/\mathcal{B}}\hat{\boldsymbol{\lambda}}_{BA}\} \\
\Rightarrow 0 &= v_B \underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}}_{BA}}_{\sin\phi} + \dot{\phi}\ell_{BA} && (\{\} \cdot \hat{\boldsymbol{n}}_{BA}) \\
\Rightarrow \dot{\phi} &= -\frac{v_B\sin\phi}{\ell_{BA}} = -\frac{v_B\sin^2\phi}{h} && (\sin\phi = \frac{h}{\ell_{AB}})
\end{aligned}$$

as we had before. The key equation was the 'three term velocity formula' eqn. (18.28) on page 959 and the observation that relative to frame $\mathcal{B}$ point A slides along the rod. Note that we never had to explicitly use the rotating coordinates associated with frame $\mathcal{B}$ to do this calculation.

You should understand the examples above, and the needed background material, before going on to the following examples.

Example: Slider Crank using geometry and trigonometry

The slider crank mechanism (fig. 18.54) was briefly introduced in the context of statics where its forces could be analyzed assuming inertial terms were negligible (see 418). But it is a commonly used mechanism (*e.g.,* in every car) and its motions are of central interest. The angle θ is the most natural configuration variable for this $n_{\text{DOF}} = 1$ system. One would like to know the position, velocity and acceleration of the slider ($x_C, \dot{x}_C$ and $\ddot{x}_C$) in terms of $\theta, \dot{\theta}$ and $\ddot{\theta}$.

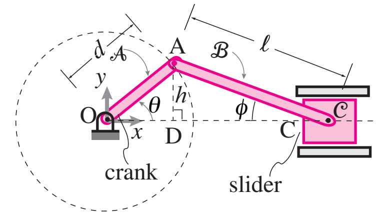

Figure 18.54: The slider crank mechanism.

$$
\begin{aligned}
x_C &= x_D + \ell_{DC} \\
&= d\cos\theta + \sqrt{\ell^2 - h^2} \\
&= d\cos\theta + \sqrt{\ell^2 - (d\sin\theta)^2} \\
&= d\left(\cos\theta + \sqrt{(\ell/d)^2 - \sin^2\theta}\right) \\
&= d\left(\cos\theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2}\right) \qquad (18.41)
\end{aligned}
$$

The positive $\sqrt{\ }$ corresponds to C being to the right of 0. The negative $\sqrt{\ }$ corresponds to point C being to the left of 0. The mechanism just doesn't work for a full revolution of the link 0A if $\ell < d$ as you can see from the picture or from that the $\sqrt{\ }$ above giving imaginary values for $\cos 2\theta$ near -1, $\sin\theta$ near 1, and θ near $\pi/2$.

To get the velocity of point C we just take the derivative of eqn. (18.41) above.

$$
\begin{aligned}
v_C &= \dot{x}_C \qquad (18.42) \\
&= \frac{d}{dt}\left\{d\left(\cos\theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2}\right)\right\} \\
&= \dot{\theta} d\left\{-\sin(\theta) - \sin(2\,\theta)\frac{1}{\sqrt{4\,\frac{l^2}{d^2} + 2\,\cos(2\,\theta) - 2}}\right\}
\end{aligned}
$$

To get the acceleration we differentiate once again. For simplicity let's assume the crank rotates at constant rate, so $\dot{\theta}$ is a constant and $\ddot{\theta} = 0$. Cranking out the derivative of eqn. (18.42), so to speak, we get

$$
\begin{aligned}
a_c = \dot{v}_C &= \frac{d}{dt}\{\text{the mess on the right of eqn. (18.42)}\} \qquad (18.43) \\
&= -d\dot{\theta}^2 \Bigg\{ \cos(\theta) \\
&\qquad + 2\,\sin^2(2\,\theta)\left(4\,\frac{l^2}{d^2} + 2\,\cos(2\,\theta) - 2\right)^{-3/2} \\
&\qquad + 2\,\cos(2\,\theta)\frac{1}{\sqrt{4\,\frac{l^2}{d^2} + 2\,\cos(2\,\theta) - 2}} \Bigg\}
\end{aligned}
$$

So we now know the position, velocity and acceleration of point C in terms of $\theta, \dot{\theta}$ and $\ddot{\theta}$. You should commit the solution eqn. (18.43) to memory. Just kidding.

