

18.4 Kinematics of 2-D mechanisms

An *ideal mechanism* or *linkage* is a collection of rigid objects constrained to move relative to the ground and each other by hinges but which still has some possible motion(s). People also use the word mechanism or linkage more loosely to include any collection of machine parts connected by any means.

The analysis of the kinematics of mechanisms is an important part of machine design. Mechanisms *synthesis*, coming up with a mechanism design which has desired motions, is obviously key in creative design, and nowadays in computer aided design.

Finally, the determination of the dynamics of a mechanism, how it will move and with what forces, is completely dependent on understanding the kinematics of the mechanism. The whole subject of mechanism kinematic analysis, although in some sense a subset of dynamics, is actually a huge and infinitely complex subject in itself^① and also a useful subject in itself. Often kinematics is the central interest in machine design, and mechanics (force and acceleration) analysis is only carried out if something that shouldn't do so, breaks or shakes. This section presents some of the basic ideas in kinematic analysis. The overall question in mechanism kinematic analysis is this:

Given a collection of parts and a description of how they are connected, in what ways can they move?

Without getting into the details of the motions yet, the first question to answer is simpler than finding the motions, but just counting them: In how many ways can the mechanism move?

Degrees of freedom (DOF)

The number of degrees of freedom (DOF) n_{DOF} of a mechanism is the number of different ways it can move. More precisely

The number of degrees of freedom n_{DOF} of a mechanism is the minimum number of configuration variables needed to describe all possible configurations of the mechanism.

The minimum number of configuration variables n_{DOF} is a property of the mechanism. The choice of what these variables are, however is not unique to a given mechanism.

Example: A particle in a plane has 2 degrees of freedom.

The set of 'configurations' of a particle in a plane is the set of positions of the particle. This is fully described by its x and y coordinates. Thus $n_{\text{DOF}} = 2$. But the configuration is also determined by the particle's polar coordinates R and θ . And there are an infinite number of other pairs of numbers that could

^①**Very advanced aside.** The general kinematics of planar linkages is as complicated as the classification of closed surfaces in any number of dimensions: spheres, donuts, spheres with two holes, etc. For every surface there is a corresponding 2D mechanism. In impenetrable math-speak: Any orientable manifold is a connected component of the configuration space of some planar linkage. This idea was proposed by Bill Thurston in the 1970's and later proved by Kapovich and Millson. We report this just so you don't feel bad that you don't know now, and never will know, all there is to know about planar mechanisms.

be used to describe the configurations (e.g., the x' and y' coordinates, the w and z coordinates with $w = e^x$ and $z = e^y$, etc). The minimum number of configuration variables, 2, is unique, but the choice of variables is not.

For planar mechanisms one can often determine the number of degrees of freedom by the following formula

$$3 \cdot (\underbrace{\# \text{ of rigid objects}}_{n_{\text{bod}}}) + 2 \cdot (\underbrace{\# \text{ of particles}}_{n_{\text{part}}}) - (\underbrace{\# \text{ of constraints}}_{n_{\text{con}}}) = n_{\text{DOF}} \quad (18.36)$$

The formula starts with the number of ways one rigid object can move (2 translations and a rotation makes 3) and one particle can move (just 2 translations) and then subtracts the restrictions to the motion. In eqn. (18.36) the number n_{con} of constraints is counted as the number of degrees of freedom restricted by the connections.

Examples of connections and their effect on n_{DOF}

See fig. 18.50 for some standard idealized connections and their number of constraints (assuming they are already constrained to a plane).

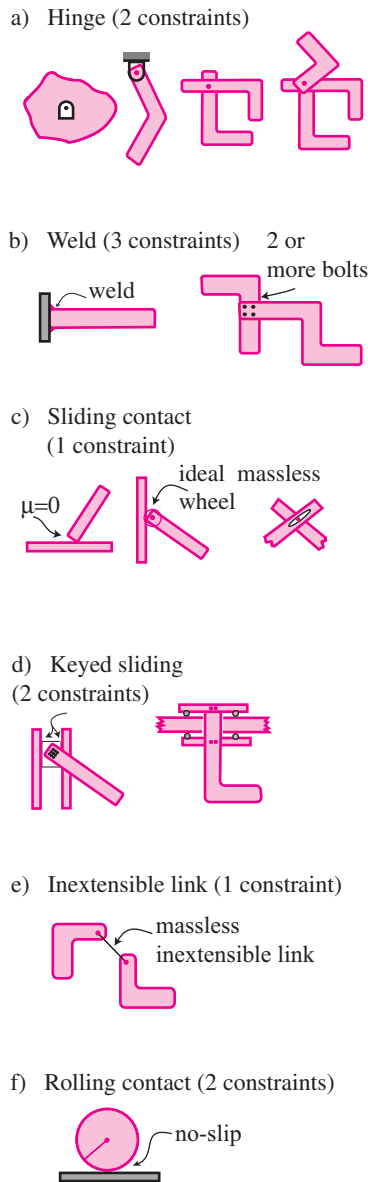


Figure 18.50: The number of degrees of freedom that are eliminated by various constraints. See text for explanation.

- a) **2** for a **pin joint**: a pin joint restricts relative motion in 2 directions but still allows relative rotation. If three objects are connected at one pin then it counts as two pin joints and thus $2 \times 2 = 4$ reductions in the number of degrees of freedom. There are 6 reductions for 4 objects connected at one pin, etc.
- b) **3** for a **welded connection**: a weld restricts relative translation in two directions as well as relative rotation ($2 + 1 = 3$). So two parts that are welded together have $2 \cdot 3 - 3 = 3$ degrees of freedom. That is, any collection of rigid objects welded together is the same as one rigid object. The word ‘weld’ is meant to include any collection of bolts, glue, string, rivets or bailing wire that prevents any relative motion.
- c) **1** for a **sliding contact**: the sliding contact restricts relative translation normal to the contact surfaces and allows translation tangent to the surfaces. Relative rotation is also allowed.
- d) **2** for a **keyed sliding contact**: allows relative translation in one direction but disallows translation in one direction as well as rotation.
- e) **1** for a **massless link hinged at its ends to two objects**: this keeps the distance between two points fixed which is one restriction (alternatively the bar adds 3 degrees of freedom and each hinge subtracts 2 ($+3 - 2 \times 2 = -1$ degree of freedom).
- f) **2** for a **rolling contact**: relative slip is not allowed nor is interpenetration.

Be warned

If some of the constraints are redundant then a system can have more degrees of freedom than eqn. (18.36) indicates. But a mechanism never has fewer degrees of freedom than eqn. (18.36) indicates.

Some simple mechanisms

Figure 18.51 shows some examples of simple mechanisms and the number of degrees of freedom. In each case we look at eqn. (18.36): $3 \cdot n_{\text{bod}} + 2 \cdot n_{\text{part}} - n_{\text{con}} = n_{\text{DOF}}$.

- a) An **object connected to ground by a hinge** has **1** degree of freedom; the set of all possible configurations can be described by the angle of the object: $n_{\text{bod}} = 1, n_{\text{con}} = 2 \Rightarrow n_{\text{DOF}} = 1$.
- b) An **unconstrained object** has **3** degrees of freedom; the set of all configurations can be described by the x and y coordinates of a reference point and by the rotation θ : $n_{\text{bod}} = 1, n_{\text{con}} = 0 \Rightarrow n_{\text{DOF}} = 3$.
- c) A **bead on a wire** has **1** degree of freedom; its configuration is fully determined by the distance the bead has advanced along the wire relative to a reference mark: $n_{\text{part}} = 1, n_{\text{con}} = 1 \Rightarrow n_{\text{DOF}} = 1$.
- d) A **statically determinate truss** has **0** degrees of freedom; a statically determinate truss has no way to deform: $n_{\text{bod}} = n_{\text{bars}}, n_{\text{part}} = 0, n_{\text{con}} = 2 \cdot n_{\text{pins}} + n_{\text{ground const}} \Rightarrow n_{\text{DOF}} = 0$. Note that the number of pin restrictions in our count here is more than twice the number of joints in the study of trusses in statics. In statics, we focussed on joints as restricted by bars. Here we look at bars as restricted by joints. A given joint counts 2, 4 or 6 pin restrictions depending on whether it connects 2, 3 or 4 bars. Thus the 11 bar truss shown has 7 joints (by truss-analysis counting) but, by the counting here it has $2 \times 2 + 2 \times 4 + 3 \times 6 = 30$ pin restrictions plus 3 ground restrictions for 33 restrictions in all. Without restrictions the 11 bars had 33 degrees of freedom. So, because $33 - 33 = 0$, and none of the restrictions is redundant, the truss has zero degrees of freedom.
- e) A **rolling wheel** has **1** degree of freedom; its configuration is fully determined either by the net angle θ it has rolled or by the x coordinate of its center: $n_{\text{bod}} = 1, n_{\text{con}} = 2 \Rightarrow n_{\text{DOF}} = 1$.
- f) A **double pendulum** or two-link robot arm has **2** degrees of freedom; its configuration is determined by the net rotations of its two links (or by the rotation of the first link and the relative rotation of the second link): $n_{\text{bod}} = 2, n_{\text{con}} = 4 \Rightarrow n_{\text{DOF}} = 2$.
- g) A **cart with two rolling wheels** has **1** degree of freedom; given the position of the cart, say x , the rotation of the wheels is determined. Similarly for a **bicycle**. For this model of a bicycle we are

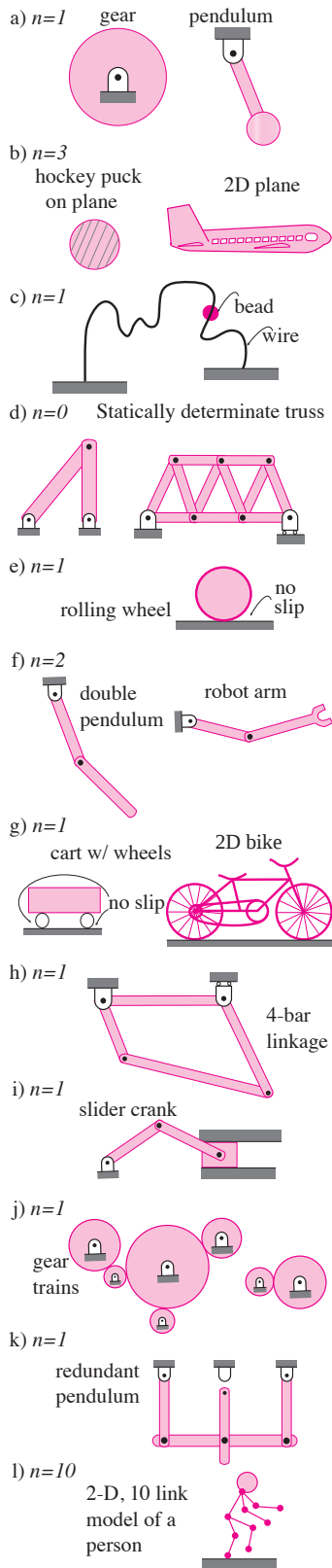


Figure 18.51: Some simple mechanisms and their number of degrees of freedom, n (called n_{DOF} in the text).

neglecting the steering degree of freedom and also the freedoms of the crank and pedals. For a vehicle with 2 wheels in 2D we have: $n_{\text{bod}} = 3, n_{\text{con}} = 4 \times 2 = 8$ (2 hinges and 2 rolling contacts) $\Rightarrow n_{\text{DOF}} = 9 - 8 = 1$.

- h) A **“four” bar linkage** has **1** degree of freedom; the angle of any one of the bars determines the angles of the others: $n_{\text{bod}} = 4, n_{\text{con}} = 2 \times 4 + 2 + 1 = 11$ (there are 4 pin joints between the bars, one pin joint to ground and one roller connection to the ground) $\Rightarrow n_{\text{DOF}} = 1$.
- i) A **slider crank** has **1** degree of freedom; the rotation of the crank determines the configuration of the system $n_{\text{bod}} = 3, n_{\text{con}} = 3 \cdot 2 + 2$ (there are 3 pins and one keyed connection) $\Rightarrow n_{\text{DOF}} = 1$.
- j) An ideal **gear train** (with all gears pinned to ground) has **1** degree of freedom; the amount of rotation of any one gear determines the rotation of all of the gears: In this case the counting formula is *wrong*. Say there are 2 gears, then $n_{\text{bod}} = 2, n_{\text{con}} = 3 \cdot 2 = 6$ (two pins and one rolling contact) $\Rightarrow n_{\text{DOF}} = 0 \neq 1$. The rolling constraint prevents interpenetration, but this was already prevented by the hinges at the center of the gears. The constraints are redundant and the system has more degrees of freedom than eqn. (18.36) indicates.
- k) A **redundant swing** with one horizontal bar suspended by 3 parallel struts has **1** degree of freedom; the angle of one upright link determines the full configuration of the mechanism. The counting formula is again wrong: $n_{\text{bod}} = 4, n_{\text{con}} = 6 \cdot 2 = 12 \Rightarrow n_{\text{DOF}} = 0$ underestimates the number of degrees of freedom because the constraints are redundant.
- l) A **2-D 10-link model of a person** with one foot on the ground has **10** degrees of freedom; the angles of the 10 links (4 arm links, 4 leg links, a body and a head) determine the full configuration of the mechanism. There are no redundant constraints so the counting formula also works using $n_{\text{bod}} = 10, n_{\text{hinges}} = 10$:

$$n_{\text{DOF}} = 3 * n_{\text{bod}} - n_{\text{con}} \quad (18.37)$$

$$= 3 * n_{\text{bod}} - 2 \cdot n_{\text{hinges}} = 30 - 2 \cdot 10 = 10 \quad (18.38)$$

What are the 10 hinges? One at the ground, 1 at each knee and elbow, 2 at the hip and 3 at the shoulders (one hinge for each object linked to the trunk).

Configuration variables

Once we know the number of degrees of freedom n_{DOF} of a system it is often useful to settle on one set of n_{DOF} configuration variables. In this book n_{DOF} will be 1, 2 or at most 3. Thus we pick 1, 2 or 3 variables.

Example: Straight line motion

Chapter 14 on straight line motion was mostly about one-degree-of-freedom systems ($n_{\text{DOF}} = 1$). These systems could all be characterized by the single configuration variable x , the displacement along the line of a reference point on the

object relative to a reference point on the ground. All the positions, velocities and accelerations of all points in the system could be found in terms of x , $\dot{x}$ and $\ddot{x}$ (in fact all points had $\vec{v} = \dot{x}\hat{i}$ and $\vec{a} = \ddot{x}\hat{i}$).

Example: Circular motion about a fixed axis

In chapters 15 and 16 we were almost entirely focussed on systems with one degree of freedom well characterized by the one configuration variable, the rotation angle θ . For such motions positions, velocities, and accelerations of all points were determined by the initial positions of the points θ , $\dot{\theta}$ and $\ddot{\theta}$ by equations which you know well by now.

For more general motions we almost always take inspiration from the two examples above. We use the translation of a conspicuous point, or we use the rotation of a conspicuous object for a configuration variable. And more of the same if the system has more than 1 degree of freedom. The natural choice of configuration variables for some simple mechanisms is given in the text discussing fig. 18.51.

Often our main kinematic task is to express the full configuration of the system as well as all the velocities and accelerations of all its parts in terms of the positions of the parts, the configuration variables, and their first and second time derivatives.

Adding relative angular velocities

One last simple kinematic fact is needed before we can plug and chug with the theory we have so far and apply it to general kinematic mechanisms. It concerns the addition of rotations and rotation rates. The following example basically tells the whole story

Example: Double pendulum and the addition of rotation rates

The commonly used configuration variables for the double pendulum shown in fig. 18.52 are θ_1 and θ_2 . To actually know the configuration of the system obviously we need to know ϕ which is given by

$$\phi = \theta_1 + \theta_2 \quad \text{so} \quad \dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 \quad \text{and so} \quad \ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2.$$

[Aside: One reason for choosing θ_2 instead of ϕ as a configuration variable is that if one was measuring or controlling the second link, say as a robotic arm, the angle θ_2 can be measured more easily than ϕ . Also, it turns out (in hindsight) that the differential equations of motion are slightly simpler using θ_2 instead of ϕ .]

Looking at the bars as being glued to reference frames ($\mathcal{F}$ for the fixed frame, $\mathcal{B}$ for bar AB, and $\mathcal{C}$ for bar BC), the above example shows that

$$\vec{\omega}_{\mathcal{C}/\mathcal{F}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}} \quad (18.39)$$

which is often written with the simple notation

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

Which can only be given strict meaning by the more elaborate eqn. (18.39) above it.

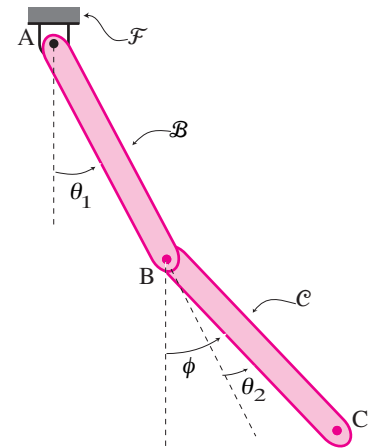


Figure 18.52: A double pendulum's configuration variables are most often taken to be θ_1 and θ_2 . But one then needs to know that $\phi = \theta_1 + \theta_2$ and that $\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2$ and $\ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2$.

