SAMPLE 18.12 Two rods connected in a one-DOF mechanism. A mechanism consists of two rods AB and CD connected together at P with a collar pinned to AB but free to slide on CD. Rod AB is driven with $\vec{\omega}_{AB} = 10\,\mathrm{rad/s}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{AB} = 4\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}$. At the instant shown, $\theta = 45^\circ$ and $\beta = 30^\circ$. The length of rod AB is $R = 0.4\,\mathrm{m}$. At the instant shown,

1. Find the angular velocity and angular acceleration of rod CD.
2. Find the velocity and acceleration of the collar with respect to rod CD.

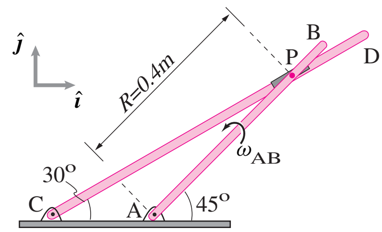

Figure 18.57

Solution Here, we are interested in instantaneous kinematics of this mechanism. Since point P is on rod AB as well as on rod CD, its velocity and acceleration can be expressed in terms of the angular motion of rod AB or that of rod CD. Let us consider rod AB first. Let $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ be basis vectors attached to rod AB that rotate with the rod. Since P is fixed on rod AB, it executes simple circular motion about A with $\vec{\omega}_{AB} = \dot{\theta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{AB} = \ddot{\theta}\hat{\boldsymbol{k}}$ where $\dot{\theta} = 10\,\mathrm{rad/s}$ and $\ddot{\theta} = 4\,\mathrm{rad/s^2}$, respectively. Then

$$\begin{aligned} \vec{\boldsymbol{v}}_{\mathrm{P}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta && (18.46)\\ \vec{\boldsymbol{a}}_{\mathrm{P}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta. && (18.47)\end{aligned}$$

Now let us consider rod CD and express the velocity and acceleration of point P in terms of motion of rod CD. Let the angular velocity and angular acceleration of rod CD be $\vec{\omega}_{CD} = \dot{\beta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{CD} = \ddot{\beta}\hat{\boldsymbol{k}}$, respectively. Let $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ be basis vectors attached to rod CD. Let the instantaneous position of point P on CD be $\vec{\boldsymbol{r}}_{\mathrm{P}} = r\hat{\boldsymbol{\lambda}}$. Since the collar can slide along CD, we can write the velocity and acceleration of point P as

$$\begin{aligned} \vec{\boldsymbol{v}}_{\mathrm{P}} &= \dot{r}\hat{\boldsymbol{\lambda}} + r\dot{\beta}\hat{\boldsymbol{n}} && (18.48)\\ \vec{\boldsymbol{a}}_{\mathrm{P}} &= (\ddot{r} - r\dot{\beta}^2)\hat{\boldsymbol{\lambda}} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{\boldsymbol{n}}. && (18.49)\end{aligned}$$

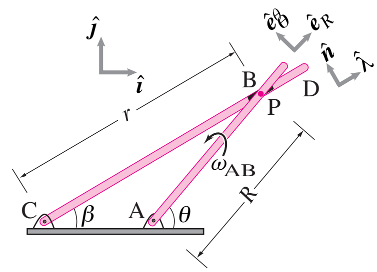

Figure 18.58: The position of point P can be written as $R\hat{\boldsymbol{e}}_R$ or $r\hat{\boldsymbol{\lambda}}$ where $r = R\sin\theta/\sin\beta$ (since $r\sin\beta = R\sin\theta$).

From eqn. (18.46) and 18.48 we get,

$$\begin{aligned} \dot{r}\hat{\boldsymbol{\lambda}} + r\dot{\beta}\hat{\boldsymbol{n}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta\\ \Rightarrow \quad \dot{r} &= R\dot{\theta}(\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}})\\ \dot{\beta} &= (1/r)R\dot{\theta}(\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{n}})\end{aligned}$$

Similarly, from eqn. (18.47) and 18.49 we get,

$$\begin{aligned} \ddot{r} - r\dot{\beta}^2 &= -R\dot{\theta}^2(\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{\lambda}}) + R\ddot{\theta}(\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}})\\ r\ddot{\beta} + 2\dot{r}\dot{\beta} &= -R\dot{\theta}^2(\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{n}}) + R\ddot{\theta}(\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{n}}).\end{aligned}$$

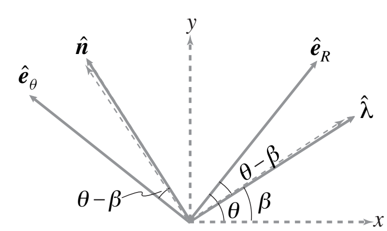

Figure 18.59: Geometry of the unit vectors.