Plots of x_C, v_C, and a_C from these equations are shown in fig. 18.55ab for two different extremes of slider crank design: one with a very long connecting rod that gives sinusoidal motion, and one with a connecting rod just barely long enough to prevent locking that gives intermittent motion.

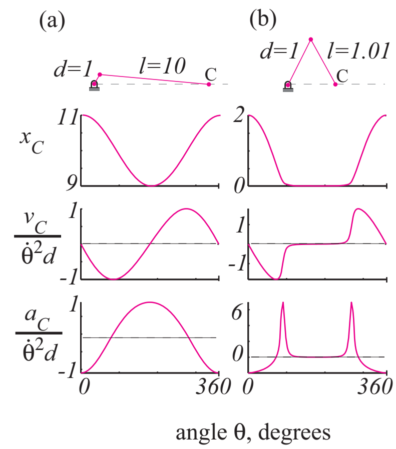

Figure 18.55: Position x_C of the slider, its velocity v_C and its acceleration a_c (Strictly, *e.g.,* $\vec{a}_C = a_C\hat{\imath}$). Two sets of curves are shown. **(a)** a long connecting rod, and **(b)** a connecting rod a hair longer than the crank. In case (a) the connecting rod is nearly horizontal at all times and the displacement of point C is almost entirely due to the horizontal displacement of the end of the crank arm. Thus point C moves with almost exactly a cosine wave with amplitude equal to the length of the crank. The velocity and acceleration curves are thus also sine and cosine waves. In case (b) the motion is close to a cosine curve approximately when $-60^\circ < \theta < 60^\circ$. That is, the displacement of C is about twice the horizontal displacement of the end of the crank arm when the end is to the right of its base bearing. When the end of the crank arm is to the left of the bearing point C is nearly stationary and is just to the right of the crank base bearing. The transition between these two cases involves a sudden large acceleration.

Unlike some more complex mechanisms, the slider crank is solvable in that one can write a formula for the position of any point of interest in terms of the single configuration variable θ. For more complex mechanisms this may not be possible. Further, even if possible the above example shows that the differentiation required to find velocity and acceleration can lead to a bit of a mess.

A different approach is to assume that at some value of the configuration variable (θ for the slider crank) that the full configuration of the system is known. That is, that the locations of all points are known. Then we can use our vector methods to find velocities and accelerations of all points of interest.

Example: Slider crank using vector methods (see previous example)
Take the slider crank of fig. 18.54 to be in some known configuration. We now try to find the velocity and acceleration of point C in terms of the positions of the points 0, B, and C as well as θ and $\dot{\theta}$.

The basic approach is to write true things, and then solve for unknowns. First work on velocities. The basic idea is to look at the *closure* condition. That is, the velocity of point C as calculated by working down the linkage from 0 to A to C has to be consistent with the velocity of C as calculated in the fixed frame.

$$\begin{aligned} \vec{\boldsymbol{v}}_\mathrm{C} &= \vec{\boldsymbol{v}}_\mathrm{C} \\ v_C\hat{\boldsymbol{\imath}} &= \vec{\boldsymbol{v}}_{\mathrm{A}/0} + \vec{\boldsymbol{v}}_{\mathrm{C/A}} \\ v_C\hat{\boldsymbol{\imath}} &= (\dot{\theta}\hat{\boldsymbol{k}})\times\vec{\boldsymbol{r}}_{\mathrm{A}/0} + (-\dot{\phi}\hat{\boldsymbol{k}})\times\vec{\boldsymbol{r}}_{\mathrm{C/A}} \end{aligned} \tag{18.44}$$

eqn. (18.44) is a 2-D vector equation in the 2 unknown scalars v_C and $\dot{\phi}$. It could be solved as a pair of equations, or solved directly by first dotting both sides with $\hat{\boldsymbol{\jmath}}$ to find $\dot{\phi}$ and dotting both sides with $\vec{\boldsymbol{r}}_{\mathrm{C/A}}$ to find v_C. These yield