Example: Double pendulum (see previous example)

Take $\bar{\omega}_{B/\mathcal{F}} = \dot{\phi}_1 \hat{k}$, $\bar{\omega}_{C/B} = \dot{\phi}_2 \hat{k}$ and $\bar{\omega}_{C/\mathcal{F}} = \dot{\phi} \hat{k}$ and eqn. (18.39) is self evident from the addition of angles.

Similarly we have

$$\bar{\alpha}_{C/\mathcal{F}} = \bar{\alpha}_{B/\mathcal{F}} + \bar{\alpha}_{C/B} \quad (18.40)$$

which is often written with the simple notation

$$\bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2.$$

For those going on to study 3-D mechanics, one should note that, unlike eqn. (18.39), eqn. (18.40) does *not* hold in 3-D.

Kinematics of mechanisms

One approach to mechanisms is to do what one can with high-school geometry and trigonometry, the laws of sines (see page 110), and so on.

Example: Rod on step using geometry and trigonometry

One end of a rod slides on the ground. The other end slides on a corner at A (see fig. 18.53). Given that $\bar{v}_B = v_B \hat{i}$ we can find $\dot{\phi}$ as follows:

$$\frac{h}{\ell_{OB}} = \tan \phi \Rightarrow \left\{ \frac{\ell_{OB}}{h} = \frac{\cos \phi}{\sin \phi} \right\}$$

$$\frac{d}{dt} \left\{ \right\} \Rightarrow \frac{\dot{\ell}_{OB}}{h} = \frac{-\dot{\phi}}{\sin^2 \phi} \Rightarrow \dot{\phi} = -\frac{v_B \sin^2 \phi}{h}$$

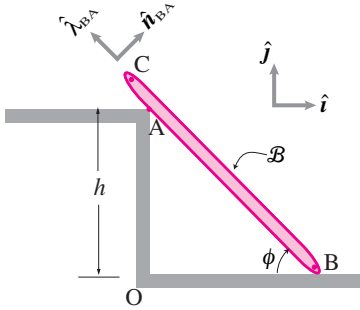


Figure 18.53: A rigid rod slides so that it always touches the ground and the corner at A. One would like to know the relation between $\dot{\phi}$ and $\bar{v}_B$.

As the above calculation shows, this problem doesn't need the heavy machinery of our moving-frame vector methods. But it provides an instructive example.

Example: Rod on step using moving-frame methods (see previous example)

We look at point A and note that we can think of it as a fixed point in the fixed frame $\mathcal{F}$ and also as a point that is moving relative to the translating and rotating frame $\mathcal{B}$. We evaluate its velocity both ways.

$$\begin{aligned} \bar{v}_A &= \bar{v}_A \\ \bar{\mathbf{0}} &= \bar{v}_B + \bar{\omega}_B \times \bar{r}_{A/B} + \bar{v}_{A/B} & (\text{eqn. (18.28)}) \\ \{\bar{\mathbf{0}}\} &= v_B \hat{i} - \dot{\phi} \hat{k} \times (\ell_{BA} \hat{\lambda}_{BA}) + v_{A/B} \hat{\lambda}_{BA} \\ \Rightarrow 0 &= v_B \underbrace{\hat{i} \cdot \hat{n}_{BA}}_{\sin \phi} + \dot{\phi} \ell_{BA} & (\{\} \cdot \hat{n}_{BA}) \\ \Rightarrow \dot{\phi} &= -\frac{v_B \sin \phi}{\ell_{BA}} = -\frac{v_B \sin^2 \phi}{h} & (\sin \phi = \frac{h}{\ell_{AB}}) \end{aligned}$$

as we had before. The key equation was the 'three term velocity formula' eqn. (18.28) on page 959 and the observation that relative to frame $\mathcal{B}$ point A slides along the rod. Note that we never had to explicitly use the rotating coordinates associated with frame $\mathcal{B}$ to do this calculation.

You should understand the examples above, and the needed background material, before going on to the following examples.

Example: Slider Crank using geometry and trigonometry

The slider crank mechanism (fig. 18.54) was briefly introduced in the context of statics where its forces could be analyzed assuming inertial terms were negligible (see 418). But it is a commonly used mechanism (e.g., in every car) and its motions are of central interest. The angle θ is the most natural configuration variable for this $n_{\text{DOF}} = 1$ system. One would like to know the position, velocity and acceleration of the slider (x_C , $\dot{x}_C$ and $\ddot{x}_C$) in terms of θ , $\dot{\theta}$ and $\ddot{\theta}$.

$$\begin{aligned}
 x_C &= x_D + \ell_{DC} \\
 &= d \cos \theta + \sqrt{\ell^2 - h^2} \\
 &= d \cos \theta + \sqrt{\ell^2 - (d \sin \theta)^2} \\
 &= d \left(\cos \theta + \sqrt{(\ell/d)^2 - \sin^2 \theta} \right) \\
 &= d \left(\cos \theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2} \right) \quad (18.41)
 \end{aligned}$$

The positive $\sqrt{\quad}$ corresponds to C being to the right of 0. The negative $\sqrt{\quad}$ corresponds to point C being to the left of 0. The mechanism just doesn't work for a full revolution of the link OA if $\ell < d$ as you can see from the picture or from that the $\sqrt{\quad}$ above giving imaginary values for $\cos 2\theta$ near -1, $\sin \theta$ near 1, and θ near $\pi/2$.

To get the velocity of point C we just take the derivative of eqn. (18.41) above.

$$\begin{aligned}
 v_C &= \dot{x}_C \quad (18.42) \\
 &= \frac{d}{dt} \left\{ d \left(\cos \theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2} \right) \right\} \\
 &= \dot{\theta} d \left\{ -\sin(\theta) - \sin(2\theta) \frac{1}{\sqrt{4 \frac{l^2}{d^2} + 2 \cos(2\theta) - 2}} \right\}
 \end{aligned}$$

To get the acceleration we differentiate once again. For simplicity let's assume the crank rotates at constant rate, so $\dot{\theta}$ is a constant and $\ddot{\theta} = 0$. Cranking out the derivative of eqn. (18.42), so to speak, we get

$$\begin{aligned}
 a_C = \ddot{x}_C &= \frac{d}{dt} \{ \text{the mess on the right of eqn. (18.42)} \} \quad (18.43) \\
 &= -d\dot{\theta}^2 \left\{ \cos(\theta) \right. \\
 &\quad \left. + 2 \sin^2(2\theta) \left(4 \frac{l^2}{d^2} + 2 \cos(2\theta) - 2 \right)^{-3/2} \right. \\
 &\quad \left. + 2 \cos(2\theta) \frac{1}{\sqrt{4 \frac{l^2}{d^2} + 2 \cos(2\theta) - 2}} \right\}
 \end{aligned}$$

So we now know the position, velocity and acceleration of point C in terms of θ , $\dot{\theta}$ and $\ddot{\theta}$. You should commit the solution eqn. (18.43) to memory. Just kidding.

Plots of x_C , v_C , and a_C from these equations are shown in fig. 18.55ab for two different extremes of slider crank design: one with a very long connecting rod that gives sinusoidal motion, and one with a connecting rod just barely long enough to prevent locking that gives intermittent motion.

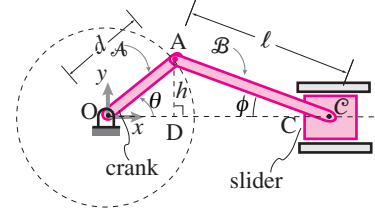


Figure 18.54: The slider crank mechanism.

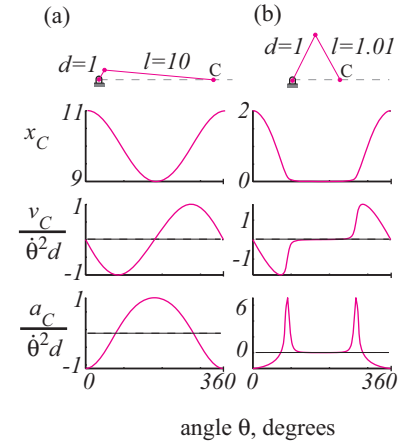


Figure 18.55: Position x_C of the slider, its velocity v_C and its acceleration a_C (Strictly, e.g., $\vec{a}_C = a_C \hat{i}$). Two sets of curves are shown. (a) a long connecting rod, and (b) a connecting rod a hair longer than the crank. In case (a) the connecting rod is nearly horizontal at all times and the displacement of point C is almost entirely due to the horizontal displacement of the end of the crank arm. Thus point C moves with almost exactly a cosine wave with amplitude equal to the length of the crank. The velocity and acceleration curves are thus also sine and cosine waves. In case (b) the motion is close to a cosine curve approximately when $-60^\circ < \theta < 60^\circ$. That is, the displacement of C is about twice the horizontal displacement of the end of the crank arm when the end is to the right of its base bearing. When the end of the crank arm is to the left of the bearing point C is nearly stationary and is just to the right of the crank base bearing. The transition between these two cases involves a sudden large acceleration.

Unlike some more complex mechanisms, the slider crank is solvable in that one can write a formula for the position of any point of interest in terms of the single configuration variable θ . For more complex mechanisms this may not be possible. Further, even if possible the above example shows that the differentiation required to find velocity and acceleration can lead to a bit of a mess.

A different approach is to assume that at some value of the configuration variable (θ for the slider crank) that the full configuration of the system is known. That is, that the locations of all points are known. Then we can use our vector methods to find velocities and accelerations of all points of interest.

Example: Slider crank using vector methods (see previous example)

Take the slider crank of fig. 18.54 to be in some known configuration. We now try to find the velocity and acceleration of point C in terms of the positions of the points O, B, and C as well as θ and $\dot{\theta}$.

The basic approach is to write true things, and then solve for unknowns. First work on velocities. The basic idea is to look at the *closure* condition. That is, the velocity of point C as calculated by working down the linkage from O to A to C has to be consistent with the velocity of C as calculated in the fixed frame.

$$\begin{aligned}\vec{v}_C &= \vec{v}_C \\ v_C \hat{i} &= \vec{v}_{A/O} + \vec{v}_{C/A} \\ v_C \hat{i} &= (\dot{\theta} \hat{k}) \times \vec{r}_{A/O} + (-\dot{\phi} \hat{k}) \times \vec{r}_{C/A}\end{aligned}\quad (18.44)$$

eqn. (18.44) is a 2-D vector equation in the 2 unknown scalars v_C and $\dot{\phi}$. It could be solved as a pair of equations, or solved directly by first dotting both sides with $\hat{j}$ to find $\dot{\phi}$ and dotting both sides with $\vec{r}_{C/A}$ to find v_C . These yield

$$\begin{aligned}\dot{\phi} &= \frac{((\dot{\theta} \hat{k}) \times \vec{r}_{A/O}) \cdot \hat{j}}{(\hat{k} \times \vec{r}_{C/A}) \cdot \hat{j}} \quad \text{and} \\ v_C &= \frac{\dot{\theta} (\hat{k} \times \vec{r}_{A/O}) \cdot \vec{r}_{C/A}}{\hat{i} \cdot \vec{r}_{C/A}}\end{aligned}$$

where everything on the right of these equations is assumed known. Without grinding out the vector products in terms, say, of components, we can just use that we know $\dot{\phi}$ and v_C .

We proceed to find the accelerations by similar means, assuming $\dot{\theta}$ is a constant so $\ddot{\alpha}_A = \vec{0}$:

$$\begin{aligned}\vec{a}_C &= \vec{a}_C \\ a_C \hat{i} &= \vec{a}_{A/O} + \vec{a}_{C/A} \\ a_C \hat{i} &= -\dot{\theta}^2 \vec{r}_{A/O} + (-\ddot{\phi} \hat{k}) \times \vec{r}_{C/A} - \dot{\phi}^2 \vec{r}_{C/A}\end{aligned}\quad (18.45)$$

eqn. (18.45) is a 2-D vector equation in the two scalar unknowns a_C and $\ddot{\phi}$. We can set this up as two equations in two unknowns. Or we can solve for $\ddot{\phi}$ directly by dotting both sides with $\hat{j}$ and we can solve for a_C directly by crossing both sides with $\vec{r}_{C/A}$ or by dotting with a vector perpendicular to $\vec{r}_{C/A}$ (e.g., $\vec{r}_{C/A}^\perp = \hat{k} \times \vec{r}_{C/A}$)

Although we have presented an algorithm rather than a formula, we have found the velocity and acceleration of C without writing any large equations of the type needed in the previous example. The shortcoming is that this method depends on knowing the full configuration at the time of interest.

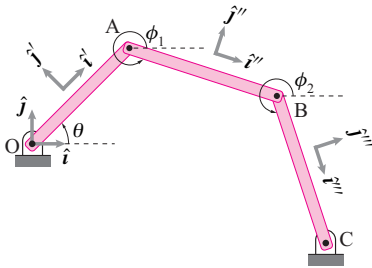


Figure 18.56: A four bar linkage.

Example: Four bar linkage using geometry and trigonometry

fig. 18.56 shows a “four-bar linkage”. Please see 417 for an introduction to 4-bar linkages in the context of statics. Four bar linkages are solvable in the sense that one can write equations for the positions of any point of interest in terms of the single configuration variable θ marked in fig. 18.56. But the formulas are really a mess. And the first and second time derivatives are an unbelievable mess.

The four-bar linkage is about as complex a system as can be solved in this sense, and it is probably too-complex for this solution to be useful in the kinematic analysis of accelerations.

For complex mechanisms one is often stuck using vector methods, like we are for practical purposes stuck with the 4-bar linkage. But the vector methods based on the current configuration are not crippled by complexity.

Example: Four-bar linkage using relative velocities and accelerations

Assuming the configuration is known (*i.e.*, that θ , ϕ_1 , and ϕ_2 are known), we can proceed with the 4-bar linkage just as we did for the slider crank. We enforce closure by picking a point and thinking about its velocity two different ways^② We could pick any point, say C. From the fixed frame we know that the velocity of C is zero. Working around the linkage, link by link, we know it is the sum of relative velocities as

$$\begin{aligned}\vec{v}_C &= \vec{v}_C \\ \vec{0} &= \vec{v}_{A/0} + \vec{v}_{B/A} + \vec{v}_{C/B} \\ &= (\dot{\theta}\hat{k}) \times \vec{r}_{A/0} + (\dot{\phi}_1\hat{k}) \times \vec{r}_{B/A} + (\dot{\phi}_2\hat{k}) \times \vec{r}_{C/B}\end{aligned}$$

which is equivalent to two scalar equations in the two unknowns $\dot{\phi}_1$ and $\dot{\phi}_2$. This equation can be solved directly for $\dot{\phi}_2$ by taking the dot product of both sides with a vector perpendicular to $\vec{r}_{B/A}$ (such as $\hat{j}''$ or $\hat{k} \times \vec{r}_{B/A}$) and for $\dot{\phi}_1$ by taking the dot product of both sides with for a vector perpendicular to $\vec{r}_{C/B}$ (such as $\hat{j}'''$ or $\hat{k} \times \vec{r}_{C/B}$) to get

$$\begin{aligned}\dot{\phi}_1 &= -\dot{\theta} \frac{(\hat{k} \times \vec{r}_{A/0}) \cdot \hat{j}'''}{(\hat{k} \times \vec{r}_{B/A}) \cdot \hat{j}'''} \quad \text{and} \\ \dot{\phi}_2 &= -\dot{\theta} \frac{(\hat{k} \times \vec{r}_{A/0}) \cdot \hat{j}''}{(\hat{k} \times \vec{r}_{C/B}) \cdot \hat{j}''}.\end{aligned}$$

The dot product with $\hat{k}$ is used to get a scalar on the top and bottom of the fraction, both vectors are already only in the $\hat{k}$ direction. Now that $\dot{\phi}_1$ and $\dot{\phi}_2$ are known the velocity of any point on the mechanism is known. For example

$$\vec{v}_B = (\dot{\phi}_2\hat{k}) \times \vec{r}_{B/C}.$$

The angular accelerations of the two links are found by the same method. For simplicity let's assume that the driving crank OA spins at constant rate so

^②**Relative vs absolute angles.** In this example, and the examples above, we use the absolute rotations of the links. However, we could also have done the calculations using the relative rotations and then used the formulas for velocity and acceleration relative to a rotating frame. For these examples this would make the calculations slightly, but not intractably, longer.

$\ddot{\theta} = 0$. Looking at the acceleration of point C two ways we have

$$\begin{aligned}
 \vec{a}_C &= \vec{a}_C \\
 \vec{0} &= \vec{a}_{A/0} + \vec{a}_{B/A} + \vec{a}_{C/B} \\
 &= \underbrace{(\ddot{\theta} \hat{k}) \times \vec{r}_{A/0}}_0 - \dot{\theta}^2 \vec{r}_{A/0} \\
 &\quad + (\ddot{\phi}_1 \hat{k}) \times \vec{r}_{B/A} - \dot{\phi}_1^2 \vec{r}_{B/A} \\
 &\quad + (\ddot{\phi}_2 \hat{k}) \times \vec{r}_{C/B} - \dot{\phi}_2^2 \vec{r}_{C/B}
 \end{aligned}$$

Because $\dot{\phi}_1$ and $\dot{\phi}_2$ are already known, this is one equation in the two unknowns $\ddot{\phi}_1$ and $\ddot{\phi}_2$. They can be solved for $\ddot{\phi}_1$ by taking the dot product of both sides with $\hat{j}'''$ and for $\ddot{\phi}_2$ by taking the dot product of both sides with $\hat{j}''$.