Thus, to find all kinematic quantities of interest, all we need now is to figure out a few dot products between the two sets of basis vectors. This is easily done by writing out $\hat{\boldsymbol{e}}_R, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{n}}$. (1) Substituting the dot products in the expressions for $\dot{r}, \dot{\beta}, \ddot{r}$, and $\ddot{\beta}$ we get

$$\begin{aligned} \dot{r} &= R\dot{\theta}\sin(\beta-\theta),\\ \dot{\beta} &= r^{-1}R\dot{\theta}\cos(\theta-\beta)\\ \ddot{r} &= r\dot{\beta}^2 - R\dot{\theta}^2\cos(\beta-\theta) + R\ddot{\theta}\sin(\beta-\theta),\\ \ddot{\beta} &= r^{-1}[-R\dot{\theta}^2\sin(\theta-\beta) + R\ddot{\theta}\cos(\theta-\beta) - 2\dot{r}\dot{\beta}].\end{aligned}$$

Substituting the given values of $\dot{R}, \dot{\theta}, \ddot{\theta}, R, \theta, \beta$, and $r = R\sin\theta/\sin\beta$, we get

$$\dot{r} = -1.04\,\mathrm{m/s},\ \dot{\beta} = 6.83\,\mathrm{rad/s},\quad \ddot{r} = -12.66\,\mathrm{m/s^2},\ \ddot{\beta} = 9.43\,\mathrm{rad/s^2}.$$

(a)	$\vec{\omega}_{CD} = 6.83\,\mathrm{rad/s}\hat{\boldsymbol{k}}$,	$\dot{\vec{\omega}}_{CD} = 9.43\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}$
(b)	$\vec{\boldsymbol{v}}_{/\mathrm{CD}} = -1.04\,\mathrm{m/s}\hat{\boldsymbol{\lambda}}$,	$\vec{\boldsymbol{a}}_{/\mathrm{CD}} = -12.66\,\mathrm{m/s^2}\hat{\boldsymbol{\lambda}}$

(1) We have,
$\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$,
$\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$,
$\hat{\boldsymbol{\lambda}} = \cos\beta\hat{\boldsymbol{\imath}} + \sin\beta\hat{\boldsymbol{\jmath}}$,
$\hat{\boldsymbol{n}} = -\sin\beta\hat{\boldsymbol{\imath}} + \cos\beta\hat{\boldsymbol{\jmath}}$.
Therefore,
$\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{\lambda}} = \cos(\beta-\theta)$,
$\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{n}} = \sin(\theta-\beta)$,
$\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}} = \sin(\beta-\theta)$,
$\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{n}} = \cos(\theta-\beta)$.

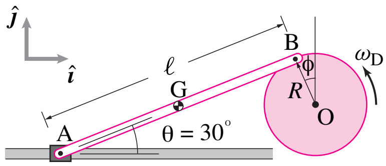

Figure 18.60:

SAMPLE 18.13 Kinematics of a Link rod in a one-DOF mechanism. In machines we often encounter mechanisms and links in which the ends of a link or a rod are constrained to move on a specified geometric path. A simplified typical link AB is shown in Fig. 18.60.

Link AB is a uniform rigid rod of length $\ell = 2\,\text{m}$. End A of the rod is attached to a collar which slides on a horizontal track. End B of the rod is attached to a uniform disk of radius $R = 0.5\,\text{m}$ which rotates about its center O. At the instant shown, when $\theta = 30^\circ$ and $\phi = \theta$, end A is observed to move at $2\,\text{m/s}$ to the left.

1. Find the angular velocity of the rod.
2. Find the angular velocity of the disk.
3. Find the velocity of the center-of-mass of the rod.

Solution Let the angular velocities of the rod and the disk be $\vec{\boldsymbol{\omega}}_{AB} = \dot{\theta}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\omega}}_D = \dot{\phi}\hat{\boldsymbol{k}}$ respectively, where $\dot{\theta}$ and $\dot{\phi}$ are unknowns. We are given $\vec{\boldsymbol{v}}_A = -v_A\hat{\boldsymbol{\imath}}$ where $v_A = 2\,\text{m/s}$.

