18.3 General expressions for velocity $\vec{v}$ and acceleration $\vec{a}$ of a moving point

Now that we have some comfort with moving frames we can develop formulas that are not so strongly attached to base vectors. That is, we take account that the base vectors rotate with the frame, but develop formulas that don't use the base vectors explicitly. Thus the formulas we develop here work equally for any frame that is glued to the rotating frame of choice, independent of its orientation.

Absolute velocity of a point moving relative to a moving frame

Imagine that you know the absolute velocity $\vec{v}_{0'/\mathcal{F}}$ of some point O' on an object $\mathcal{B}$, say the center of a car tire and the angular velocity of the tire, $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$. Finally, imagine you also know the relative velocity of point P, $\vec{v}_{\mathrm{P}/\mathcal{B}}$, say of a bug crawling on the tire.

If the frame $\mathcal{B}$ is translating or rotating, the velocity of particle P relative to the frame $\vec{v}_{\mathrm{P}/\mathcal{B}}$ is not the absolute velocity (the velocity relative to a Newtonian frame). The absolute velocity in this case is $\vec{v}_{\mathrm{P}/\mathcal{F}}$, or more simply $\vec{v}_{\mathrm{P}}$, or more simply still, just $\vec{v}$. We want to know the relationship between the relative velocity $\vec{v}_{\mathrm{P}/\mathcal{B}}$ and the absolute velocity $\vec{v}_{\mathrm{P}/\mathcal{F}}$ (otherwise known as just $\vec{v}$).

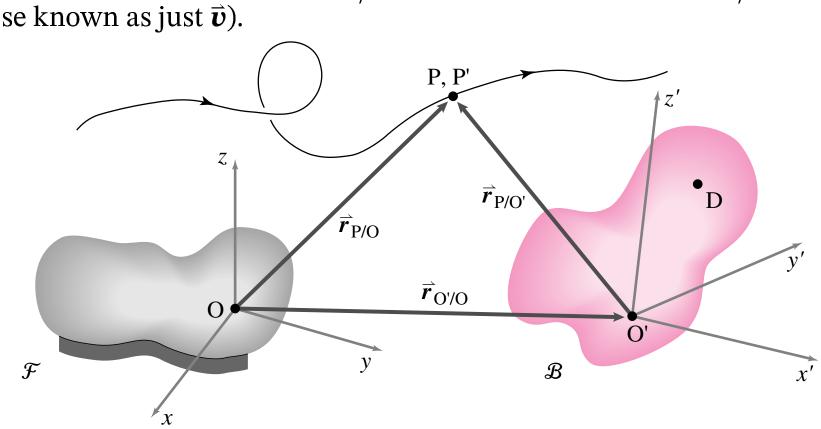


Figure 18.32: The position of point P relative to the origin O' of moving frame $\mathcal{B}$ is $\vec{r}_{\mathrm{P/O'}}$. The position of the origin of the frame $\mathcal{B}$ relative to the origin O of the fixed frame $\mathcal{F}$ is $\vec{r}_{\mathrm{O'/O}}$. The position of point P relative to O is the sum of $\vec{r}_{\mathrm{P/O'}}$ and $\vec{r}_{\mathrm{O'/O}}$. The motion of point P relative to the fixed frame may be complicated.

Let's start by looking at the position. The position of a point P that is moving is:

$$\vec{r}_{\mathrm{P/O}} = \vec{r}_{\mathrm{O'/O}} + \vec{r}_{\mathrm{P/O'}}$$

where O' is the origin of a coordinate system which is glued to the rigid object, as shown in fig. 18.32.

To find the absolute velocity of point P we will use the $\dot{\vec{Q}}$ formula, equation 18.19, for computing the rate of change of a vector. The velocity of P is the rate of change of its position. Here, we use $\vec{Q} = \vec{r}_{P/O'}$

$$\begin{aligned}
\vec{v}_{P/\mathcal{F}} &= {}^{\mathcal{F}}\dot{\vec{r}}_{P/O} \\
&= {}^{\mathcal{F}}\dot{\vec{r}}_{O'/O} + {}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} \\
&= \vec{v}_{O'/\mathcal{F}} + \overbrace{\underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}}^{{}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}}
\end{aligned}$$

The $\dot{\vec{Q}}$ formula 18.19 was used in the calculation to compute ${}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}$

$$ {}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} = {}^{\mathcal{B}}\dot{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}. \tag{18.27}$$

Thus, the 'three term velocity formula'.