At this point you know θ , $\dot{\theta}$, ϕ_1 , $\dot{\phi}_1$, $\ddot{\phi}_1$, ϕ_2 , $\dot{\phi}_2$, and $\ddot{\phi}_2$ and can thus calculate the position, velocity and acceleration of any point in the mechanism.

SAMPLE 18.12 Two rods connected in a one-DOF mechanism. A mechanism consists of two rods AB and CD connected together at P with a collar pinned to AB but free to slide on CD. Rod AB is driven with $\bar{\omega}_{AB} = 10 \text{ rad/s} \hat{k}$ and $\dot{\bar{\omega}}_{AB} = 4 \text{ rad/s}^2 \hat{k}$. At the instant shown, $\theta = 45^\circ$ and $\beta = 30^\circ$. The length of rod AB is $R = 0.4 \text{ m}$. At the instant shown,

1. Find the angular velocity and angular acceleration of rod CD.
2. Find the velocity and acceleration of the collar with respect to rod CD.

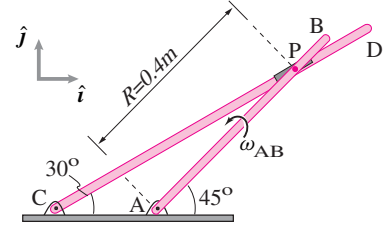


Figure 18.57

Solution Here, we are interested in instantaneous kinematics of this mechanism. Since point P is on rod AB as well as on rod CD, its velocity and acceleration can be expressed in terms of the angular motion of rod AB or that of rod CD. Let us consider rod AB first. Let $\hat{e}_R$ and $\hat{e}_\theta$ be basis vectors attached to rod AB that rotate with the rod. Since P is fixed on rod AB, it executes simple circular motion about A with $\bar{\omega}_{AB} = \dot{\theta} \hat{k}$ and $\dot{\bar{\omega}}_{AB} = \ddot{\theta} \hat{k}$ where $\dot{\theta} = 10 \text{ rad/s}$ and $\ddot{\theta} = 4 \text{ rad/s}^2$, respectively. Then

$$\bar{v}_P = R\dot{\theta}\hat{e}_\theta \quad (18.46)$$

$$\bar{a}_P = -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta. \quad (18.47)$$

Now let us consider rod CD and express the velocity and acceleration of point P in terms of motion of rod CD. Let the angular velocity and angular acceleration of rod CD be $\bar{\omega}_{CD} = \dot{\beta} \hat{k}$ and $\dot{\bar{\omega}}_{CD} = \ddot{\beta} \hat{k}$, respectively. Let $\hat{\lambda}$ and $\hat{n}$ be basis vectors attached to rod CD. Let the instantaneous position of point P on CD be $\bar{r}_P = r\hat{\lambda}$. Since the collar can slide along CD, we can write the velocity and acceleration of point P as

$$\bar{v}_P = \dot{r}\hat{\lambda} + r\dot{\beta}\hat{n} \quad (18.48)$$

$$\bar{a}_P = (\ddot{r} - r\dot{\beta}^2)\hat{\lambda} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{n}. \quad (18.49)$$

From eqn. (18.46) and 18.48 we get,

$$\begin{aligned} \dot{r}\hat{\lambda} + r\dot{\beta}\hat{n} &= R\dot{\theta}\hat{e}_\theta \\ \Rightarrow \dot{r} &= R\dot{\theta}(\hat{e}_\theta \cdot \hat{\lambda}) \\ \dot{\beta} &= (1/r)R\dot{\theta}(\hat{e}_\theta \cdot \hat{n}) \end{aligned}$$

Similarly, from eqn. (18.47) and 18.49 we get,

$$\begin{aligned} \ddot{r} - r\dot{\beta}^2 &= -R\dot{\theta}^2(\hat{e}_R \cdot \hat{\lambda}) + R\ddot{\theta}(\hat{e}_\theta \cdot \hat{\lambda}) \\ r\ddot{\beta} + 2\dot{r}\dot{\beta} &= -R\dot{\theta}^2(\hat{e}_R \cdot \hat{n}) + R\ddot{\theta}(\hat{e}_\theta \cdot \hat{n}). \end{aligned}$$

Thus, to find all kinematic quantities of interest, all we need now is to figure out a few dot products between the two sets of basis vectors. This is easily done by writing out $\hat{e}_R$, $\hat{e}_\theta$, $\hat{\lambda}$, and $\hat{n}$. ① Substituting the dot products in the expressions for $\dot{r}$, $\dot{\beta}$, $\ddot{r}$, and $\ddot{\beta}$ we get

$$\begin{aligned} \dot{r} &= R\dot{\theta} \sin(\beta - \theta), \\ \dot{\beta} &= r^{-1}R\dot{\theta} \cos(\theta - \beta) \\ \ddot{r} &= r\dot{\beta}^2 - R\dot{\theta}^2 \cos(\beta - \theta) + R\ddot{\theta} \sin(\beta - \theta), \\ \ddot{\beta} &= r^{-1}[-R\dot{\theta}^2 \sin(\theta - \beta) + R\ddot{\theta} \cos(\theta - \beta) - 2\dot{r}\dot{\beta}]. \end{aligned}$$

Substituting the given values of $\dot{R}$, $\dot{\theta}$, $\ddot{\theta}$, R , θ , β , and $r = R \sin \theta / \sin \beta$, we get

$$\dot{r} = -1.04 \text{ m/s}, \quad \dot{\beta} = 6.83 \text{ rad/s}, \quad \ddot{r} = -12.66 \text{ m/s}^2, \quad \ddot{\beta} = 9.43 \text{ rad/s}^2.$$

$$\begin{aligned} \text{(a)} \quad \bar{\omega}_{CD} &= 6.83 \text{ rad/s} \hat{k}, & \dot{\bar{\omega}}_{CD} &= 9.43 \text{ rad/s}^2 \hat{k} \\ \text{(b)} \quad \bar{v}_{/CD} &= -1.04 \text{ m/s} \hat{\lambda}, & \bar{a}_{/CD} &= -12.66 \text{ m/s}^2 \hat{\lambda} \end{aligned}$$

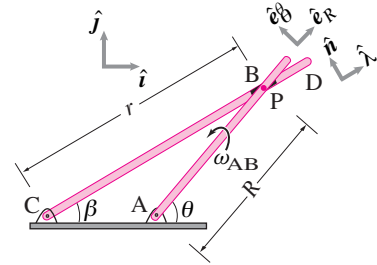


Figure 18.58: The position of point P can be written as $R\hat{e}_R$ or $r\hat{\lambda}$ where $r = R \sin \theta / \sin \beta$ (since $r \sin \beta = R \sin \theta$).

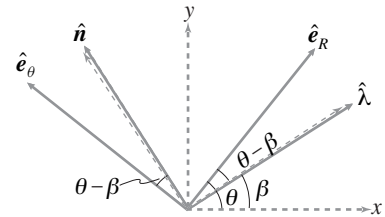


Figure 18.59: Geometry of the unit vectors.

① We have,
 $\hat{e}_R = \cos \theta \hat{i} + \sin \theta \hat{j}$,
 $\hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}$,
 $\hat{\lambda} = \cos \beta \hat{i} + \sin \beta \hat{j}$,
 $\hat{n} = -\sin \beta \hat{i} + \cos \beta \hat{j}$.
 Therefore,
 $\hat{e}_R \cdot \hat{\lambda} = \cos(\beta - \theta)$,
 $\hat{e}_R \cdot \hat{n} = \sin(\theta - \beta)$,
 $\hat{e}_\theta \cdot \hat{\lambda} = \sin(\beta - \theta)$,
 $\hat{e}_\theta \cdot \hat{n} = \cos(\theta - \beta)$.

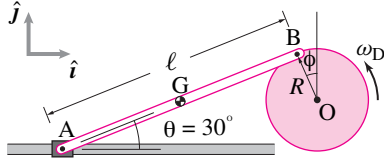


Figure 18.60:

SAMPLE 18.13 Kinematics of a Link rod in a one-DOF mechanism.

In machines we often encounter mechanisms and links in which the ends of a link or a rod are constrained to move on a specified geometric path. A simplified typical link AB is shown in Fig. 18.60.

Link AB is a uniform rigid rod of length $\ell = 2$ m. End A of the rod is attached to a collar which slides on a horizontal track. End B of the rod is attached to a uniform disk of radius $R = 0.5$ m which rotates about its center O. At the instant shown, when $\theta = 30^\circ$ and $\phi = \theta$, end A is observed to move at 2 m/s to the left.

1. Find the angular velocity of the rod.
2. Find the angular velocity of the disk.
3. Find the velocity of the center-of-mass of the rod.

Solution Let the angular velocities of the rod and the disk be $\vec{\omega}_{AB} = \dot{\theta}\hat{k}$ and $\vec{\omega}_D = \dot{\phi}\hat{k}$ respectively, where $\dot{\theta}$ and $\dot{\phi}$ are unknowns. We are given $\vec{v}_A = -v_A\hat{i}$ where $v_A = 2$ m/s.

1. Point B is on the rod as well as the disk. Hence, the velocity of point B can be found by considering either the motion of the rod or the disk. Considering the motion of the rod we write,

$$\begin{aligned}
 \vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\
 &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\
 &= -v_A\hat{i} + \dot{\theta}\hat{k} \times \ell(\cos\theta\hat{i} + \sin\theta\hat{j}) \\
 &= -(v_A + \dot{\theta}\ell\sin\theta)\hat{i} + \dot{\theta}\ell\cos\theta\hat{j}.
 \end{aligned} \tag{18.50}$$

Now considering the motion of the disk (and noting that $\phi = \theta$), we write,

$$\begin{aligned}
 \vec{v}_B &= \vec{\omega}_D \times \vec{r}_{B/O} \\
 &= \dot{\phi}\hat{k} \times R(-\sin\theta\hat{i} + \cos\theta\hat{j}) \\
 &= -\dot{\phi}R\sin\theta\hat{j} - \dot{\phi}R\cos\theta\hat{i}.
 \end{aligned} \tag{18.51}$$

But $\vec{v}_B = \vec{v}_B$, therefore, from equations (18.50) and (18.51) we get

$$-(v_A + \dot{\theta}\ell\sin\theta)\hat{i} + \dot{\theta}\ell\cos\theta\hat{j} = -\dot{\phi}R\sin\theta\hat{j} - \dot{\phi}R\cos\theta\hat{i}$$

By equating the $\hat{i}$ and $\hat{j}$ components of the above equation we get

$$-(v_A + \dot{\theta}\ell\sin\theta) = -\dot{\phi}R\cos\theta, \tag{18.52}$$

$$\text{and} \quad \dot{\theta}\ell\cos\theta = -\dot{\phi}R\sin\theta$$

$$\Rightarrow \dot{\theta} = -\dot{\phi}\frac{R}{\ell}\tan\theta. \tag{18.53}$$

Dividing Eqn. (18.52) by (18.53) we get

$$\begin{aligned}
 -\left(\frac{v_A}{\dot{\theta}} + \ell\sin\theta\right) &= \frac{\ell\cos\theta}{\tan\theta} = \ell\frac{\cos^2\theta}{\sin\theta} \\
 \Rightarrow -\frac{v_A}{\dot{\theta}} &= \ell\left(\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right) = \ell\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta}\right) = \frac{\ell}{\sin\theta} \\
 \Rightarrow \dot{\theta} &= -\frac{v_A}{\ell}\sin\theta = -\frac{v_A}{\ell}\sin 30^\circ \\
 &= -\frac{2\text{ m/s}}{2\text{ m}} \cdot \frac{1}{2} = -0.5\text{ rad/s}.
 \end{aligned}$$

$$\vec{\omega}_{AB} = -0.5\text{ rad/s}\hat{k}$$

2. From eqn. (18.53)

$$\begin{aligned}\dot{\phi} &= -\frac{\dot{\theta}\ell}{R \tan \theta} = -\frac{\overbrace{-\frac{v_A}{\ell} \sin \theta \ell}^{\dot{\theta}}}{R \frac{\sin \theta}{\cos \theta}} = \frac{v_A}{R} \cos \theta \\ &= -\frac{2 \text{ m/s} \sqrt{3}}{0.5 \text{ m} \cdot 2} = 2\sqrt{3} \text{ rad/s}.\end{aligned}$$

Thus $\vec{\omega}_D = \dot{\phi} \hat{k} = 3.46 \text{ rad/s} \hat{k}$.

$$\vec{\omega}_D = 3.46 \text{ rad/s} \hat{k}$$

[At this point, it is a good idea to check our algebra by substituting the values of $\dot{\theta}$ and $\dot{\phi}$ in equations (18.50) and (18.51) to calculate $\vec{v}_B$.] ②

3. Now we can calculate the velocity of the center-of-mass of the rod by considering either point A or point B as a reference:

$$\begin{aligned}\vec{v}_G &= \vec{v}_A + \vec{v}_{G/A} \\ &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{G/A} \\ &= -v_A \hat{i} + \dot{\theta} \hat{k} \times \frac{\ell}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= -(v_A + \dot{\theta} \frac{\ell}{2} \sin \theta) \hat{i} + \dot{\theta} \frac{\ell}{2} \cos \theta \hat{j} \\ &= -(2 \text{ m/s} - 0.5 \text{ rad/s} \cdot \frac{2 \text{ m}}{2} \cdot \frac{1}{2}) \hat{i} + (-0.5 \text{ rad/s} \cdot \frac{2 \text{ m}}{2} \cdot \frac{\sqrt{3}}{2}) \hat{j} \\ &= -\frac{7}{4} \text{ m/s} \hat{i} - \frac{\sqrt{3}}{4} \text{ m/s} \hat{j}.\end{aligned}$$

$$\vec{v}_G = -(1.75 \hat{i} + 0.43 \hat{j}) \text{ m/s}$$

We could easily check our calculation by taking point B as a reference and writing

$$\begin{aligned}\vec{v}_G &= \vec{v}_B + \vec{v}_{G/B} \\ &= \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{G/B}\end{aligned}$$

By plugging in appropriate values we get, of course, the same value as above.

Comment: We used the standard basis vectors $\hat{i}$ and $\hat{j}$ for all our vector calculations in this sample. We can shorten these calculations by choosing other appropriate basis vectors as we show in the following samples.

② Substituting $\dot{\theta} = -0.5 \text{ rad/s}$ in Eqn. (18.50) and plugging in the given values of other variables we get

$$\vec{v}_B = -(\frac{3}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j}) \text{ m/s}.$$

Similarly, substituting $\dot{\phi} = 2\sqrt{3} \text{ rad/s}$ in Eqn. (18.50) and plugging in the other given values we get

$$\vec{v}_B = -(\frac{3}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j}) \text{ m/s},$$

which checks with the $\vec{v}_B$ found above.

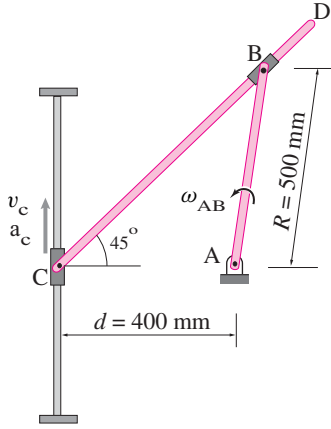


Figure 18.61

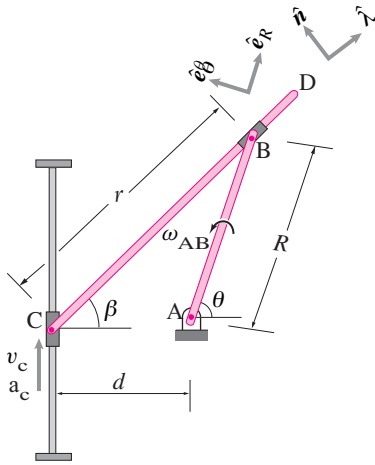


Figure 18.62

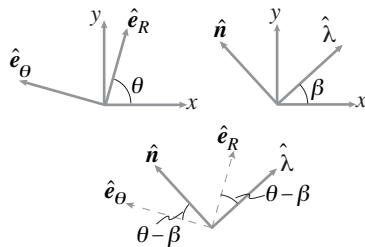


Figure 18.63

SAMPLE 18.14 A two-DOF mechanism. A two degree-of-freedom mechanism made of three rods and two sliders is shown in the figure. At the instant shown, the crank AB is rotating with angular velocity $\bar{\omega}_{AB} = 12 \text{ rad/s} \hat{k}$ and angular acceleration $\dot{\bar{\omega}}_{AB} = 10 \text{ rad/s}^2 \hat{k}$. At the same instant, the collar at end C of the link rod CD is sliding on the vertical rod with velocity $\bar{v}_C = 0.5 \text{ m/s} \hat{j}$ and acceleration $\bar{a}_C = 10 \text{ m/s}^2 \hat{j}$. Find the angular velocity and angular acceleration of the link rod CD.