$$\begin{aligned} \dot{\phi} &= \frac{\left((\dot{\theta}\hat{\boldsymbol{k}})\times\vec{\boldsymbol{r}}_{\mathrm{A}/0}\right)\cdot\hat{\boldsymbol{\jmath}}}{\left(\hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{C/A}}\right)\cdot\hat{\boldsymbol{\jmath}}} \quad \text{and} \\ v_C &= \frac{\dot{\theta}\left(\hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{A}/0}\right)\cdot\vec{\boldsymbol{r}}_{\mathrm{C/A}}}{\hat{\boldsymbol{\imath}}\cdot\vec{\boldsymbol{r}}_{\mathrm{C/A}}} \end{aligned}$$

where everything on the right of these equations is assumed known. Without grinding out the vector products in terms, say, of components, we can just use that we know $\dot{\phi}$ and v_C.

We proceed to find the accelerations by similar means, assuming $\dot{\theta}$ is a constant so $\vec{\boldsymbol{\alpha}}_{\mathcal{A}} = \vec{\boldsymbol{0}}$:

$$\begin{aligned} \vec{\boldsymbol{a}}_\mathrm{C} &= \vec{\boldsymbol{a}}_\mathrm{C} \\ a_C\hat{\boldsymbol{\imath}} &= \vec{\boldsymbol{a}}_{\mathrm{A}/0} + \vec{\boldsymbol{a}}_{\mathrm{C/A}} \\ a_C\hat{\boldsymbol{\imath}} &= -\dot{\theta}^2\vec{\boldsymbol{r}}_{\mathrm{A}/0} + (-\ddot{\phi}\hat{\boldsymbol{k}})\times\vec{\boldsymbol{r}}_{\mathrm{C/A}} - \dot{\phi}^2\vec{\boldsymbol{r}}_{\mathrm{C/A}} \end{aligned} \tag{18.45}$$

eqn. (18.45) is a 2-D vector equation in the two scalar unknowns a_c and $\ddot{\phi}$. We can set this up as two equations in two unknowns. Or we can solve for $\ddot{\phi}$ directly by dotting both sides with $\hat{\boldsymbol{\jmath}}$ and we can solve for a_C directly by crossing both sides with $\vec{\boldsymbol{r}}_{\mathrm{C/A}}$ or by dotting with a vector perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{C/A}}$ (e.g., $\vec{\boldsymbol{r}}^{\perp}_{C/A} = \hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{C/A}}$)

Although we have presented an algorithm rather than a formula, we have found the velocity and acceleration of C without writing any large equations of the type needed in the previous example. The shortcoming is that this method depends on knowing the full configuration at the time of interest.

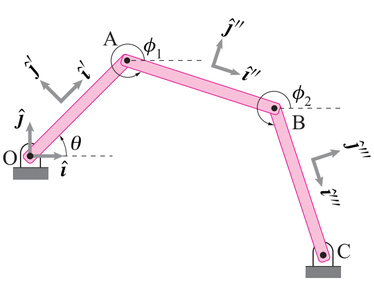

Figure 18.56: A four bar linkage.

Example: Four bar linkage using geometry and trigonometry
fig. 18.56 shows a "four-bar linkage". Please see 417 for an introduction to 4-bar linkages in the context of statics. Four bar linkages are solvable in the sense that one can write equations for the positions of any point of interest in terms of the single configuration variable θ marked in fig. 18.56. But the formulas are really a mess. And the first and second time derivatives are an unbelievable mess.

The four-bar linkage is about as complex a system as can be solved in this sense, and it is probably too-complex for this solution to be useful in the kinematic analysis of accelerations.

For complex mechanisms one is often stuck using vector methods, like we are for practical purposes stuck with the 4-bar linkage. But the vector methods based on the current configuration are not crippled by complexity.