Absolute velocity of point P

2. Velocity of point P relative to moving frame $\mathcal{B}$

$$\overbrace{\vec{v}_P} = \underbrace{\vec{v}_{O'}} + \overbrace{\vec{v}_{P/\mathcal{B}}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}} \tag{18.28}$$

1. Velocity of reference point O' on object $\mathcal{B}$

3. Velocity relative to point O' of the point P' glued to object $\mathcal{B}$ which is instantaneously coincident with point P

Another way to write the formula for absolute velocity is as

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where P' is a point glued to $\mathcal{B}$ which is instantaneously coincident with P, so the absolute velocity of P' is

$$\vec{v}_{P'} = \vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}. \tag{18.29}$$

Reconsider the bug crawling on the tire, object $\mathcal{B}$, in fig. 18.33. To find the absolute velocity of the bug, we need to be concerned with how the bug moves relative to the tire *and* how the tire moves relative to the ground.

Example: Absolute velocity of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

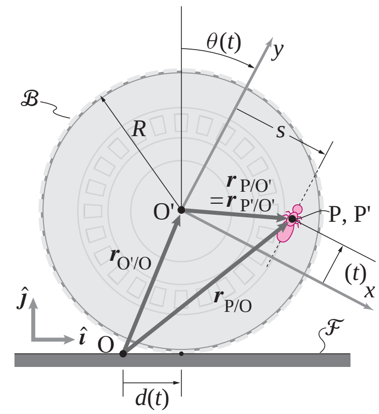


Figure 18.33

Referring to equation 18.27 on page 2, the absolute velocity of the bug is

$$\begin{aligned}
\vec{v}_{P/\mathcal{F}} &= \vec{v}_{O'/\mathcal{F}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} \\
&= R\dot{\theta}\hat{\imath} \quad + \quad \dot{\ell}\,\hat{\jmath}' \quad + \quad (-\dot{\theta}\hat{k}') \times (s\hat{\imath}' + \ell\,\hat{\jmath}') \\
&= R\dot{\theta}\hat{\imath} \quad + \quad \ell\dot{\theta}\hat{\imath}' \quad + \quad (\dot{\ell} - s\dot{\theta})\hat{\jmath}'.
\end{aligned}$$

At the instant of interest, the direction of the bug's absolute velocity depends upon the relative magnitudes of $\dot{\ell}$ and $\dot{\theta}$ as well as the orientation of $\hat{\imath}'$ and $\hat{\jmath}'$.

As we noted earlier, another way to write the formula for absolute velocity is

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where, in the example above, $\vec{v}_{P'} = R\dot{\theta}\hat{\imath} + \dot{\theta}(\ell\,\hat{\imath}' - s\,\hat{\jmath}')$ and $\vec{v}_{P/\mathcal{B}} = \dot{\ell}\,\hat{\jmath}'$. At the instant of concern, we can think of the absolute velocity of the bug as the velocity of the mark labeled P' under the bug *plus* the velocity of the bug relative to the tire.

Acceleration

We would like to find acceleration of a point using information about its motion relative to a moving frame. The result, the 'five term acceleration formula' is the most complicated formula in this book. (For reference, it is in Table II, 5c).

Acceleration relative to an object or frame

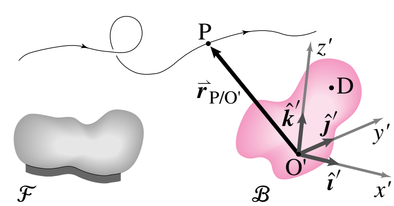


Figure 18.34: An object or reference frame $\mathcal{B}$ and a point P move around. We are interested in the acceleration of point P.

The acceleration of a point relative to an object or frame is the acceleration you would calculate if you were looking at the particle while you translated and rotated with the frame and took no account of the outside world. That is, if the position of a particle P relative to the origin O' of a coordinate system in a moving frame $\mathcal{B}$ is given by:

$$\vec{r}_{P/O'} = \underbrace{r_{Px'/O'}}_{x'}\,\hat{\imath}' + \underbrace{r_{Py'/O'}}_{y'}\,\hat{\jmath}' + \underbrace{r_{Pz'/O'}}_{z'}\,\hat{k}',$$

then the acceleration of the particle P relative to the frame is:

$$\vec{a}_{P/\mathcal{B}} = \underbrace{\ddot{r}_{Px'/O'}}_{\ddot{x}'}\,\hat{\imath}' + \underbrace{\ddot{r}_{Py'/O'}}_{\ddot{y}'}\,\hat{\jmath}' + \underbrace{\ddot{r}_{Pz'/O'}}_{\ddot{z}'}\,\hat{k}'.$$

That is, the acceleration relative to the frame takes no account of (a) the motion of the frame or of (b) the rotation of the base vectors with the frame to which they are fixed.