Solution Once again, we are interested in instantaneous kinematics — we wish to find the angular velocity and acceleration of rod CD at the given instant. This problem is just like the previous sample problem except that end C of the link rod CD is not fixed but free to slide on the vertical bar. But the velocity and acceleration of point C is given; so it is exactly like the previous sample (there, the velocity and acceleration of point C was identically zero). So, we adopt the same line of attack. We figure out the velocity and acceleration of point B using the kinematics of rod AB. We then write the velocity and acceleration of the same point using the kinematics of rod CD (this will involve the unknown angular velocity and acceleration of CD that we are interested in). Equate the two and solve for the unknowns we are interested in.

Let the angular velocity and acceleration of rod CD be $\bar{\omega}_{CD} = \beta \hat{k}$ and $\dot{\bar{\omega}}_{CD} = \beta \dot{\hat{k}}$, respectively. Let $\hat{e}_R$ and $\hat{e}_\theta$ be base vectors rotating with rod AB, and $\hat{\lambda}$ and $\hat{n}$ be the base vectors rotating with rod CD (see fig. 18.62). Considering rod AB, we have

$$\bar{v}_B = R \dot{\theta} \hat{e}_\theta \quad (18.54)$$

$$\bar{a}_B = -R \dot{\theta}^2 \hat{e}_R + R \ddot{\theta} \hat{e}_\theta. \quad (18.55)$$

Considering rod CD, we have

$$\bar{v}_B = \bar{v}_C + \bar{v}_{B/C} = v_C \hat{j} + \dot{r} \hat{\lambda} + r \dot{\beta} \hat{n} \quad (18.56)$$

$$\bar{a}_B = \bar{a}_C + \bar{a}_{B/C} = a_C \hat{j} + (\ddot{r} - r \dot{\beta}^2) \hat{\lambda} + (2\dot{r} \dot{\beta} + r \ddot{\beta}) \hat{n}. \quad (18.57)$$

Now equating eqn. (18.54) and (18.56), and dotting both sides with $\hat{\lambda}$ and $\hat{n}$, we get

$$\dot{r} = R \dot{\theta} (\hat{e}_\theta \cdot \hat{\lambda}) - v_C (\hat{j} \cdot \hat{\lambda}) = -R \dot{\theta} \sin(\theta - \beta) - v_C \sin \beta \quad (18.58)$$

$$r \dot{\beta} = R \dot{\theta} (\hat{e}_\theta \cdot \hat{n}) - v_C (\hat{j} \cdot \hat{n}) = R \dot{\theta} \cos(\theta - \beta) - v_C \cos \beta \quad (18.59)$$

where the dot products among the basis vectors are easily found from either their geometry (see fig. 18.63) or from their component representation (see previous sample). Following exactly the same procedure, we get, from eqn. (18.55) and 18.57,

$$\ddot{r} = -R \dot{\theta}^2 \cos(\theta - \beta) + R \ddot{\theta} (-\sin(\theta - \beta)) - a_C \sin \beta + r \dot{\beta}^2 \quad (18.60)$$

$$r \ddot{\beta} = -R \dot{\theta}^2 \sin(\theta - \beta) + R \ddot{\theta} \cos(\theta - \beta) - a_C \cos \beta - 2\dot{r} \dot{\beta}. \quad (18.61)$$

Now, note that although we are only interested in finding $\dot{\beta}$ and $\ddot{\beta}$. So, we only need eqn. (18.59) and eqn. (18.61). But, eqn. (18.61) requires $\ddot{r}$ on the right hand side and, therefore, we do need eqn. (18.58). We can, however, happily ignore eqn. (18.60).

Now, to find the numerical values of $\dot{\beta}$ and $\ddot{\beta}$, we need to find r and θ in addition to all other given values. Consider triangle ABC in fig. 18.62. Using the law of sines ($\frac{R}{\sin \beta} = \frac{r}{\sin \theta} = \frac{d}{\sin(\theta - \beta)}$), we get $r = 0.7 \text{ m}$ and $\theta = 79.45^\circ$. Now, substituting all known numerical values in eqns. (18.58), (18.59), and (18.61), we get

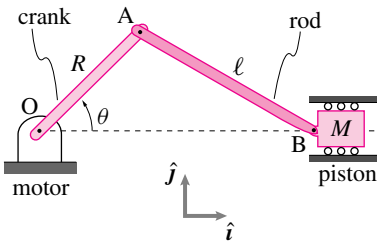
$$\dot{r} = -3.75 \text{ m/s}, \quad \dot{\beta} = 6.61 \text{ rad/s}, \quad \ddot{\beta} = 8.43 \text{ rad/s}^2.$$

$$\bar{\omega}_{CD} = (6.61 \text{ rad/s}) \hat{k}, \quad \dot{\bar{\omega}}_{CD} = (8.43 \text{ rad/s}^2) \hat{k}$$

18.4 Kinematics of 2-D mechanisms

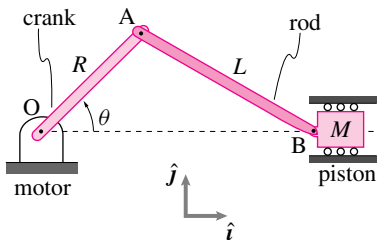
18.4.1 Slider crank kinematics (No FBD required!). 2-D. Assume $R, \ell, \theta, \dot{\theta}, \ddot{\theta}$ are given. The crank mechanism parts move on the xy plane with the x direction being along the piston. Vectors should be expressed in terms of $\hat{i}, \hat{j}$, and $\hat{k}$ components.

- What is the angular velocity of the crank OA? *
- What is the angular acceleration of the crank OA? *
- What is the velocity of point A? *
- What is the acceleration of point A? *
- What is the angular velocity of the connecting rod AB? [Geometry fact: $\vec{r}_{AB} = \sqrt{\ell^2 - R^2 \sin^2 \theta} \hat{i} - R \sin \theta \hat{j}$] *
- For what values of θ is the angular velocity of the connecting rod AB equal to zero (assume $\dot{\theta} \neq 0$)? (you need not answer part (e) correctly to answer this question correctly.) *



Problem 18.4.1

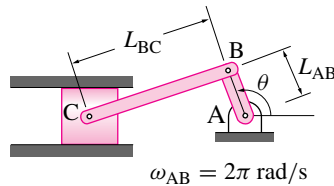
18.4.2 Slider-Crank. Consider a slider-crank mechanism. Given $\theta, \dot{\theta}, \ddot{\theta}, L$, and R , can you find the velocity and acceleration of B? There are many ways to do this problem.



Problem 18.4.2

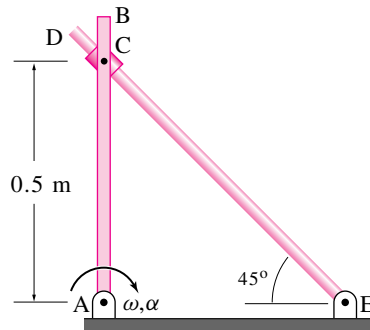
18.4.3 The crank AB with length $L_{AB} = 1$ inch in the crank mechanism shown rotates at a constant rate $\dot{\theta}_{AB} = \omega_{AB} = 2\pi$ rad/s counter-clockwise. The initial angle of rotation is $\theta = 0$ at $t = 0$. The connecting rod BC has a length of $L_{BC} = 2$ in.

- What is the velocity of point B at the end of the crank when $\theta = \pi/2$ rad?
- What is the velocity of point C at the end of the connecting rod when $\theta = \pi/2$ rad?
- What is the angular velocity of the connecting rod BC when $\theta = \pi/2$ rad?
- What is the angular velocity of the connecting rod BC as a function of time?



Problem 18.4.3

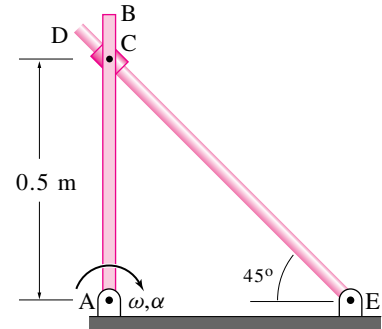
18.4.4 The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3$ rad/s and angular acceleration $\alpha = 2$ rad/s². Find the angular velocity of rod DE. *



Problem 18.4.4

18.4.5 Reconsider problem 18.4.4. The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless

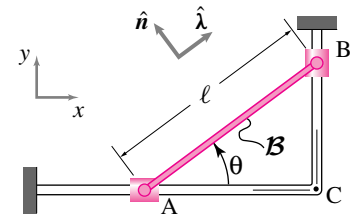
rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3$ rad/s and angular acceleration $\alpha = 2$ rad/s². Find the angular acceleration of rod DE. *



Problem 18.4.5

18.4.6 Collar A is constrained to slide on a horizontal rod to the right at constant speed v_A . It is connected by a pin joint to one end of a rigid bar AB with length ℓ which makes an angle θ with the horizontal at the instant of interest. A second collar B connected by a pin joint to the other end of the rigid bar AB slides on a vertical rod.

- Find the velocity of point B. Answer in terms of v_A, θ, ℓ , and $\hat{j}$.
- Find the angular speed $\dot{\theta}$ of the rod? Answer in terms of v_A, θ, ℓ , and $\hat{j}$.

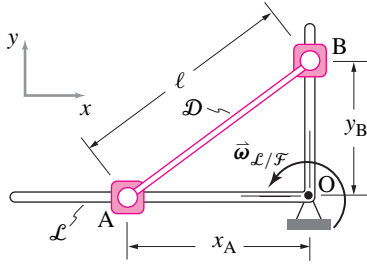


Problem 18.4.6

18.4.7 A bar of length $\ell = 5$ ft, body $\mathcal{D}$, connects sliders A and B on an L-shaped frame, body $\mathcal{L}$, which itself is rotating at constant speed about an axle perpendicular to the plane of the figure through the point O and relative to a fixed frame $\mathcal{F}$, $\dot{\omega}_{\mathcal{L}/\mathcal{F}} = 0.5$ rad/sk. At the instant shown, body $\mathcal{L}$ is aligned with the $x - y$ axes, slider A is $x_A = 4$ ft from point O, slider B is $y_B = 3$ ft from O. The speed and acceleration of slider B relative to the frame $\mathcal{L}$ are $\vec{v}_{B/\mathcal{L}} = -2$ ft/s $\hat{j}$ and $\vec{a}_{B/\mathcal{L}} = -2$ ft/s² $\hat{j}$, respectively.

Determine:

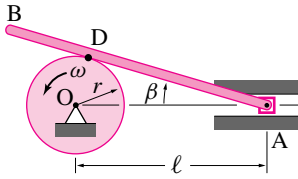
- a) The absolute velocity of the slider A, $\vec{v}_{A/\mathcal{F}}$, and
 b) The absolute angular velocity of bar AB, body $\mathcal{D}$, $\vec{\omega}_{\mathcal{D}/\mathcal{F}}$.



Problem 18.4.7

18.4.8 The link AB is supported by a wheel at D and its end A is constrained so that it only has horizontal velocity. No slipping occurs between the wheel and the link. The wheel has an angular velocity ω and radius r . The distance $OA = \ell$.

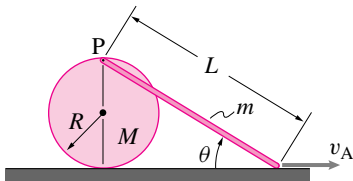
Given: ω , r , and ℓ . ($\beta = \sin^{-1} \frac{r}{\ell}$). Determine the angular velocity of the link AB and velocity of the point A.



Problem 18.4.8

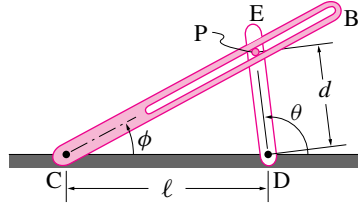
18.4.9 A solid cylinder of radius R and mass M rolls without slip along the ground. A thin rod of mass m and length L is attached by a frictionless pin P to the cylinder's rim and its right end is dragged at a constant speed $\vec{v}_A$ along the (frictionless) horizontal ground.

- a) For the position shown (where P is directly above the contact point), find P's velocity $\vec{v}_P$ and the rod's angular velocity $\vec{\omega}$.
 b) Find P's acceleration $\vec{a}_P$ and the angular acceleration $\vec{\alpha}$ of the rod.



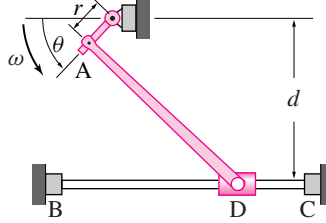
Problem 18.4.9

18.4.10 The slotted link CB is driven in an oscillatory motion by the link ED which rotates about D with constant angular velocity $\dot{\theta} = \omega_D$. The pin P is attached to ED at fixed radius d and engages the slot on CB as shown. Find the angular velocity and acceleration $\dot{\phi}$ and $\ddot{\phi}$ of CB when $\theta = \pi/2$.



Problem 18.4.10

18.4.11 The rod of radius $r = 50$ mm shown has a constant angular velocity of $\omega = 30$ rad/s counterclockwise. Knowing that rod AD is 250 mm long and distance $d = 150$ mm, determine the acceleration of collar D when $\theta = 90^\circ$.

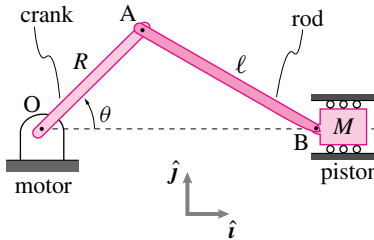


Problem 18.4.11

18.4.1 Slider crank kinematics (No FBD required!). 2-D. Assume $R, \ell, \theta, \dot{\theta}, \ddot{\theta}$ are given. The crank mechanism parts move on the xy plane with the x direction being along the piston. Vectors should be expressed in terms of $\hat{i}, \hat{j}$, and $\hat{k}$ components.

- What is the angular velocity of the crank OA?
- What is the angular acceleration of the crank OA?
- What is the velocity of point A?
- What is the acceleration of point A?
- What is the angular velocity of the connecting rod AB? [Geometry fact: $\vec{r}_{AB} = \sqrt{\ell^2 - R^2 \sin^2 \theta} \hat{i} - R \sin \theta \hat{j}$]
- For what values of θ is the angular velocity of the connecting rod

AB equal to zero (assume $\dot{\theta} \neq 0$)? (you need not answer part (e) correctly to answer this question correctly.)



Problem 18.1

Answer:

- $\vec{\omega}_{OA} = \dot{\theta} \hat{k}$.
- $\vec{\alpha}_{OA} = \ddot{\theta} \hat{k}$.
- $\vec{v}_A = -R\dot{\theta} \sin \theta \hat{i} + R\dot{\theta} \cos \theta \hat{j}$.
- $\vec{a}_A = -(R\ddot{\theta} \cos \theta + R\dot{\theta}^2 \sin \theta) \hat{i} + (R\ddot{\theta} \sin \theta - R\dot{\theta}^2 \cos \theta) \hat{j}$.
- $\vec{\omega}_{AB} = \frac{-R\dot{\theta} \cos \theta}{\sqrt{\ell^2 - R^2 \sin^2 \theta}} \hat{k}$.
- $\theta = (2n + 1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \dots$

7.5 Slider crank. p.6

Given $R, \ell, \theta, \dot{\theta}, \ddot{\theta}$

(a) Find ω_{OA} . $\omega_{OA} = \dot{\theta} \hat{k}$

(b) Find α_{OA} . $\alpha_{OA} = \ddot{\theta} \hat{k}$

(c) Find $\vec{v}_A$: $\vec{v}_A = \vec{\omega}_{OA} \times \vec{r}_{AO} = \dot{\theta} \hat{k} \times (R \cos \theta \hat{i} + R \sin \theta \hat{j}) = R\dot{\theta}(\cos \theta \hat{j} - \sin \theta \hat{i}) = \vec{v}_A$

(d) $\vec{a}_A = \dot{\vec{v}}_A = \dot{\vec{\omega}}_{OA} \times \vec{r}_{AO} + \vec{\omega}_{OA} \times (\vec{\omega}_{OA} \times \vec{r}_{AO})$
 $\vec{a}_A = \ddot{\theta} \hat{k} \times (R \cos \theta \hat{i} + R \sin \theta \hat{j}) - \dot{\theta}^2 (R \cos \theta \hat{i} + R \sin \theta \hat{j})$
 $\vec{a}_A = -R(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \hat{i} + R(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \hat{j}$

(e) Find ω_{AB} . Note $\omega_{AB} = \omega_{AB} \hat{k}$, $\vec{v}_B = \vec{v}_A + \omega_{AB} \times \vec{r}_{AB}$ since pt. B constrained to move horizontally only.
 $\vec{v}_B \hat{i} = \omega_{AB} \hat{k} \times (\sqrt{\ell^2 - R^2 \sin^2 \theta} \hat{i} - R \sin \theta \hat{j}) + R\dot{\theta}(\cos \theta \hat{j} - \sin \theta \hat{i})$
 $\vec{v}_B \hat{i} = -R\omega_{AB} \sin \theta \hat{i} - R\dot{\theta} \sin \theta \hat{i} + \omega_{AB} \sqrt{\ell^2 - R^2 \sin^2 \theta} \hat{j} + R\dot{\theta} \cos \theta \hat{j}$
 dot the above w/ $\hat{i}$ & $\hat{j}$:
 $v_B = -R\omega_{AB} \sin \theta - R\dot{\theta} \sin \theta$
 $0 = \omega_{AB} \sqrt{\ell^2 - R^2 \sin^2 \theta} + R\dot{\theta} \cos \theta$
 so, from the 2nd eqn, $\omega_{AB} = \frac{-R\dot{\theta} \cos \theta}{\sqrt{\ell^2 - R^2 \sin^2 \theta}} \hat{k}$

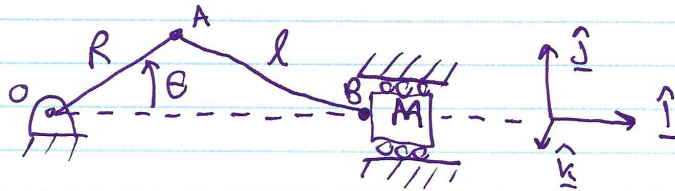
(f) $\omega_{AB} = 0$ at $\theta = \pi/2, 3\pi/2, \dots$ because the connecting rod is undergoing strictly linear motion. ($\vec{v}_A + \vec{v}_B$ both in x -direction)

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

18.4.1

M.R. Buche 5/6/21

Given $R, l, \theta, \dot{\theta}, \ddot{\theta}$ for a crank mechanism in 2D.

a) Find $\underline{\omega}_{OA}$.

$$\underline{\omega}_{OA} = \dot{\theta} \hat{k}$$

b) Find $\underline{\alpha}_{OA}$.