Example: Four-bar linkage using relative velocities and accelerations
Assuming the configuration is known (*i.e.,* that θ, ϕ_1, and ϕ_2 are known), we can proceed with the 4-bar linkage just as we did for the slider crank. We enforce closure by picking a point and thinking about its velocity two different ways②
We could pick any point, say C. From the fixed frame we know that the velocity of C is zero. Working around the linkage, link by link, we know it is the sum of relative velocities as

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{C}} &= \vec{\boldsymbol{v}}_{\mathrm{C}} \\
\vec{\mathbf{0}} &= \vec{\boldsymbol{v}}_{\mathrm{A}/0} + \vec{\boldsymbol{v}}_{\mathrm{B}/\mathrm{A}} + \vec{\boldsymbol{v}}_{\mathrm{C}/\mathrm{B}} \\
&= \left(\dot{\theta}\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{A}/0} + (\dot{\phi}_1\hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}} + (\dot{\phi}_2\hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{B}}
\end{aligned}$$

②**Relative *vs* absolute angles.** In this example, and the examples above, we use the absolute rotations of the links. However, we could also have done the calculations using the relative rotations and then used the formulas for velocity and acceleration relative to a rotating frame. For these examples this would make the calculations slightly, but not intractably, longer.

which is equivalent to two scalar equations in the two unknowns $\dot{\phi}_1$ and $\dot{\phi}_2$. This equation can be solved directly for $\dot{\phi}_2$ by taking the dot product of both sides with a vector perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}$ (such as $\hat{\boldsymbol{\jmath}}''$ or $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}$) and for $\dot{\phi}_1$ by taking the dot product of both sides with for a vector perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{B}}$ (such as $\hat{\boldsymbol{\jmath}}'''$ or $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{B}}$) to get

$$\begin{aligned}
\dot{\phi}_1 &= -\dot{\theta}\frac{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{A}/0}\right) \cdot \hat{\boldsymbol{\jmath}}'''}{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}\right) \cdot \hat{\boldsymbol{\jmath}}'''} \quad \text{and} \\
\dot{\phi}_2 &= -\dot{\theta}\frac{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{A}/0}\right) \cdot \hat{\boldsymbol{\jmath}}''}{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{B}}\right) \cdot \hat{\boldsymbol{\jmath}}''}.
\end{aligned}$$

The dot product with $\hat{\boldsymbol{k}}$ is used to get a scalar on the top and bottom of the fraction, both vectors are already only in the $\hat{\boldsymbol{k}}$ direction. Now that $\dot{\phi}_1$ and $\dot{\phi}_2$ are known the velocity of any point on the mechanism is known. For example

$$\vec{\boldsymbol{v}}_{\mathrm{B}} = (\dot{\phi}_2\hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{C}}.$$

The angular accelerations of the two links are found by the same method. For simplicity let's assume that the driving crank 0A spins at constant rate so

$\ddot{\theta} = 0$. Looking at the acceleration of point C two ways we have

$$
\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{C}} &= \vec{\boldsymbol{a}}_{\mathrm{C}} \\
\vec{\mathbf{0}} &= \vec{\boldsymbol{a}}_{\mathrm{A/0}} + \vec{\boldsymbol{a}}_{\mathrm{B/A}} + \vec{\boldsymbol{a}}_{\mathrm{C/B}} \\
&= \underbrace{\left(\ddot{\theta}\hat{\boldsymbol{k}}\right)}_{0} \times \vec{\boldsymbol{r}}_{\mathrm{A/0}} - \dot{\theta}^2 \vec{\boldsymbol{r}}_{\mathrm{A/0}} \\
&\quad + (\ddot{\phi}_1 \hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{B/A}} - \dot{\phi}_1^2 \vec{\boldsymbol{r}}_{\mathrm{B/A}} \\
&\quad + (\ddot{\phi}_2 \hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{C/B}} - \dot{\phi}_2^2 \vec{\boldsymbol{r}}_{\mathrm{C/B}}
\end{aligned}
$$

Because $\dot{\phi}_1$ and $\dot{\phi}_2$ are already known, this is one equation in the two unknowns $\ddot{\phi}_1$ and $\ddot{\phi}_2$. They can be solved for $\ddot{\phi}_1$ by taking the dot product of both sides with $\hat{\boldsymbol{\jmath}}'''$ and for $\ddot{\phi}_2$ by taking the dot product of both sides with $\hat{\boldsymbol{\jmath}}''$.

At this point you know $\theta, \dot{\theta}, \phi_1, \dot{\phi}_1, \ddot{\phi}_1, \phi_2, \dot{\phi}_2$, and $\ddot{\phi}_2$ and can thus calculate the position, velocity and acceleration of any point in the mechanism.