Reconsider the bug labeled point P crawling on the tire, object $\mathcal{B}$, in fig. 18.35.

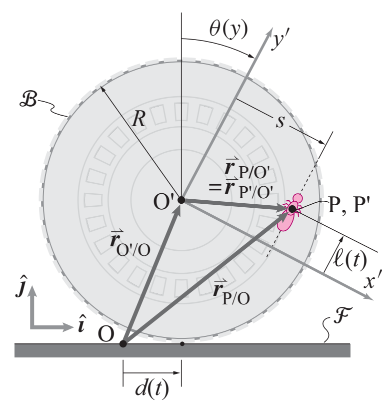


Figure 18.35: A bug crawling on a tire at point P. Point P' is on the tire and instantaneously corresponds to point P.

Example: Acceleration relative to a frame (2-D): Bug crawling on a tire, again
If we are sitting on the tire, all that we see is the bug crawling in a straight line at non-constant rate relative to us. Thus, its acceleration relative to the tire is

$$\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} = \ddot{\ell}\,\hat{\boldsymbol{\jmath}}'.$$

So, at the instant of interest, the bug has an acceleration relative to the tire frame parallel to the y'-axis.

Absolute acceleration of a point P′ glued to a moving frame

Imagine that you know the absolute acceleration of some point O' at the center of a frame $\mathcal{B}$, say the center of a car tire. Imagine you also know the angular velocity of the tire, $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$, and the angular acceleration, $\vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}}$. Then, you can find the absolute acceleration of a piece of gum labeled point D stuck to the sidewall (see fig. 18.37). If we start with the equation 18.29 for the absolute velocity of a point glued to a moving frame on page 2 and differentiate with respect to time, we get the absolute acceleration of a point D fixed in a moving frame $\mathcal{B}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{D}} &= \frac{d}{dt}[\vec{\boldsymbol{v}}_{\mathrm{O}'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'}] \\
&= \vec{\boldsymbol{a}}_{\mathrm{O}'/\mathrm{O}} + [\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'})] \\
&= \vec{\boldsymbol{a}}_{\mathrm{O}'/\mathrm{O}} + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'}) \qquad (18.30)
\end{aligned}$$

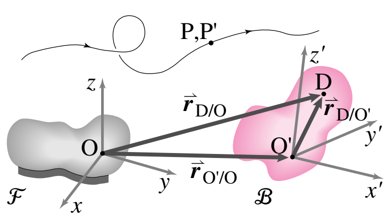


Figure 18.36: Tracking a moving particle from two different reference frames.

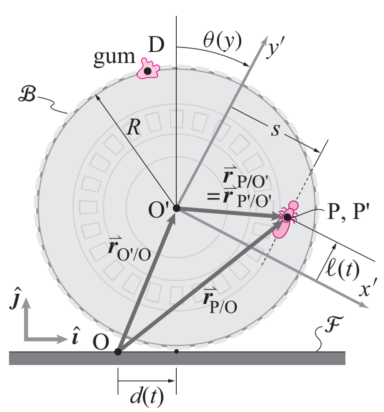


Figure 18.37: Gum is glued to a rolling tire at D. The point P′ glued to the tire is just where the point P happens to be passing at the instant of interest.

Example: Absolute acceleration of a point glued to a moving frame (2-D): Bug crawling on a tire, again
Here, the acceleration of point P′ glued to the tire, relative to the tire is zero, $\vec{\boldsymbol{a}}_{\mathrm{P}'/\mathcal{B}} = 0$ (see fig. 18.37). The angular velocity of the wheel with respect to the ground is $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} = -\dot{\theta}\hat{\boldsymbol{k}} = -\dot{\theta}\hat{\boldsymbol{k}}'$. The angular speed is increasing at a rate $\ddot{\theta}$. Thus, $\vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}} = -\ddot{\theta}\hat{\boldsymbol{k}} = -\ddot{\theta}\hat{\boldsymbol{k}}'$. The position of P' relative to O' is $\vec{\boldsymbol{r}}_{\mathrm{P}'/\mathrm{O}'} = s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}'$.
Using equation 18.30 on page 4, we get the absolute acceleration of point P' to be