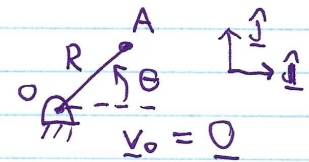
$$\underline{\alpha}_{OA} = \ddot{\theta} \hat{k}$$

c) Find $\underline{v}_A$.

$$\underline{v}_A = \underline{\omega}_{OA} \times \underline{r}_{A/O}$$

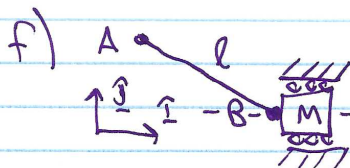
$$\underline{r}_{A/O} = R (\hat{i} \cos \theta + \hat{j} \sin \theta)$$

$$\underline{v}_A = R \dot{\theta} (\hat{j} \cos \theta - \hat{i} \sin \theta)$$

d) Find $\underline{a}_A$.

$$\underline{a}_A = \underline{\alpha}_{OA} \times \underline{r}_{A/O} - \omega_{OA}^2 \underline{r}_{A/O}$$

$$\underline{a}_A = R \ddot{\theta} (\hat{j} \cos \theta - \hat{i} \sin \theta) - R \dot{\theta}^2 (\hat{i} \cos \theta + \hat{j} \sin \theta)$$



Since $\underline{v}_B \cdot \hat{j} = 0$,
we have $\underline{v}_{AB} = 0$
when $\underline{v}_A \cdot \hat{j} = 0$

$$\underline{v}_A \cdot \hat{j} = R \dot{\theta} \cos \theta$$

Assuming $\dot{\theta} \neq 0$ and $R > 0$,
 $\underline{v}_{AB} = 0$ when $\theta = \pi/2$ and $3\pi/2$.

Page 1 of 2

18.4.1

M. R. Buche 5/6/21

e) Find $\underline{\omega}_{AB}$, use $\underline{r}_{B/A} = \hat{i}\sqrt{l^2 - R^2 \sin^2 \theta} - \hat{j}R \sin \theta$.

We know $\underline{v}_A$ (part c) and $\underline{v}_B = |\underline{v}_B| \hat{i}$.

Also: $\underline{v}_B = \underline{v}_A + \underline{\omega}_{AB} \times \underline{r}_{B/A}$, $\underline{\omega}_{AB} = |\underline{\omega}_{AB}| \hat{k}$

Laurie Anderson: "let $x = x$ " ($\underline{v}_B = \underline{v}_B$),

$$|\underline{v}_B| \hat{i} = \underline{v}_A + \underline{\omega}_{AB} \times \underline{r}_{B/A}$$

↳ does this spark joy? (dot with $\hat{j}$)

$$\underline{v}_A \cdot \hat{j} = R \dot{\theta} \cos \theta$$

$$\underline{\omega}_{AB} \times \underline{r}_{B/A} = |\underline{\omega}_{AB}| \left(\hat{i} R \sin \theta + \hat{j} \sqrt{l^2 - R^2 \sin^2 \theta} \right)$$

$$(\underline{\omega}_{AB} \times \underline{r}_{B/A}) \cdot \hat{j} = |\underline{\omega}_{AB}| \sqrt{l^2 - R^2 \sin^2 \theta}$$

$$\text{Solving } \underline{v}_A \cdot \hat{j} + (\underline{\omega}_{AB} \times \underline{r}_{B/A}) \cdot \hat{j} = 0,$$

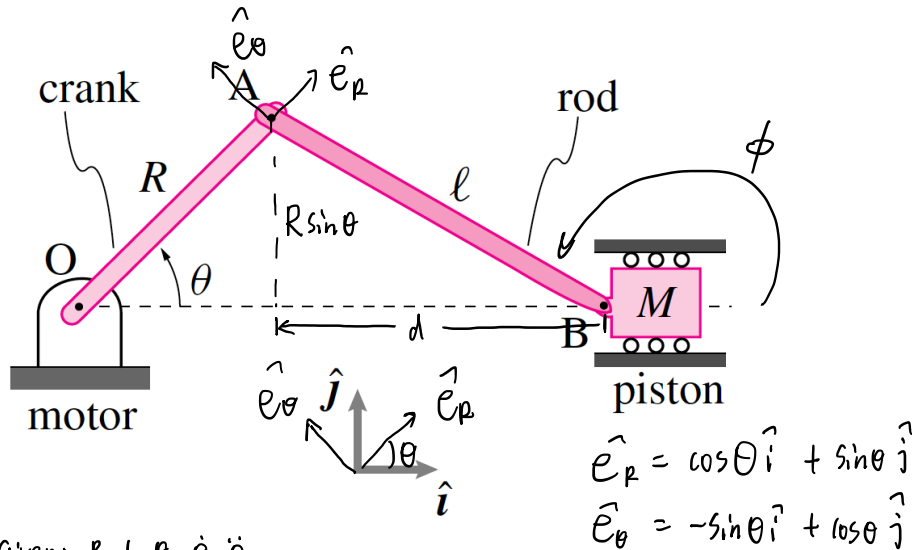
we see

$$\underline{\omega}_{AB} = \frac{-\dot{\theta} \cos \theta \hat{k}}{\sqrt{(l/R)^2 - \sin^2 \theta}}$$

Observe that for $l=R$, we receive singular configurations (also called dead positions) at $\theta=0$ and π . More generally, we receive them at $\pi - \arcsin(l/R)$ for $l \leq R$.

17.4.1

Saturday, May 4, 2019 5:29 PM

Given: $R, l, \theta, \dot{\theta}, \ddot{\theta}$

Find :

a) angular velocity of OA

$$\vec{\omega}_{OA} = \dot{\theta} \hat{k}$$

b) angular acceleration of OA

$$\vec{\alpha}_{OA} = \ddot{\theta} \hat{k}$$

c) velocity of A

$$\vec{v}_A = \dot{\theta} R \hat{e}_\theta = \dot{\theta} R (-\sin\theta \hat{i} + \cos\theta \hat{j})$$

d) Acceleration of A

$$\begin{aligned} \vec{a}_A &= \dot{\vec{v}}_A = \ddot{\theta} R \hat{e}_\theta - \dot{\theta}^2 R \hat{e}_\rho \\ &= \ddot{\theta} R (-\sin\theta \hat{i} + \cos\theta \hat{j}) - \dot{\theta}^2 R (\cos\theta \hat{i} + \sin\theta \hat{j}) \\ &= (-\ddot{\theta} R \sin\theta - \dot{\theta}^2 R \cos\theta) \hat{i} + (\ddot{\theta} R \cos\theta - \dot{\theta}^2 R \sin\theta) \hat{j} \end{aligned}$$

e) angular velocity of AB ?

$$\vec{\omega}_{AB} = \omega_{AB} \hat{k}$$

$$\vec{v}_A = \vec{v}_B$$

$$\begin{aligned}
 & \left\{ \begin{aligned} \dot{\theta} \hat{k} \hat{e}_\theta &= V_B \hat{i} + \dot{\phi} \hat{k} \times \vec{r}_{BA} \\ \Rightarrow \underbrace{\dot{\theta} R \hat{e}_\theta \cdot \hat{j}}_{\cos \theta} &= 0 + (\dot{\phi} \hat{k} \times \vec{r}_{BA}) \cdot \hat{j} \end{aligned} \right. \\
 & \quad \quad \quad \left\{ \begin{aligned} \vec{r}_{BA} &= -d \hat{i} + R \sin \theta \hat{j} \\ & \quad \quad \quad \sqrt{l^2 - R^2 \sin^2 \theta} \end{aligned} \right. \\
 & \Rightarrow \dot{\phi} = \frac{-\dot{\theta} R \cos \theta}{\sqrt{l^2 - R^2 \sin^2 \theta}} \\
 & \quad \quad \quad \vec{\omega}_{AB} = \dot{\phi} \hat{k} \\
 & \left\{ \begin{aligned} \dot{\theta} R (-\sin \theta \hat{i} + \cos \theta \hat{j}) &= \vec{V}_B - \omega_{AB} \sqrt{l^2 - R^2 \sin^2 \theta} \hat{j} - \omega_{AB} R \sin \theta \hat{i} \end{aligned} \right. \\
 & \{ \} \cdot \hat{j} \Rightarrow \dot{\theta} R \cos \theta = -\omega_{AB} \sqrt{l^2 - R^2 \sin^2 \theta} \\
 & \Rightarrow \omega_{AB} = -\frac{\dot{\theta} R \cos \theta}{\sqrt{l^2 - R^2 \sin^2 \theta}} \\
 & \therefore \vec{\omega}_{AB} = -\frac{\dot{\theta} R \cos \theta}{\sqrt{l^2 - R^2 \sin^2 \theta}} \hat{k}
 \end{aligned}$$

f) $\theta = ?$ makes the angular velocity of AB equal to zero ($\dot{\theta} \neq 0$)

$$\theta = \pm 90^\circ.$$

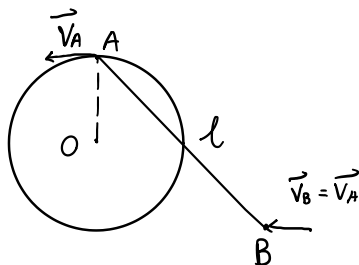
physically, when $\theta = \pm 90^\circ$, $\vec{V}_A = \vec{\omega}_{AB} \times \vec{r}_{AB} \parallel \hat{i}$

thus also parallel to the velocity of $\vec{V}_B$.

Thus at this instant, the rod AB is not rotating, just translating. (No need to get (e) correctly).

Mathematically from part e)

$$\text{Set } \omega_{AB} = 0 \Rightarrow \cos \theta = 0 \quad \theta = \pm 90^\circ.$$



Sec 205

You Won Park

TAM 2030

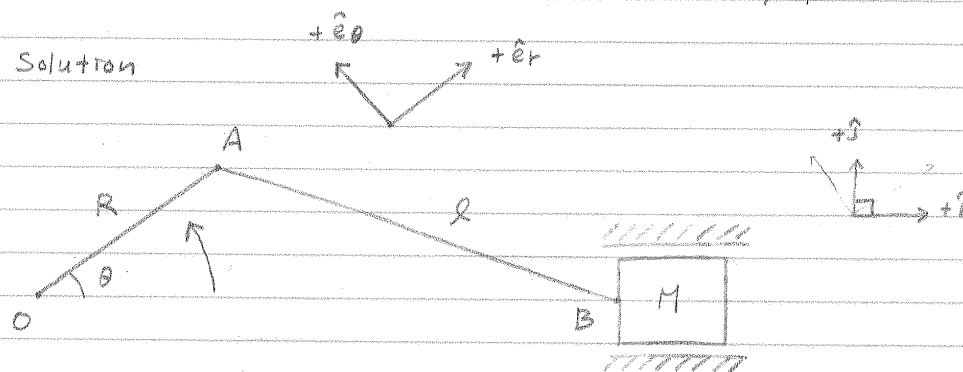
10:10 AM Sec 205

TA: Pranav Bhounsule

HW 22, Apr. 14, 2009

15.29

Solution

 $R, l, \theta, \dot{\theta}, \ddot{\theta}$ given

a) Angular velocity of OA?

$$\vec{\omega}_{OA} = \dot{\theta} \hat{k}$$

b) Angular acceleration of OA?

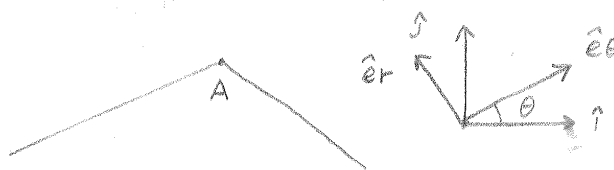
$$\vec{\alpha}_{OA} = \ddot{\theta} \hat{k}$$

c) Velocity of A?

$$\vec{v}_A = \dot{\theta} R \hat{e}_\theta = \dot{\theta} R (-\sin \theta \hat{i} + \cos \theta \hat{j})$$

d) Acceleration of A?

$$\vec{a}_A = \dot{\vec{v}}_A = (-\ddot{\theta} R \sin \theta - \dot{\theta}^2 R \cos \theta) \hat{i} + (\ddot{\theta} R \cos \theta - \dot{\theta}^2 R \sin \theta) \hat{j}$$



e) Angular velocity of AB?

$$\vec{r}_{AB} = \sqrt{L^2 - R^2 \sin^2 \theta} \hat{i} - R \sin \theta \hat{j}$$

$$\vec{v}_A = \vec{r}_{AB} \times \vec{\omega}_{AB}$$

$$\vec{v}_A = (\sqrt{L^2 - R^2 \sin^2 \theta} \hat{i} - R \sin \theta \hat{j}) \times \omega_{AB} \hat{k}$$

$$\{-R\dot{\theta} \sin \theta \hat{i} + R\dot{\theta} \cos \theta \hat{j}\} = -\omega_{AB} \sqrt{L^2 - R^2 \sin^2 \theta} \hat{j}$$

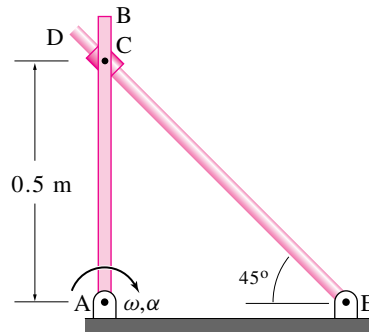
$$-R\dot{\theta} \sin \theta \hat{i} + R\dot{\theta} \cos \theta \hat{j} = -\omega_{AB} \sqrt{L^2 - R^2 \sin^2 \theta} \hat{j}$$

$$\Rightarrow R\dot{\theta} \cos \theta = -\omega_{AB} \sqrt{L^2 - R^2 \sin^2 \theta}$$

$$\therefore \vec{\omega}_{AB} = -\frac{R\dot{\theta} \cos \theta}{\sqrt{L^2 - R^2 \sin^2 \theta}} \hat{k}$$

f) for $\theta = \pm 90^\circ$

18.4.4 The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3 \text{ rad/s}$ and angular acceleration $\alpha = 2 \text{ rad/s}^2$. Find the angular velocity of rod DE.



Problem 18.4

Answer:

$$\bar{\omega}_{DE} = -\frac{\omega}{2} \hat{k} = -1.5 \text{ rad/s} \hat{k}$$

8.22 Given: $\omega_{AB} = 3 \text{ rad/s}$,
Find ω_{DE} .

First, find $\underline{v}_C$ from motion of rod AB:
 $\underline{v}_C = \omega_{AB} r_{CA} \hat{i}$ (1)

Now, attach a frame B to rod DE. This frame rotates with rod DE.
 Choose a coordinate system $x'y'z'$ for this rod with its origin at pt. E. The $x'y'z'$ coordinate system is aligned with the xyz coordinate system at this instant in time.

Then: $\underline{v}_C = \underline{v}_C' + \underline{v}_{rel}$
 where $\underline{v}_{rel} = v_{C/E} \hat{\lambda}_{CE} = v_{rel} \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}} \right)$ } we know the direction of $\underline{v}_{rel}$ since the collar slides along the rod.