1. Point B is on the rod as well as the disk. Hence, the velocity of point B can be found by considering either the motion of the rod or the disk. Considering the motion of the rod we write,

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{AB} \times \vec{\boldsymbol{r}}_{B/A} \\
&= -v_A\hat{\boldsymbol{\imath}} + \dot{\theta}\hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= -(v_A + \dot{\theta}\ell\sin\theta)\hat{\boldsymbol{\imath}} + \dot{\theta}\ell\cos\theta\hat{\boldsymbol{\jmath}}. \qquad (18.50)
\end{aligned}$$

Now considering the motion of the disk (and noting that $\phi = \theta$), we write,

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{\omega}}_D \times \vec{\boldsymbol{r}}_{B/O} \\
&= \dot{\phi}\hat{\boldsymbol{k}} \times R(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \\
&= -\dot{\phi}R\sin\theta\hat{\boldsymbol{\jmath}} - \dot{\phi}R\cos\theta\hat{\boldsymbol{\imath}}. \qquad (18.51)
\end{aligned}$$

But $\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_B$, therefore, from equations (18.50) and (18.51) we get

$$-(v_A + \dot{\theta}\ell\sin\theta)\hat{\boldsymbol{\imath}} + \dot{\theta}\ell\cos\theta\hat{\boldsymbol{\jmath}} = -\dot{\phi}R\sin\theta\hat{\boldsymbol{\jmath}} - \dot{\phi}R\cos\theta\hat{\boldsymbol{\imath}}$$

By equating the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components of the above equation we get

$$\begin{aligned}
-(v_A + \dot{\theta}\ell\sin\theta) &= -\dot{\phi}R\cos\theta, \qquad (18.52) \\
\text{and} \quad \dot{\theta}\ell\cos\theta &= -\dot{\phi}R\sin\theta \\
\Rightarrow \quad \dot{\theta} &= -\dot{\phi}\frac{R}{\ell}\tan\theta. \qquad (18.53)
\end{aligned}$$

Dividing Eqn. (18.52) by (18.53) we get

$$\begin{aligned}
-\left(\frac{v_A}{\dot{\theta}} + \ell\sin\theta\right) &= \frac{\ell\cos\theta}{\tan\theta} = \ell\frac{\cos^2\theta}{\sin\theta} \\
\Rightarrow \quad -\frac{v_A}{\dot{\theta}} &= \ell\left(\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right) = \ell\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta}\right) = \frac{\ell}{\sin\theta} \\
\Rightarrow \quad \dot{\theta} &= -\frac{v_A}{\ell}\sin\theta = -\frac{v_A}{\ell}\sin 30^\circ \\
&= -\frac{2\,\text{m/s}}{2\,\text{m}} \cdot \frac{1}{2} = -0.5\,\text{rad/s}.
\end{aligned}$$

$$\vec{\boldsymbol{\omega}}_{AB} = -0.5\,\text{rad/s}\hat{\boldsymbol{k}}$$

2. From eqn. (18.53)

$$\begin{aligned}\dot\phi &= -\frac{\dot\theta\ell}{R\tan\theta} = -\frac{\overbrace{-\frac{v_A}{\ell}}^{\dot\theta}\sin\theta\,\ell}{R\frac{\sin\theta}{\cos\theta}} = \frac{v_A}{R}\cos\theta \\ &= -\frac{2\,\mathrm{m/s}}{0.5\,\mathrm{m}}\frac{\sqrt{3}}{2} = 2\sqrt{3}\,\mathrm{rad/s}.\end{aligned}$$

Thus $\vec{\omega}_D = \dot\phi\hat{\boldsymbol{k}} = 3.46\,\mathrm{rad/s}\hat{\boldsymbol{k}}$.

$$\boxed{\vec{\omega}_D = 3.46\,\mathrm{rad/s}\hat{\boldsymbol{k}}}$$

[At this point, it is a good idea to check our algebra by substituting the values of $\dot\theta$ and $\dot\phi$ in equations (18.50) and (18.51) to calculate $\vec{\boldsymbol{v}}_B$.] 2