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P}'/\mathcal{F}} &= \vec{\boldsymbol{a}}_{\mathrm{O}'/\mathcal{F}} + \underbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{\mathrm{P}'/\mathrm{O}'})}_{\text{centripetal term}} + \underbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{\mathrm{P}'/\mathrm{O}'}}_{\text{tangential term}} \\
&= \underbrace{R\ddot{\theta}\hat{\boldsymbol{\imath}}}_{\text{acceleration of origin of moving frame}} \overbrace{-\dot{\theta}^2(s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}')}^{\text{centripetal term}} + \overbrace{R\ddot{\theta}(s\,\hat{\boldsymbol{\jmath}}' - \ell\,\hat{\boldsymbol{\imath}}')}^{\text{tangential term}}
\end{aligned}$$

In this example, the absolute acceleration P' is due to:

1. the increase in the translational speed of the tire relative to the ground (acceleration of origin of moving frame),
2. its going in circles at a non-constant rate about point O' relative to the ground (‘tangential term’), and

3. ‘centripetal term’ towards the origin of the moving frame. (In three-dimensional problems, this term is directed towards an axis through $\vec{\omega}$ that goes through O’).

Absolute acceleration of a point moving relative to a moving frame

If we start with the equation for absolute velocity 18.27 on page 2 and differentiate with respect to time we get the absolute acceleration of a point P using a moving frame $\mathcal{B}$. To do this calculation we need to use the product rule of differentiation. Refer to the $\dot{\vec{Q}}$ formula, eqn. (18.24) on page 953. Here is the calculation:

$$\begin{aligned}
\vec{a}_{\mathrm{P}} &= \frac{d}{dt}[\vec{v}_{\mathrm{O'/O}} + \vec{v}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}] \\
&= \vec{a}_{\mathrm{O'/O}} + (\vec{a}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}) \\
&\quad + \ [\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}})] \\
&= \underbrace{\vec{a}_{\mathrm{O'/O}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}) + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}}_{\vec{a}_{\mathrm{P'}}} + \vec{a}_{\mathrm{P}/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}.
\end{aligned}$$

The collection of terms $\vec{a}_{\mathrm{P'}}$ is the acceleration of a point P' which is glued to body $\mathcal{B}$ and is instantaneously coincident with P. It is the same as $\vec{a}_{\mathrm{D}}$ using $D = P'$ in equation 18.30. To repeat, the result is

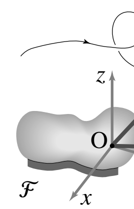


Figure 18.3
which insta
point P’ whi

The ‘five term’ acceleration formula

$$\underbrace{\vec{a}_{\mathrm{P}}}_{\text{The absolute acceleration}} = \overbrace{\vec{a}_{\mathrm{O'/O}}}^{\text{1. acceleration of reference point } O' \text{ on object } \mathcal{B}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}})}_{\text{2. centripetal acceleration: acceleration relative to } O' \text{ of point glued to object, coincident with P, and going in circles around } O' \text{ at constant rate}} + \overbrace{\vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}}^{\text{3. tangential acceleration: acceleration relative to } O' \text{ of point glued to object, coincident with P, due to non-constant } \vec{\omega}} + \underbrace{\vec{a}_{\mathrm{P}/\mathcal{B}}}_{\text{4. acceleration of P relative to moving frame } \mathcal{B}} + \underbrace{2\,\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}}_{\text{5. Coriolis accel.}} \qquad (18.31)$$

$$= \underbrace{\vec{a}_{\mathrm{P'}}}_{\text{The first 3 terms above}} + \vec{a}_{\mathrm{P}/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}} \qquad (18.32)$$

This ‘five-term-acceleration’ formula is both famous and infamous. It’s famous because it is given a lot of emphasis by some instructors(1), and

① **True story.** Once a teacher wrote eqn. (18.31) in big letters on a fresh clean black board, including the explanations in boxes. Then he talked it through, telling the class that this was what they had been building up to. That after all the vector notation, the more involved kinematic development and the messy notation that this was finally the climax. There was still silence in the room as, so the teacher was hoping, the students could feel satisfied with the accomplishment. After time for a slow breath a student asked "Was it as good for you as it was for us?"