$\underline{v}_C' = \underline{v}_{C'/E} + \omega_B \times \underline{r}_{C'/E}$
 O: pt. E' does not move relative to the fixed point E
 $\Rightarrow \underline{v}_C' = \omega_B \hat{k} \times (-0.5\text{m} \hat{i} + 0.5\text{m} \hat{j})$
 thus $\underline{v}_C = (-0.5\text{m})\omega_B (\hat{j} + \hat{i}) + \frac{v_{rel}}{\sqrt{2}} (\hat{i} - \hat{j})$ (2)

now: $\underline{v}_C = \underline{v}_C$ (equate (1) & (2)) continued

8.22 continued p.10

$\omega_{AB} r_{CA} \hat{i} = (-0.5\text{m})\omega_B (\hat{j} + \hat{i}) + \frac{v_{rel}}{\sqrt{2}} (\hat{i} - \hat{j})$ (3)

$\hat{i} \cdot (3) \Rightarrow 1.5 \text{ m/s} = -0.5\text{m}(\omega_B) + \frac{v_{rel}}{\sqrt{2}}$ (4)

$\hat{j} \cdot (3) \Rightarrow 0 = -0.5\text{m}(\omega_B) - \frac{v_{rel}}{\sqrt{2}}$ (5)

(4) + (5) $\Rightarrow 1.5 \text{ m/s} = -1\text{m}(\omega_B) \Rightarrow \omega_B = -1.5 \text{ rad/s}$
 thus, since $\omega_B = \omega_{DE}$, $\omega_{DE} = -1.5 \text{ rad/s} \hat{k}$

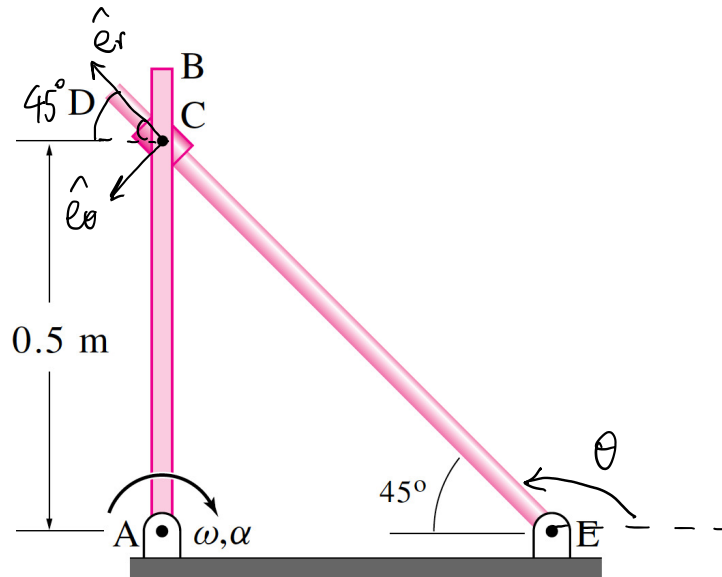
(Note: (4)-(5) $\Rightarrow 1.5 \text{ m/s} = \frac{2v_{rel}}{\sqrt{2}} \Rightarrow v_{rel} = 0.75\sqrt{2} \text{ m/s}$
 $\Rightarrow \underline{v}_{rel} = v_{rel} \hat{\lambda}_{CE} = 0.75\sqrt{2} \text{ m/s} \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}} \right) = 0.75 \text{ m/s} (\hat{i} - \hat{j}) = \underline{v}_{rel}$ * (used in prob 8.30)

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

17.4.4

Saturday, May 4, 2019

6:11 PM



C is pinned to AB but free on DE .

At this instant, given:

angular velocity of AB : $\vec{\omega}_{AB} = \omega \hat{k}$ $\omega = -3 \text{ rad/s}$

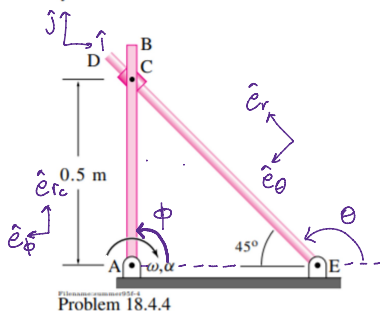
angular acceleration of AB: $\vec{\alpha}_{AB} = \alpha \hat{k}$ $\alpha = -2 \text{ rad/s}^2$

Find: angular velocity of DE. $\vec{\omega}_{DE} = \dot{\theta} \hat{k}$

$$\begin{aligned} \vec{v}_C &= \vec{v}_E \\ \{\vec{\omega}_{AB} \times \vec{r}_{C/A} &= \vec{\omega}_{DE} \times \vec{r}_{D/E} + v_{Cr} \hat{e}_r\} \\ \{ \} \cdot \hat{e}_\theta &\Rightarrow \omega \hat{k} \times 0.5 \hat{j} \cdot \hat{e}_\theta = (\dot{\theta} \hat{k} \times \frac{0.5}{\sin 45^\circ} \hat{e}_r + v_{Cr} \hat{e}_r) \cdot \hat{e}_\theta \\ 0.5 \omega \cdot \sin 45^\circ &= \dot{\theta} \frac{0.5}{\sin 45^\circ} \\ \Rightarrow \dot{\theta} &= \frac{1}{2} \omega = -1.5 \text{ [rad/s]} \\ \therefore \vec{\omega}_{DE} &= \dot{\theta} \hat{k} = -1.5 \hat{k} \text{ [rad/s]} \end{aligned}$$

#69 Solution Michelle deSouza

18.4.4 The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3 \text{ rad/s}$ and angular acceleration $\alpha = 2 \text{ rad/s}^2$. Find the angular velocity of rod DE.



Problem 18.4.4

Given: $r_{C/A} = 0.5 \text{ m}$

Find: ω_{DE}

$$\vec{\omega}_{AB} = -3\hat{k} \text{ rad/s} = -\omega_{AB}\hat{k}$$

$$\vec{\alpha}_{AB} = -2\hat{k} \text{ rad/s}^2 = -\alpha_{AB}\hat{k}$$

$$\theta = 135^\circ$$

$$\phi = 90^\circ$$

Instantaneous unit vectors:

$$\hat{e}_r = \cos\phi\hat{i} + \sin\phi\hat{j} = \hat{j}$$

$$\hat{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j} = -\hat{i}$$

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

Note: $\dot{\phi} = -\omega$, $\ddot{\phi} = -\alpha$

(ω_{AB}, α_{AB} defined as CW+. We will use $\dot{\phi}$ defined CCW+, so $\vec{\omega}_{AB} = -\omega_{AB}\hat{k} = \dot{\phi}\hat{k}$.)

From AB: C is pinned to AB

From DE: C is free to slide, $r_{C/D} \neq \text{const.}$

Rigid object kinematics:

$$\vec{v}_C = \vec{v}_A + \vec{v}_{C/A}$$

$$\vec{v}_C = \vec{v}_A + \dot{\phi}\hat{k} \times \vec{r}_{C/A}$$

$$\vec{v}_C = -\omega_{AB}\hat{k} \times r_{C/A}\hat{e}_r$$

$$\vec{v}_C = -\omega_{AB}r_{C/A}\hat{e}_\phi$$

$$\vec{r}_C = \vec{r}_E + r_{C/E}\hat{e}_r$$

$$\vec{v}_C = \vec{v}_E + \dot{r}_{C/E}\hat{e}_r + r_{C/E}\omega_{DE}\hat{e}_\theta$$

$$\vec{v}_C = \vec{v}_C$$

$$-\omega_{AB}r_{C/A}\hat{e}_\phi = \dot{r}_{C/E}\hat{e}_r + r_{C/E}\omega_{DE}\hat{e}_\theta \quad \text{dot with } \hat{e}_\theta \text{ to isolate } \omega_{DE}$$

$$\hat{e}_\phi \cdot \hat{e}_\theta - \omega_{AB}r_{C/A}(\hat{e}_\phi \cdot \hat{e}_\theta) = r_{C/E}\omega_{DE}$$

$$\omega_{DE} = -\omega_{AB} \frac{r_{C/A}}{r_{C/E}} (\hat{e}_\phi \cdot \hat{e}_\theta)$$

$$r_{C/E} = \frac{r_{C/A}}{\sin\theta}$$

$$\hat{e}_\phi \cdot \hat{e}_\theta = -\hat{i} \cdot \hat{e}_\theta = \sin\theta$$

$$\omega_{DE} = -\omega_{AB} \frac{r_{C/A}}{r_{C/A}/\sin\theta} \sin\theta = -\omega_{AB} \sin^2\theta = -3\left(\frac{1}{2}\right) = -1.5 \text{ rad/s}$$

$$\omega_{DE} = \omega_{AB} \frac{r_{C/A}}{r_{C/E}} (\hat{e}_\phi \cdot \hat{e}_\theta) \quad |\omega_{DE}| \text{ trends:}$$

- ↑ for more // rods
- ↑ for higher ω_{AB}
- ↑ for longer $r_{C/A}$
- ↓ for longer $r_{C/D}$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

15.32, 15.38, 16.1, 16.12, 16.21

Kim Dunphy

HW #1-5 due 12/2/08

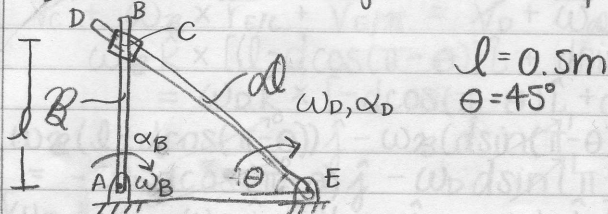
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Section 2 at 1:25

TA: Rong Long

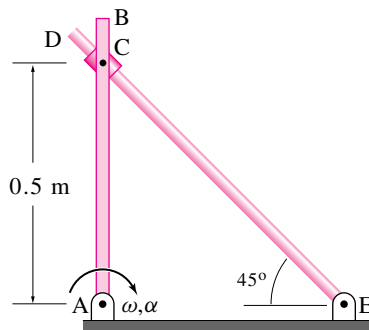
1) 15.32 → SOLUTION

Two rods AB & DE connected at collar C (pinned to AB, can slide on DE frictionlessly); AB rotating CCW, $\omega_B = 3 \text{ rad/s}$, $\alpha_B = 2 \text{ rad/s}^2$, find angular velocity of DE



$$\begin{aligned} \vec{V}_C &= \vec{V}_C \\ \vec{V}_A + \vec{\omega}_B \times \vec{r}_{CA} + \vec{V}_{C/B} &= \vec{V}_E + \vec{\omega}_D \times \vec{r}_{CE} + \vec{V}_{C/D} \\ (-\omega_B \hat{k}) \times (l \hat{j}) &= \omega_D \hat{k} \times (-l \hat{i} + l \hat{j}) + V_{C/D} (\cos \theta \hat{i} - \sin \theta \hat{j}) \\ -\omega_B l \hat{i} &= -\omega_D l (\hat{i} + \hat{j}) + V_{C/D} (\cos \theta \hat{i} - \sin \theta \hat{j}) \\ -(3)(0.5) \hat{i} &= -\omega_D (0.5 \hat{i} + 0.5 \hat{j}) + V_{C/D} (\cos 45^\circ \hat{i} - \sin 45^\circ \hat{j}) \\ \{-1.5 \hat{i} &= -\omega_D (0.5 \hat{i} + 0.5 \hat{j}) + V_{C/D} (\frac{\sqrt{2}}{2} \hat{i} - \frac{\sqrt{2}}{2} \hat{j})\} \\ \{3 \cdot \hat{i} &\Rightarrow -1.5 = -0.5 \omega_D + \frac{\sqrt{2}}{2} V_{C/D} \\ \omega_D &= 3 + \sqrt{2} V_{C/D} \\ \{3 \cdot \hat{j} &\Rightarrow 0 = -0.5 \omega_D - \frac{\sqrt{2}}{2} V_{C/D} \\ \omega_D &= -\sqrt{2} V_{C/D} \\ V_{C/D} &= -\frac{1}{\sqrt{2}} \omega_D \\ \omega_D &= 3 + \sqrt{2} \left(-\frac{1}{\sqrt{2}} \omega_D \right) \\ \omega_D &= 3 - \omega_D \\ 2\omega_D &= 3 \\ \omega_D &= 1.5 \text{ rad/s in } -\hat{k} \text{ direction} \\ \boxed{\omega_D = -1.5 \text{ rad/s } \hat{k}} \end{aligned}$$

18.4.5 Reconsider problem 18.4.4. The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3 \text{ rad/s}$ and angular acceleration $\alpha = 2 \text{ rad/s}^2$. Find the angular acceleration of rod DE.



Problem 18.5

Answer:

$$\ddot{\alpha}_{DE} = 3.5 \text{ rad/s}^2 \hat{k}$$

8.30 Same setup as prob 8.22. use same frame. B attached to rod DE from the kinematics of rod AB:

$$\dot{\mathbf{r}}_C = \dot{\mathbf{r}}_{AB} \times \dot{\mathbf{r}}_{C/A} + \dot{\mathbf{r}}_{AB} \times \dot{\mathbf{r}}_{C/A} = -\dot{\mathbf{r}}_{C/A} \dot{\omega}_{AB} \hat{j} + \dot{\mathbf{r}}_{C/A} \alpha_{AB} \hat{i}$$

$$\Rightarrow \dot{\mathbf{r}}_C = -4.5 \text{ m/s}^2 \hat{j} + 1 \text{ m/s}^2 \hat{i}$$

considering frame B (attached to rod DE):

$$\dot{\mathbf{r}}_C = \dot{\mathbf{r}}_{C/E} + \dot{\mathbf{r}}_B \times (\dot{\mathbf{r}}_B \times \dot{\mathbf{r}}_{C/E}) + \alpha_B \times \dot{\mathbf{r}}_{C/E} + \dot{\mathbf{r}}_{C/B} + \dot{\mathbf{r}}_{C/B}$$

$\dot{\mathbf{r}}_{C/E}$ immobile $2 \dot{\mathbf{r}}_B \times \dot{\mathbf{r}}_{C/B}$

8.30 continued p.12

$$\Rightarrow \dot{\mathbf{r}}_C = \dot{\mathbf{r}}_C + (-1.5 \text{ rad/s} \hat{k} \times (-1.5 \text{ rad/s} \hat{k} \times (-0.5 \text{ m} \hat{i} + 0.5 \text{ m} \hat{j})) + \alpha_B \hat{k} \times (-0.5 \text{ m} \hat{i} + 0.5 \text{ m} \hat{j}) + \dot{\mathbf{r}}_{C/B} \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}} \right) + 2(-1.5 \text{ rad/s} \hat{k} \times 0.75 \text{ m/s} (\hat{i} - \hat{j}))$$

where I have used:

$$\dot{\mathbf{r}}_B = -1.5 \text{ rad/s} \hat{k} \text{ (from prob 8.22)}$$

$$\dot{\mathbf{r}}_{C/B} = 0.75 \text{ m/s} (\hat{i} - \hat{j}) \text{ (from prob 8.22)}$$

$$\dot{\mathbf{r}}_{C/B} = \dot{\mathbf{r}}_{C/B} \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}} \right) \leftarrow \text{collar must accelerate along rod DE.}$$

$\Rightarrow$ using $\dot{\mathbf{r}}_C$ (calc. from rod AB) = $\dot{\mathbf{r}}_C$ (from rod DE) and doing some cross-products yields

$$(-4.5 \hat{j} + 1 \hat{i}) \text{ m/s}^2 = 1.125 \text{ m/s}^2 (\hat{i} - \hat{j}) - 0.5 \text{ m} \alpha_B (\hat{i} + \hat{j}) + \frac{\dot{\mathbf{r}}_{C/B}}{\sqrt{2}} (\hat{i} - \hat{j}) - 2.25 \text{ m/s}^2 (\hat{i} + \hat{j})$$

dotting the above w/ $\hat{i}$ & $\hat{j}$:

$$\hat{i} \cdot () \Rightarrow 1 \text{ m/s}^2 = 1.125 \text{ m/s}^2 - 0.5 \text{ m} \alpha_B + \frac{\dot{\mathbf{r}}_{C/B}}{\sqrt{2}} - 2.25 \text{ m/s}^2$$

$$\hat{j} \cdot () \Rightarrow -4.5 \text{ m/s}^2 = -1.125 \text{ m/s}^2 - 0.5 \text{ m} \alpha_B - \frac{\dot{\mathbf{r}}_{C/B}}{\sqrt{2}} - 2.25 \text{ m/s}^2$$

adding these two eqns together yields:

$$-3.5 \text{ m/s}^2 = -1 \text{ m} \alpha_B - 4.5 \text{ m/s}^2$$

$$\Rightarrow \alpha_B = -1 \text{ rad/s}^2 \Rightarrow \ddot{\alpha}_B = -1 \text{ rad/s}^2 \hat{k}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

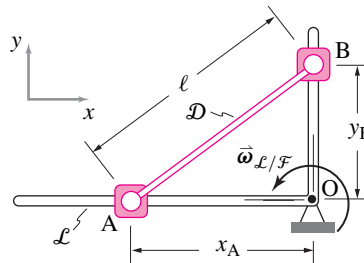
18.4.7 A bar of length $\ell = 5$ ft, body $\mathcal{D}$, connects sliders A and B on an L-shaped frame, body $\mathcal{L}$, which itself is rotating at constant speed about an axle perpendicular to the plane of the figure through the point O and relative to a fixed frame $\mathcal{F}$, $\dot{\omega}_{\mathcal{L}/\mathcal{F}} = 0.5 \text{ rad/sk}$. At the instant shown, body $\mathcal{L}$ is aligned with the $x - y$ axes, slider A is $x_A = 4$ ft from point O, slider B is $y_B = 3$ ft from O. The speed and acceleration of slider B relative to the frame $\mathcal{L}$ are $\vec{v}_{B/\mathcal{L}} = -2 \text{ ft/sj}$ and $\vec{a}_{B/\mathcal{L}} = -2 \text{ ft/s}^2\mathbf{j}$, respectively.

Determine:

- a) The absolute velocity of the slider

A, $\vec{v}_{A/\mathcal{F}}$, and

- b) The absolute angular velocity of bar AB, body $\mathcal{D}$, $\dot{\omega}_{\mathcal{D}/\mathcal{F}}$.