3. Now we can calculate the velocity of the center-of-mass of the rod by considering either point A or point B as a reference:

$$\begin{aligned}\vec{\boldsymbol{v}}_G &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{G/A} \\ &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{AB}\times\vec{\boldsymbol{r}}_{G/A} \\ &= -v_A\hat{\boldsymbol{\imath}} + \dot\theta\hat{\boldsymbol{k}}\times\frac{\ell}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\ &= -(v_A + \dot\theta\frac{\ell}{2}\sin\theta)\hat{\boldsymbol{\imath}} + \dot\theta\frac{\ell}{2}\cos\theta\hat{\boldsymbol{\jmath}} \\ &= -(2\,\mathrm{m/s} - 0.5\,\mathrm{rad/s}\cdot\frac{2\,\mathrm{m}}{2}\cdot\frac{1}{2})\hat{\boldsymbol{\imath}} + (-0.5\,\mathrm{rad/s}\cdot\frac{2\,\mathrm{m}}{2}\cdot\frac{\sqrt{3}}{2})\hat{\boldsymbol{\jmath}} \\ &= -\frac{7}{4}\,\mathrm{m/s}\hat{\boldsymbol{\imath}} - \frac{\sqrt{3}}{4}\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_G = -(1.75\hat{\boldsymbol{\imath}} + 0.43\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}}$$

We could easily check our calculation by taking point B as a reference and writing

$$\begin{aligned}\vec{\boldsymbol{v}}_G &= \vec{\boldsymbol{v}}_B + \vec{\boldsymbol{v}}_{G/B} \\ &= \vec{\boldsymbol{v}}_B + \vec{\boldsymbol{\omega}}_{AB}\times\vec{\boldsymbol{r}}_{G/B}\end{aligned}$$

By plugging in appropriate values we get, of course, the same value as above.

Comment: We used the standard basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ for all our vector calculations in this sample. We can shorten these calculations by choosing other appropriate basis vectors as we show in the following samples.

2 Substituting $\dot\theta = -0.5\,\mathrm{rad/s}$ in Eqn. (18.50) and plugging in the given values of other variables we get

$$\vec{\boldsymbol{v}}_B = -(\frac{3}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}.$$

Similarly, substituting $\dot\phi = 2\sqrt{3}\,\mathrm{rad/s}$ in Eqn. (18.50) and plugging in the other given values we get

$$\vec{\boldsymbol{v}}_B = -(\frac{3}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s},$$

which checks with the $\vec{\boldsymbol{v}}_B$ found above.

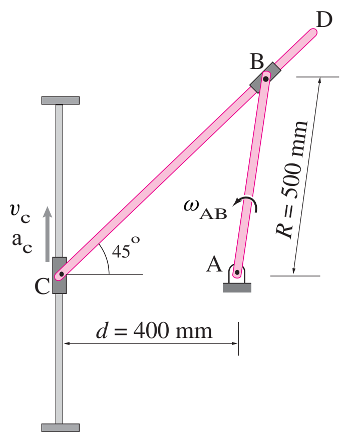

Figure 18.61

SAMPLE 18.14 **A two-DOF mechanism.** A two degree-of-freedom mechanism made of three rods and two sliders is shown in the figure. At the instant shown, the crank AB is rotating with angular velocity $\vec{\omega}_{AB} = 12\,\mathrm{rad/s}\hat{\boldsymbol{k}}$ and angular acceleration $\dot{\vec{\omega}}_{AB} = 10\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}$. At the same instant, the collar at end C of the link rod CD is sliding on the vertical rod with velocity $\vec{v}_C = 0.5\,\mathrm{m/s}\hat{\boldsymbol{j}}$ and acceleration $\vec{a}_C = 10\,\mathrm{m/s^2}\hat{\boldsymbol{j}}$. Find the angular velocity and angular acceleration of the link rod CD.

Solution Once again, we are interested in instantaneous kinematics — we wish to find the angular velocity and acceleration of rod CD at the given instant. This problem is just like the previous sample problem except that end C of the link rod CD is not fixed but free to slide on the vertical bar. But the velocity and acceleration of point C is given; so it is exactly like the previous sample (there, the velocity and acceleration of point C was identically zero). So, we adopt the same line of attack. We figure out the velocity and acceleration of point B using the kinematics of rod AB. We then write the velocity and acceleration of the same point using the kinematics of rod CD (this will involve the unknown angular velocity and acceleration of CD that we are interested in). Equate the two and solve for the unknowns we are interested in.