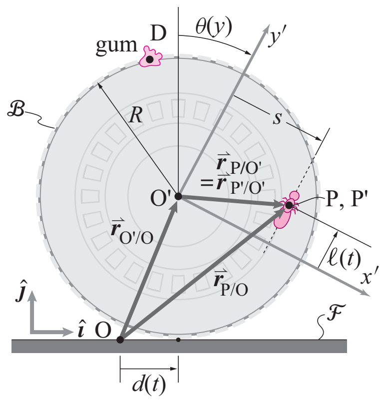


Figure 18.39

infamous because it takes some getting used to. Eqn. 18.32 is the 'three term acceleration formula'. It combines the first three terms in the 5-term formula and interprets them as the acceleration of the point P′ on $\mathcal{B}$ that instantaneously coincides with P. The best way to get used to the five term acceleration formula is to find situations where some of the terms drop out.

Reconsider the bug labeled point P crawling on the tire, object $\mathcal{B}$, in fig. 18.37. To find the absolute acceleration of the bug we need to think about how the bug moves relative to the tire *and* how the tire moves relative to the ground.

Example: Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

From the previous bug examples on page 3 and page 4 we know that

$$\vec{\boldsymbol{v}}_{P/\mathcal{B}} = \dot{\ell}\,\hat{\boldsymbol{\jmath}}', \quad \text{and } \vec{\boldsymbol{a}}_{P/\mathcal{B}} = \ddot{\ell}\,\hat{\boldsymbol{\jmath}}'.$$

Referring to the five term acceleration formula, equation 18.31 on page 5, the absolute acceleration of the bug is

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P/\mathcal{F}} \quad &= \quad \vec{\boldsymbol{a}}_{O'/O} \\
&\quad + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'}) \\
&\quad + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'} \\
&\quad + \vec{\boldsymbol{a}}_{P/\mathcal{B}} \\
&\quad + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}} \\
&= \underbrace{R\ddot{\theta}\hat{\boldsymbol{\imath}} - \dot{\theta}^2(s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}') + \ddot{\theta}(\ell\hat{\boldsymbol{\imath}}' - s\,\hat{\boldsymbol{\jmath}}')}_{\vec{\boldsymbol{a}}_{P'/\mathcal{F}}} + \ddot{\ell}\,\hat{\boldsymbol{\jmath}}' + 2\dot{\theta}\dot{\ell}\hat{\boldsymbol{\imath}}' \\
&= R\ddot{\theta}\hat{\boldsymbol{\imath}} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\theta}\dot{\ell})\hat{\boldsymbol{\imath}}' + (-s\ddot{\theta} + \ddot{\ell} - \ell\dot{\theta}^2)\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

So, at the instant of interest, the bug's absolute acceleration is due to:

1. the translational acceleration of the tire, $\vec{\boldsymbol{a}}_{O'/O} = R\ddot{\theta}\hat{\boldsymbol{\imath}}$
2. the centripetal acceleration of going in circles of radius $\sqrt{s^2 + \ell^2}$ about the center of the tire as it rolls, $-\dot{\theta}^2(s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}')$, pointing at the center of the tire,
3. the tangential acceleration of going in circles about the center of the tire as the tire rolls at non-constant rate, $\ddot{\theta}(\ell\hat{\boldsymbol{\imath}}' - s\,\hat{\boldsymbol{\jmath}}')$,
4. the acceleration of the bug relative to the tire as it crawls on the line, $\vec{\boldsymbol{a}}_{P/\mathcal{B}} = \ddot{\ell}\,\hat{\boldsymbol{\jmath}}'$, and
5. the Coriolis acceleration caused, in part, by the change in direction, relative to the ground, of the velocity of the bug relative to the tire, $2\dot{\theta}\dot{\ell}\hat{\boldsymbol{\imath}}'$.

Items 1, 2 and 3 sum to be the acceleration of point P' *on* the tire but instantaneously coinciding with moving point P.