Problem 18.7

8.21 Consider the moving frame $\mathcal{L}$, which is glued to the bent rod AOB; its coordinate axes $x'y'z'$ are coincident with the fixed axes x, y, z at this instant.

Given: $\dot{\omega}_{\mathcal{L}/\mathcal{F}}$, $\vec{v}_{B/\mathcal{L}}$, $\dot{\omega}_{\mathcal{D}/\mathcal{F}}$

Find $\vec{v}_{A/\mathcal{F}}$, $\dot{\omega}_{\mathcal{D}/\mathcal{F}}$

'3-term velocity formula': $\vec{v}_A = \vec{v}_O + \vec{v}_{rel}$
 Find $\vec{v}_{rel} = \vec{v}_{A/\mathcal{L}}$ by noting that length of bar AB = const: thus $x_A^2 + y_B^2 = \text{const}$.
 Differentiate: $-x_A \dot{x}_A + y_B \dot{y}_B = 0$.
 We're given: $x_A = 4 \text{ ft}$, $y_B = 3 \text{ ft}$, $\dot{y}_B = v_{B/\mathcal{L}} = -2 \text{ ft/s}$
 $\Rightarrow$ thus $\dot{x}_A = v_{A/\mathcal{L}} = 1.5 \text{ ft/s} \Rightarrow \vec{v}_{A/\mathcal{L}} = 1.5 \text{ ft/s } \hat{i}$
 so $\vec{v}_A = \vec{v}_O + \dot{\omega}_{\mathcal{L}/\mathcal{F}} \times \vec{r}_{A/O} + \vec{v}_{A/\mathcal{L}}$
 $\vec{v}_A = \mathbf{0} + (0.5 \text{ rad/s } \hat{k}) \times (-4 \text{ ft } \hat{i}) + 1.5 \text{ ft/s } \hat{i}$
 (pt. O does not move) $\Rightarrow \vec{v}_A = -2 \text{ ft/s } \hat{j} + 1.5 \text{ ft/s } \hat{i}$

8.21 continued p. 9

b) find $\dot{\omega}_{\mathcal{D}/\mathcal{F}} \Rightarrow$ note that

$(\vec{v}_A = \vec{v}_B + \vec{v}_{A/B})$ relative to frame $\mathcal{L}$

$(\vec{v}_A = \vec{v}_B + \dot{\omega}_{\mathcal{D}/\mathcal{F}} \times \vec{r}_{A/B})$ relative to $\mathcal{L}$

$\vec{v}_{A/\mathcal{L}} = \vec{v}_{B/\mathcal{L}} + \dot{\omega}_{\mathcal{D}/\mathcal{F}} \times \vec{r}_{A/B}$ $\leftarrow \dot{\omega}_{\mathcal{D}/\mathcal{F}}$ is the only unknown

putting in #3:

$(-1.5 \text{ ft/s } \hat{i}) = (-2 \text{ ft/s } \hat{j}) + \dot{\omega}_{\mathcal{D}/\mathcal{F}} \hat{k} \times (-3 \text{ ft } \hat{i} - 4 \text{ ft } \hat{j})$
 $\Rightarrow -1.5 \text{ ft/s } \hat{i} = -2 \text{ ft/s } \hat{j} + \dot{\omega}_{\mathcal{D}/\mathcal{F}} (3 \text{ ft } \hat{i} - 4 \text{ ft } \hat{j})$
 Dotting the above with either $\hat{i}$ or $\hat{j}$ yields $\dot{\omega}_{\mathcal{D}/\mathcal{F}} = -0.5 \text{ rad/s } \hat{k}$
 Now $\dot{\omega}_{\mathcal{D}/\mathcal{F}} = \dot{\omega}_{\mathcal{D}/\mathcal{L}} + \dot{\omega}_{\mathcal{L}/\mathcal{F}} = (-0.5 + 0.5) \text{ rad/s } \hat{k}$
 $\Rightarrow \dot{\omega}_{\mathcal{D}/\mathcal{F}} = \mathbf{0}$

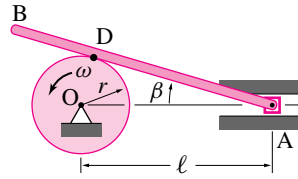
Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

18.4.8 The link AB is supported by a wheel at D and its end A is constrained so that it only has horizontal velocity. No slipping occurs between the wheel and the link. The wheel has an angular velocity ω and radius r . The distance $OA = \ell$.

Given: ω , r , and ℓ . ($\beta = \sin^{-1} \frac{r}{\ell}$).

Determine the angular velocity of the

link AB and velocity of the point A .



Problem 18.8

B.4

find ω_{AB} .
Note:
 $\sin \beta = \cos \alpha = \frac{r}{L}$
 $\cos \beta = \sin \alpha = \frac{\sqrt{L^2 - r^2}}{L}$

looking at the disk:
 $\mathbf{v}_D = \omega \times \mathbf{r} = \omega \hat{k} \times r \left(\frac{r}{L} \hat{i} + \frac{\sqrt{L^2 - r^2}}{L} \hat{j} \right)$
 $\mathbf{v}_D = \frac{\omega r}{L} (r \hat{j} - \sqrt{L^2 - r^2} \hat{i})$

Link AB :
 $\mathbf{v}_A = \mathbf{v}_D + \mathbf{v}_{A/D} \Rightarrow$ we know that $\mathbf{v}_A = v_A \hat{i}$
 $\mathbf{v}_A = \frac{\omega r}{L} (r \hat{j} - \sqrt{L^2 - r^2} \hat{i}) + \omega_{AB} \hat{k} \times \mathbf{r}_{A/D}$
where $\mathbf{r}_{A/D} = \sqrt{L^2 - r^2} (\cos \beta \hat{i} - \sin \beta \hat{j})$
Thus:
 $\hat{i} \cdot \mathbf{v}_A = \frac{\omega r}{L} (r \hat{j} - \sqrt{L^2 - r^2} \hat{i}) \cdot \hat{i} + \omega_{AB} \sqrt{L^2 - r^2} (\cos \beta \hat{j} - \sin \beta \hat{i}) \cdot \hat{i}$
 $\hat{j} \cdot \mathbf{v}_A = 0 = \frac{\omega r}{L} (r \hat{j} - \sqrt{L^2 - r^2} \hat{i}) \cdot \hat{j} + \omega_{AB} \sqrt{L^2 - r^2} (\cos \beta \hat{j} - \sin \beta \hat{i}) \cdot \hat{j}$
 $\hat{i} \cdot \mathbf{v}_A \Rightarrow 0 = \frac{\omega r}{L} (-\sqrt{L^2 - r^2}) + \omega_{AB} \sqrt{L^2 - r^2} \cos \beta$
 $\Rightarrow \omega_{AB} = \frac{-\omega r}{\sqrt{L^2 - r^2} \cos \beta} = \frac{-\omega r}{\sqrt{L^2 - r^2}} \frac{L}{r} = \frac{-\omega L}{\sqrt{L^2 - r^2}}$
 $\hat{j} \cdot \mathbf{v}_A \Rightarrow v_A = \frac{\omega r}{L} (r - \sqrt{L^2 - r^2} \sin \beta) + \omega_{AB} \sqrt{L^2 - r^2} \sin \beta$
 $v_A = \frac{\omega r}{L} (r - \sqrt{L^2 - r^2} \frac{r}{L}) - \frac{\omega r L}{\sqrt{L^2 - r^2}} \left(\frac{r}{L} \right) =$ continued

B.4 continued

$v_A = \frac{-\omega r (L^2 - r^2) - \omega r^3}{L \sqrt{L^2 - r^2}} = \frac{-\omega r L}{\sqrt{L^2 - r^2}} \Rightarrow$
 $\mathbf{v}_A = \frac{-\omega r L}{\sqrt{L^2 - r^2}} \hat{i}$

b. Instantaneous center of rotation:
There is some point (call it O) (which may change with time) about which all the points on rod ADB rotate. This is a fictitious point that may not lie on rod ADB at all; see sketch $\Rightarrow$

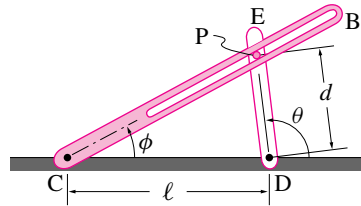
At this instant, point D behaves like it's going in circles of radius OD about pt. O , while pt. A behaves like it's going in circles of radius OA about O .

We will focus our efforts on triangle ODA .
From the kinematics of the disk, $v_D = r\omega$.
thus, $\omega_{AB} = \frac{v_D}{OD}$. Find OD using Law of Sines:
 $\frac{OD}{\sin \alpha} = \frac{AD}{\sin \beta} \Rightarrow \frac{(OD)(L)}{\sqrt{L^2 - r^2}} = \frac{L\sqrt{L^2 - r^2}}{r}$
 $\Rightarrow OD = \frac{L^2 - r^2}{r}$ thus $\omega_{AB} = \frac{v_D}{OD} = \frac{r\omega(r)}{L^2 - r^2} = \frac{r^2\omega}{L^2 - r^2}$ Noting that $\omega_{AB} = -\omega_{AB} \hat{k}$,
 $\omega_{AB} = \frac{-r^2\omega}{L^2 - r^2} \hat{k}$ (same as before)

Now, note that $\mathbf{v}_A = R_{A/O} \omega_{AB}$. Find $R_{A/O} = OA$:
 $\frac{OA}{\sin \theta_{AO}} = \frac{AD}{\sin \beta} \Rightarrow OA = \frac{L\sqrt{L^2 - r^2}}{r}$
thus, $\mathbf{v}_A = (OA)(\omega_{AB}) = \frac{L\sqrt{L^2 - r^2}}{r} \left(\frac{r^2\omega}{L^2 - r^2} \right)$
so $v_A = \frac{rL\omega}{\sqrt{L^2 - r^2}}$
note that $\mathbf{v}_A = -v_A \hat{i}$: $\mathbf{v}_A = \frac{-rL\omega}{\sqrt{L^2 - r^2}} \hat{i}$

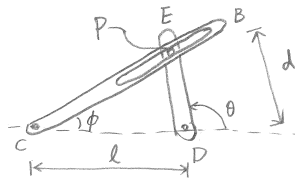
Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

18.4.10 The slotted link CB is driven in an oscillatory motion by the link ED which rotates about D with constant angular velocity $\dot{\theta} = \omega_D$. The pin P is attached to ED at fixed radius d and engages the slot on CB as shown. Find the angular velocity and acceleration $\dot{\phi}$ and $\ddot{\phi}$ of CB when $\theta = \pi/2$.



Problem 18.10

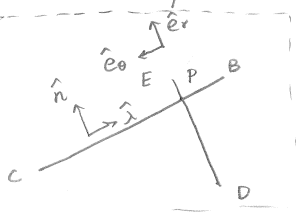
15-38



P is pinned on DE and can slide along CB, $\dot{\theta} = \omega_D$

Find $\dot{\phi}$, $\ddot{\phi}$ when $\theta = \frac{\pi}{2}$

i) Find $\dot{\phi}$



Build up two local coordinate system $\hat{e}_0, \hat{e}_r$ and $\hat{\lambda}, \hat{n}$.

Call the moving frame attaching to CB, B
the moving frame attaching to DE, D

(a)

$$\vec{V}_P = \vec{V}_P$$

$$\vec{V}_C^0 + \vec{V}_{P/B} + \vec{\omega}_B \times \vec{r}_{P/C} = \vec{V}_D^0 + \vec{V}_{P/D} + \vec{\omega}_D \times \vec{r}_{P/D}$$

$$\Rightarrow V_{P/B} \hat{\lambda} + \dot{\phi} \hat{k} \times r_{P/C} \hat{\lambda} = \omega_D \hat{k} \times d \hat{e}_r$$

$$\text{when } \theta = \frac{\pi}{2}, \quad r_{P/C} = \frac{l}{\cos \phi}$$

$$\Rightarrow \left\{ V_{P/B} \hat{\lambda} + \dot{\phi} \frac{l}{\cos \phi} \hat{n} = \omega_D d \hat{e}_0 \right\}$$

$$\{ \} \cdot \hat{n} \Rightarrow \dot{\phi} \frac{l}{\cos \phi} = \omega_D d (\hat{e}_0 \cdot \hat{n})$$

$$\{ \} \cdot \hat{\lambda} \Rightarrow V_{P/B} = \omega_D d (\hat{e}_0 \cdot \hat{\lambda})$$

$$\text{when } \theta = \frac{\pi}{2}, \quad \hat{e}_0 \cdot \hat{n} = \sin \phi$$

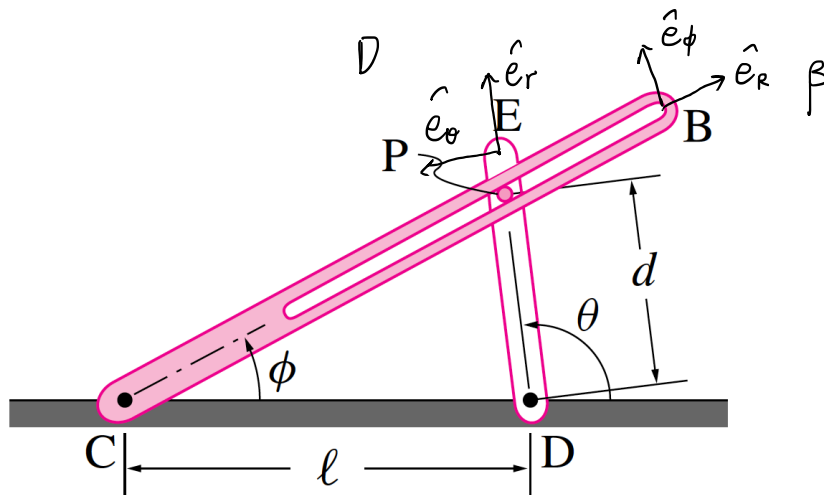
$$\hat{e}_0 \cdot \hat{\lambda} = -\cos \phi$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

17.4.10

Saturday, May 4, 2019

6:38 PM



p is attached to DE

p can slide along CB

DE rotates with const angular velocity $\dot{\theta} = \omega_0$

Given: $d, l, \dot{\theta} = \omega_0, \theta = \frac{\pi}{2}$

Find a) angular velocity of CB : $\dot{\phi}$ when $\theta = \frac{\pi}{2}$.
b) angular acceleration of CB : $\ddot{\phi}$

Build up two local body fixed frame β and $\mathcal{D}$
as shown in the figure

$$\vec{v}_p = \vec{v}_p$$

$$\vec{v}_c^{\mathcal{D}} + \vec{v}_{p/\beta} + \vec{\omega}_\beta \times \vec{r}_{p/c} = \vec{v}_d^{\mathcal{D}} + \vec{v}_{p/\mathcal{D}} + \vec{\omega}_\mathcal{D} \times \vec{r}_{p/d}$$

$$\Rightarrow v_{p/\beta} \hat{e}_\phi + \dot{\phi} \hat{k} \times \vec{r}_{p/c} \hat{e}_\phi = \dot{\theta} \hat{k} \times d \hat{e}_r$$

when $\theta = \frac{\pi}{2}$ $r_{p/c} = \frac{l}{\cos \phi}$ $\dot{\theta} = \omega_0$ plug in

$$\left\{ v_{P/\beta} \hat{e}_r + \dot{\phi} \frac{l}{\cos \phi} \hat{e}_\phi = \omega_0 d \hat{e}_\theta \right\}$$

$$\{ \} \cdot \hat{e}_\phi \Rightarrow \dot{\phi} \frac{l}{\cos \phi} = \omega_0 d (\hat{e}_\theta \cdot \hat{e}_\phi)$$

When $\theta = \frac{\pi}{2}$ $\hat{e}_\theta \cdot \hat{e}_\phi = \sin \phi$

$$\therefore \dot{\phi} = \omega_0 \frac{d}{l} \sin \phi \cos \phi$$

Since $\cos \phi = \frac{d}{\sqrt{l^2 + d^2}}$ $\sin \phi = \frac{l}{\sqrt{l^2 + d^2}}$

$$\therefore \dot{\phi} = \omega_0 \frac{d^2}{d^2 + l^2} \quad \text{when } \theta = \frac{\pi}{2}$$

b) $\vec{a}_P = \vec{a}_P$

$$\begin{aligned} & \vec{a}_P^0 + \vec{a}_{P/\beta} + \vec{\omega}_\beta \times \vec{r}_{P/\beta} - \omega_\beta^2 \vec{r}_{P/\beta} + 2 \vec{\omega}_\beta \times \vec{v}_{P/\beta} \\ &= \vec{a}_P^0 + \vec{a}_{P/D}^0 + \vec{\omega}_D \times \vec{r}_{P/D} - \omega_D^2 \vec{r}_{P/D} + 2 \omega_D \times \vec{v}_{P/D}^0 \\ & \vec{\omega}_D = \vec{0} \quad \text{since } \dot{\theta} = \omega_0 = \text{const} \end{aligned}$$

$$\vec{a}_{P/\beta} + \vec{\omega}_\beta \times \vec{r}_{P/\beta} - \omega_\beta^2 \vec{r}_{P/\beta} + 2 \vec{\omega}_\beta \times \vec{v}_{P/\beta} = -\omega_0^2 \vec{r}_{P/D}$$

$$\Rightarrow \left\{ a_{P/\beta} \hat{e}_r + \dot{\phi} \hat{k} \times \frac{l}{\cos \phi} \hat{e}_r - \phi^2 \frac{l}{\cos \phi} \hat{e}_r + 2 \dot{\phi} \hat{k} \times (-\omega_0 d \cos \phi) \hat{e}_r = -\omega_0^2 d \hat{e}_r \right\}$$

$$\{ \} \cdot \hat{e}_\phi \Rightarrow \ddot{\phi} \frac{l}{\cos \phi} - 2 \omega_0 \dot{\phi} d \cos \phi = -\omega_0^2 d (\hat{e}_r \cdot \hat{e}_\phi)$$

When $\theta = \frac{\pi}{2}$ $\hat{e}_r \cdot \hat{e}_\phi = \cos \phi$

$$\therefore \ddot{\phi} = \frac{2 \omega_0 \dot{\phi} d \cos^2 \phi}{l} - \omega_0^2 \frac{d}{l} \cos^2 \phi$$

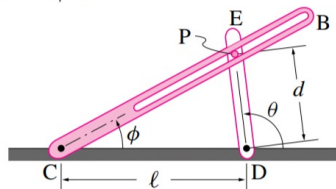
$$= \omega_0^2 \frac{d l (d^2 - l^2)}{(l^2 + d^2)^2}$$

Thus angular acceleration of CB is $\ddot{\phi} = \omega_0^2 \frac{d l (d^2 - l^2)}{(l^2 + d^2)^2}$ when $\theta = \frac{\pi}{2}$

HW 13 P70 18.4.10

Teaya Yong (YV449)

18.4.10 The slotted link CB is driven in an oscillatory motion by the link ED which rotates about D with constant angular velocity $\dot{\theta} = \omega_D$. The pin P is attached to ED at fixed radius d and engages the slot on CB as shown. Find the angular velocity and acceleration $\dot{\phi}$ and $\ddot{\phi}$ of CB when $\theta = \pi/2$.