Let the angular velocity and acceleration of rod CD be $\vec{\omega}_{CD} = \dot{\beta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{CD} = \ddot{\beta}\hat{\boldsymbol{k}}$, respectively. Let $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ be base vectors rotating with rod AB, and $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ be the base vectors rotating with rod CD (see fig. 18.62). Considering rod AB, we have

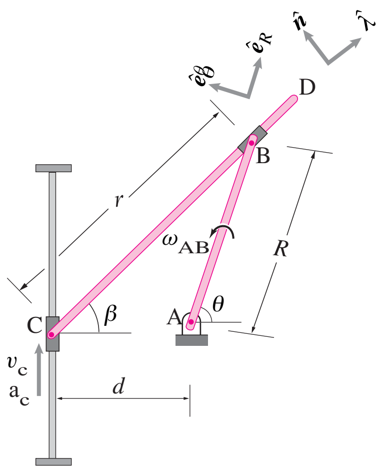

Figure 18.62

$$\begin{aligned}
\vec{v}_B &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta && (18.54)\\
\vec{a}_B &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta. && (18.55)
\end{aligned}$$

Considering rod CD, we have

$$\begin{aligned}
\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} &= v_C\hat{\boldsymbol{j}} + \dot{r}\hat{\boldsymbol{\lambda}} + r\dot{\beta}\hat{\boldsymbol{n}} && (18.56)\\
\vec{a}_B = \vec{a}_C + \vec{a}_{B/C} &= a_C\hat{\boldsymbol{j}} + (\ddot{r} - r\dot{\beta}^2)\hat{\boldsymbol{\lambda}} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{\boldsymbol{n}}. && (18.57)
\end{aligned}$$

Now equating eqn. (18.54) and (18.56), and dotting both sides with $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$, we get

$$\begin{aligned}
\dot{r} &= R\dot{\theta}\overbrace{(\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}})}^{-\sin(\theta-\beta)} - v_C\overbrace{(\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\lambda}})}^{\sin\beta} = -R\dot{\theta}\sin(\theta-\beta) - v_C\sin\beta && (18.58)\\
r\dot{\beta} &= R\dot{\theta}\underbrace{(\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{n}})}_{\cos(\theta-\beta)} - v_C\underbrace{(\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{n}})}_{\cos\beta} = R\dot{\theta}\cos(\theta-\beta) - v_C\cos\beta && (18.59)
\end{aligned}$$

where the dot products among the basis vectors are easily found from either their geometry (see fig. 18.63) or from their component representation (see previous sample). Following exactly the same procedure, we get, from eqn. (18.55) and 18.57,

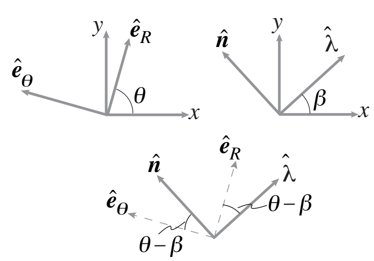

Figure 18.63

$$\begin{aligned}
\ddot{r} &= -R\dot{\theta}^2\cos(\theta-\beta) + R\ddot{\theta}(-\sin(\theta-\beta)) - a_C\sin\beta + r\dot{\beta}^2 && (18.60)\\
r\ddot{\beta} &= -R\dot{\theta}^2\sin(\theta-\beta) + R\ddot{\theta}\cos(\theta-\beta) - a_C\cos\beta - 2\dot{r}\dot{\beta}. && (18.61)
\end{aligned}$$

Now, note that although we are only interested in finding $\dot{\beta}$ and $\ddot{\beta}$. So, we only need eqn. (18.59) and eqn. (18.61). But, eqn. (18.61) requires $\dot{r}$ on the right hand side and, therefore, we do need eqn. (18.58). We can, however, happily ignore eqn. (18.60).

Now, to find the numerical values of $\dot{\beta}$ and $\ddot{\beta}$, we need to find r and θ in addition to all other given values. Consider triangle ABC in fig. 18.62. Using the law of sines ($\frac{R}{\sin\beta} = \frac{r}{\sin\theta} = \frac{d}{\sin(\theta-\beta)}$), we get $r = 0.7\,\mathrm{m}$ and $\theta = 79.45^\circ$. Now, substituting all known numerical values in eqns. (18.58), (18.59), and (18.61), we get

$$\dot{r} = -3.75\,\mathrm{m/s}, \quad \dot{\beta} = 6.61\,\mathrm{rad/s}, \quad \ddot{\beta} = 8.43\,\mathrm{rad/s^2}.$$

$$\vec{\omega}_{CD} = (6.61\,\mathrm{rad/s})\hat{\boldsymbol{k}}, \quad \dot{\vec{\omega}}_{CD} = (8.43\,\mathrm{rad/s^2})\hat{\boldsymbol{k}}$$