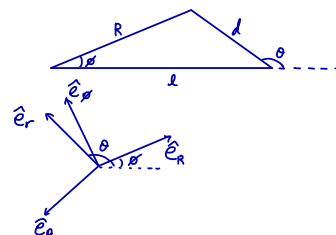
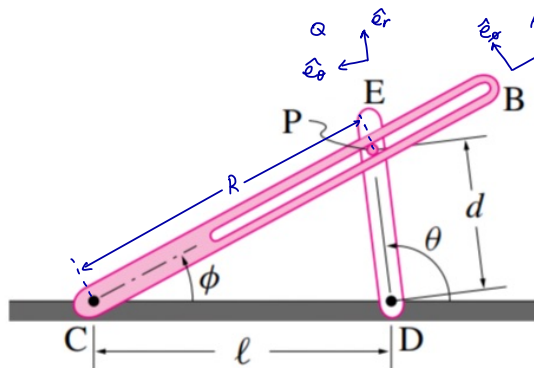
Problem 18.4.10

Given: link ED rotates at $\dot{\theta} = \omega_D$
 P is attached to ED with radius d
 P slides along BC

Find: a) $\dot{\phi}(\theta = \pi/2)$ of CB
 b) $\ddot{\phi}(\theta = \pi/2)$ of CB

Define one reference frame for each object:

- ① F is the fixed frame
- ② A is the body frame for link CB (unit vectors $\hat{e}_R, \hat{e}_\theta$)
- ③ Q is the body frame for link DE (unit vectors $\hat{e}_r, \hat{e}_\theta$)



a) Finding $\dot{\phi}(\theta = \pi/2)$

$\vec{V}_{P/F} = \vec{V}_{P/F}$ * using velocity of pin P as constraint

$$\frac{d}{dt}^F(\vec{r}_{P/C}) = \frac{d}{dt}^F(\vec{r}_{P/D} + \vec{r}_{D/C})$$

This $\dot{R}$ is the key for part b)

$$\vec{V}_{P/A} + \vec{\omega}_{A/F} \times \vec{r}_{P/C} = \vec{V}_{P/Q} + \vec{\omega}_{Q/F} \times \vec{r}_{P/D} + \vec{V}_{D/F}$$

$$\dot{R} \hat{e}_R + \dot{\phi} R \hat{e}_\theta = \dot{R} \hat{e}_r + \dot{\theta} d \hat{e}_\theta + \dot{\theta} d \hat{e}_\theta$$

$$\{ \dot{R} \hat{e}_R + \dot{\phi} R \hat{e}_\theta = \dot{\theta} d \hat{e}_\theta \}$$

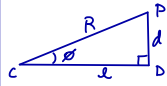
$$\{ \} \cdot \hat{e}_\theta : \dot{\phi} R = \dot{\theta} d (\hat{e}_\theta \cdot \hat{e}_\theta)$$

To solve $\dot{\phi}$, need to first find R and $\hat{e}_\theta \cdot \hat{e}_\theta$:

↳ BUT the general case for these are not so pretty so let's only consider what R and $\hat{e}_\theta \cdot \hat{e}_\theta$ would be for the orientation at $\theta = \pi/2$:

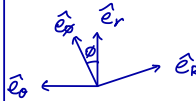
(next page)

Finding $R(\theta = \frac{\pi}{2})$:



$$R = \sqrt{d^2 + e^2}$$

Finding $\hat{e}_\theta \cdot \hat{e}_\phi$ at $\theta = \frac{\pi}{2}$:



$$\hat{e}_\theta \cdot \hat{e}_\phi = \sin\theta = \frac{d}{\sqrt{d^2 + e^2}}$$

plugging in : $\dot{\theta} \sqrt{d^2 + l^2} = \omega_0 d \frac{d}{\sqrt{d^2 + l^2}}$

$$\dot{\phi}(\theta = \frac{\pi}{2}) = \frac{\omega_0 d^2}{d^2 + l^2}$$

b) Finding $\ddot{\phi}(\theta = \frac{\pi}{2})$

$$\vec{a}_{P/F} = \vec{a}_{P/F} \quad * \text{ using acceleration of pin P as constraint.}$$

Using 5-term acceleration formula:

$$\begin{aligned} & \vec{a}_{B/O}^0 + \vec{a}_{P/A} + \vec{\omega}_{A/F} \times \vec{r}_{P/C} - \omega_{A/F}^2 \vec{r}_{P/C} + 2\vec{\omega}_{A/F} \times \vec{v}_{P/A} \\ = & \vec{a}_{B/O}^0 + \vec{a}_{P/Q}^0 + \vec{\omega}_{A/Q} \times \vec{r}_{P/D} - \omega_{A/Q}^2 \vec{r}_{P/D} + 2\vec{\omega}_{A/Q} \times \vec{v}_{P/Q}^0 \\ & \text{L } \vec{\omega}_{A/Q}^2 \text{ since } \dot{\theta} = \omega_0 = \text{const.} \end{aligned}$$

$$\vec{a}_{P/A} + \dot{\vec{\omega}}_{A/F} \times \vec{r}_{P/C} - \omega_{A/F}^2 \vec{r}_{P/C} + 2\vec{\omega}_{A/F} \times \vec{v}_{P/A} = -\omega_{A/F}^2 \vec{r}_{PD}$$

$$(\ddot{R} \hat{e}_R) + (\ddot{\theta} \hat{k}) \times (R \hat{e}_R) - \dot{\theta}^2 (R \hat{e}_R) + 2(\dot{\theta} \hat{k}) \times (\dot{R} \hat{e}_R) = -\omega_0^2 (d \hat{e}_r)$$

$$\{ \ddot{R} \hat{e}_R + \ddot{\phi} R \hat{e}_\phi - \dot{\phi}^2 R \hat{e}_R + 2\dot{\phi} \dot{R} \hat{e}_\phi = -\omega_D^2 d \hat{e}_r \}$$

$$\{ \} \cdot \hat{e}_\phi : \ddot{\phi} R + 2\dot{\phi} \dot{R} = -\omega_D^2 d(\hat{e}_r \cdot \hat{e}_\phi)$$

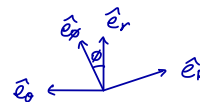
Now to solve for $\ddot{\phi}$, we need R , $\dot{R}$, $\dot{\phi}$ (answer to a), and $\hat{e}_r \cdot \hat{e}_{\phi}$

Recall at $\theta = \frac{\pi}{2}$:

$$\sin \phi = \frac{d}{\sqrt{e^2 + d^2}}$$

$$\begin{aligned}\sin \theta &= 1 \\ \cos \theta &= 0\end{aligned}$$

$$\hat{e}_r \cdot \hat{e}_\phi = \cos \phi = \frac{l}{\sqrt{l^2 + d^2}}$$

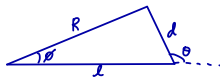


$$R = \sqrt{l^2 + d^2}$$

$$\dot{\phi}(\theta = \frac{\pi}{2}) = \frac{\omega_0 d^2}{d^2 + l^2}$$

We are left with $\dot{R}$, which needs a bit of work to get:

Law of Sines: $\frac{d}{\sin \theta} = \frac{R}{\sin(\pi - \theta)} = \frac{R}{\sin \theta}$



$$R = d \frac{\sin \theta}{\sin \phi} \quad (\text{general expression for } R \text{ at any } \theta, \phi)$$

Differentiate to find $\dot{R}$:

$$\dot{R} = d \frac{\dot{\theta} \cos \theta - \dot{\theta} \cos \theta}{\sin^2 \theta} = - \frac{d \dot{\theta} \cos \theta}{\sin^2 \theta} = - d \dot{\theta} \left(\frac{1}{d^2} \sqrt{l^2 + d^2} \right) \Rightarrow \dot{R} = - \frac{\omega d l}{\sqrt{l^2 + d^2}}$$

$$\ddot{\phi} R + 2\dot{\phi} \dot{R} = -\omega_0^2 d(\hat{e}_r \cdot \hat{e}_\phi)$$

$$\ddot{\theta} R + 2(\omega_D \frac{d^2}{R^2}) \left(-\frac{\omega_D dl}{R} \right) = -\omega_D^2 \frac{dl}{R}$$

$$\ddot{\phi} = 2\omega_D^2 \frac{d^3 l}{R^4} - \omega_D^2 \frac{ld}{R^2}$$

$$= \omega_D^2 \left(\frac{2d^3l - l d R^2}{R^4} \right) = \omega_D^2 \left(\frac{2d^3l - l d (d^2 + l^2)}{R^4} \right) = \omega_D^2 \left(\frac{d^3l - l^3d}{R^4} \right)$$

$$\ddot{\theta}(\theta = \frac{\pi}{2}) = \omega_D^2 \frac{dl(d^2 - l^2)}{(l^2 + d^2)^2}$$

Recall that we simplified many terms in this problem by only considering the case where $\theta = \frac{\pi}{2}$. The general case solution would be much more complicated.

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

$$\dot{\phi} = \omega_0 \frac{d}{l} \cos\phi \sin\phi, \quad \vec{V}_{P/B} = -\omega_0 d \cos\phi \hat{\lambda}$$

$$\cos\phi = \frac{l}{\sqrt{d^2 + l^2}}, \quad \sin\phi = \frac{d}{\sqrt{d^2 + l^2}} \quad \text{when } \theta = \frac{\pi}{2}, \Rightarrow \boxed{\dot{\phi} = \omega_0 \frac{d^2}{d^2 + l^2}}$$

$$(b) \quad \vec{a}_P = \vec{a}_P \quad \text{(Note: } \dot{\vec{\omega}}_D = 0 \text{ because } \dot{\phi} = \omega_0 = \text{constant})$$

$$\begin{aligned} \vec{a}_C^0 + \vec{a}_{P/B} + \dot{\vec{\omega}}_B \times \vec{r}_{P/C} &= \vec{a}_D^0 + \vec{a}_{P/D} + \dot{\vec{\omega}}_D \times \vec{r}_{P/D} \\ -\omega_B^2 \vec{r}_{P/C} + 2\vec{\omega}_B \times \vec{V}_{P/B} &= -\omega_D^2 \vec{r}_{P/D} + 2\vec{\omega}_D \times \vec{V}_{P/D} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad a_{P/B} \hat{\lambda} + \ddot{\phi} \hat{k} \times \frac{l}{\cos\phi} \hat{\lambda} &= -\omega_D^2 d \hat{e}_r \\ -\dot{\phi}^2 \frac{l}{\cos\phi} \hat{\lambda} + 2\dot{\phi} \hat{k} \times (-\omega_0 d \cos\phi) \hat{\lambda} \end{aligned}$$

$$\Rightarrow \{ a_{P/B} \hat{\lambda} + \ddot{\phi} \frac{l}{\cos\phi} \hat{n} - \dot{\phi}^2 \frac{l}{\cos\phi} \hat{\lambda} - 2\omega_0 \dot{\phi} d \cos\phi \hat{n} = -\omega_D^2 d \hat{e}_r \}$$

$$\{ \} \cdot \hat{n} \Rightarrow \ddot{\phi} \frac{l}{\cos\phi} - 2\omega_0 \dot{\phi} d \cos\phi = -\omega_D^2 d (\hat{e}_r \cdot \hat{n})$$

$$\hat{e}_r \cdot \hat{n} = \cos\phi$$

$$\ddot{\phi} = \frac{2\omega_0 \dot{\phi} d \cos^2\phi}{l} - \omega_D^2 \frac{d}{l} \cos^2\phi$$

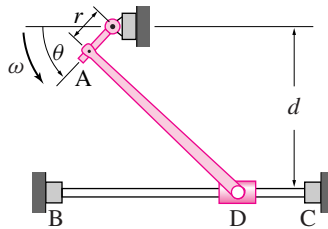
$$= \frac{2\omega_0 d}{l} \frac{l^2}{l^2 + d^2} \omega_0 \frac{d^2}{d^2 + l^2} - \omega_D^2 \frac{d}{l} \frac{l^2}{l^2 + d^2}$$

$$= \omega_D^2 \frac{d l (d^2 - l^2)}{(l^2 + d^2)^2}$$

$$\therefore \text{Angular acceleration of CB when } \theta = \frac{\pi}{2} \text{ is } \ddot{\phi} = \omega_D^2 \frac{d l (d^2 - l^2)}{(l^2 + d^2)^2}$$

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18.4.11 The rod of radius $r = 50$ mm shown has a constant angular velocity of $\omega = 30$ rad/s counterclockwise. Knowing that rod AD is 250 mm long and distance $d = 150$ mm, determine the acceleration of collar D when $\theta = 90^\circ$.



Problem 18.11

8.14 Find $\ddot{a}_D$ when $\theta = 90^\circ$
 First find $\dot{V}_A$ & $\ddot{a}_A$ from disk:
 when $\theta = 90^\circ$,
 $\dot{V}_A = r\omega \hat{i}$
 $\ddot{a}_A = r\omega^2 \hat{j}$
 Attach frame $\mathcal{B}$ to rod AD.
 The frame translates and rotates with the rod. Let its coordinate system $(\hat{i}', \hat{j}', \hat{k}')$ be coincident with $(\hat{i}, \hat{j}, \hat{k})$ at this instant.
 Then $\dot{V}_D = \dot{V}_D' + \dot{V}_{rel}$
 thus $\dot{V}_D = \dot{V}_D' + \omega \times \vec{r}_{D/O'}$
 We know $\dot{V}_A$, so let's take pt. A to be pt O':
 $\dot{V}_D' = R\omega \hat{i}$
 $\Rightarrow \dot{V}_D \hat{i} = r\omega \hat{i} + \omega \hat{k} \times (\sqrt{L^2 - (d-r)^2} \hat{i} - (d-r) \hat{j})$
 $\Rightarrow \dot{V}_D \hat{i} = r\omega \hat{i} + \omega (\sqrt{L^2 - (d-r)^2} \hat{j} + (d-r) \hat{i})$ * contd

8.14 continued p.8
 $\hat{j} \cdot (*) \Rightarrow 0 = \omega_B (\sqrt{L^2 - (d-r)^2}) \Rightarrow \omega_B = 0$
 rod undergoes straight line motion at this instant in time.
 Acceleration:
 $\ddot{a}_D = \ddot{a}_D' + \ddot{a}_{rel} + \ddot{a}_{cgr}$
 $\ddot{a}_D' = \ddot{a}_D' + \ddot{\omega}_B \times \vec{r}_{D/O'} + \ddot{\omega}_B \times \vec{r}_{D/O'}$
 given let pt. O' = point A: $\ddot{\omega}_B = 0$ unknown
 $\ddot{a}_D \hat{i} = r\omega^2 \hat{j} + \ddot{\omega}_B \hat{k} \times (\sqrt{L^2 - (d-r)^2} \hat{i} - (d-r) \hat{j})$
 $\ddot{a}_D \hat{i} = r\omega^2 \hat{j} + \ddot{\omega}_B (\sqrt{L^2 - (d-r)^2} \hat{j} + (d-r) \hat{i})$ **
 $\hat{j} \cdot (***) \Rightarrow \ddot{\omega}_B = \frac{-r\omega^2}{\sqrt{L^2 - (d-r)^2}}$
 $\hat{i} \cdot (***) \Rightarrow \ddot{a}_D = \ddot{\omega}_B (d-r) = \frac{-r\omega^2 (d-r)}{\sqrt{L^2 - (d-r)^2}}$
 plugging in #s: $\ddot{a}_D = -19.6 \text{ m/s}^2 \hat{i}$

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