Motion relative to a point versus motion relative to a frame

We can now give a different interpretation of the expressions we have been using $\vec{\boldsymbol{v}}_{B/A}$ and $\vec{\boldsymbol{a}}_{B/A}$. Rather than thinking of $\vec{\boldsymbol{v}}_{B/A}$ as the difference between $\vec{\boldsymbol{v}}_B$ and $\vec{\boldsymbol{v}}_A$ we can think of $\vec{\boldsymbol{v}}_{B/A}$ as the $\vec{\boldsymbol{v}}_{B/\mathcal{A}}$ where $\mathcal{A}$ is a frame with origin that moves with point A and which has no rotation rate relative to $\mathcal{F}$. That is

$$\vec{\boldsymbol{v}}_{\mathrm{B/A}} \quad \text{means} \quad \vec{\boldsymbol{v}}_{\mathrm{B}/\mathcal{A}}$$

Similarly,

$$\vec{\boldsymbol{a}}_{\mathrm{B/A}} \quad \text{means} \quad \vec{\boldsymbol{a}}_{\mathrm{B}/\mathcal{A}}.$$

18.2 Relation between moving frame formulae and polar coordinate formulae

A similarity exists between the polar coordinate velocity formula

$$\vec{\boldsymbol{v}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$$

and the second two terms in the 'three-term' velocity formula

$$\vec{\boldsymbol{v}}_{\mathrm{P}} = \vec{\boldsymbol{v}}_{\mathrm{O'/O}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O'}} + \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}}.$$

In fact, we have tried to build your understanding of moving frames by means of that connection.

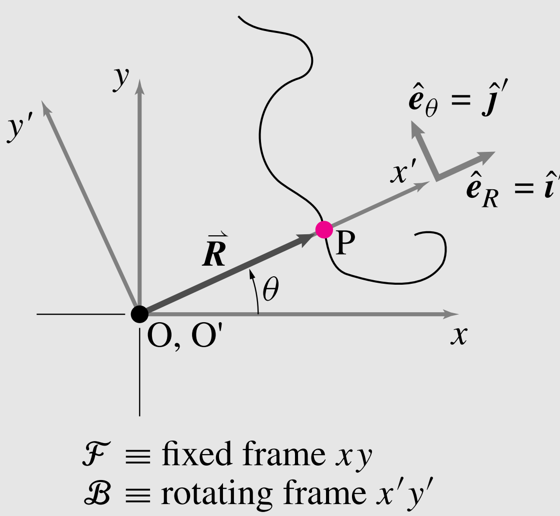
Similarly, the polar coordinate formula for acceleration

$$\vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta$$

is somehow closely linked to the last four terms of the '5-term' acceleration formula

$$\begin{aligned}\vec{\boldsymbol{a}}_{\mathrm{P}} &= \vec{\boldsymbol{a}}_{\mathrm{O'/O}} + \vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O'}}) \\ &\quad + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O'}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}}.\end{aligned}$$

Let's make these connections explicit. Imagine a particle P moving around on the xy-plane.



Let's create a moving frame $\mathcal{B}$ with rotating coordinate system $x'y'$ attached to it whose origin O' is coincident with origin O of a coordinate system xy attached to a fixed frame $\mathcal{F}$. Let this frame rotate in exactly such a way so that the particle is always on the x'-axis. So, in this frame, $\vec{\boldsymbol{r}}_{\mathrm{P/O'}} = R\hat{\boldsymbol{\imath}}'$, $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} = \dot{R}\hat{\boldsymbol{\imath}}'$, and $\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} = \ddot{R}\hat{\boldsymbol{\imath}}'$. Also, the frame motion is characterized by $\vec{\boldsymbol{v}}_{\mathrm{O'/O}} = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{a}}_{\mathrm{O'/O}} = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{k}}$, and $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} = \ddot{\theta}\hat{\boldsymbol{k}}$. So, if we plug in the three-term velocity formula, we get

$$\begin{aligned}\vec{\boldsymbol{v}}_{\mathrm{P}} &= \vec{\boldsymbol{v}}_{\mathrm{O'/O}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O'}} + \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} \\ &= \vec{\boldsymbol{0}} + (\dot{\theta}\hat{\boldsymbol{k}}) \times (R\hat{\boldsymbol{\imath}}') + \dot{R}\hat{\boldsymbol{\imath}}' \\ &= R\dot{\theta}\hat{\boldsymbol{\jmath}}' + \dot{R}\hat{\boldsymbol{\imath}}' \\ &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{R}\hat{\boldsymbol{e}}_r \qquad (\text{because } \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{e}}_r \text{ and } \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{e}}_\theta)\end{aligned}$$

which is the polar coordinate velocity formula.

Similarly, if we plug into the five-term acceleration formula, we get

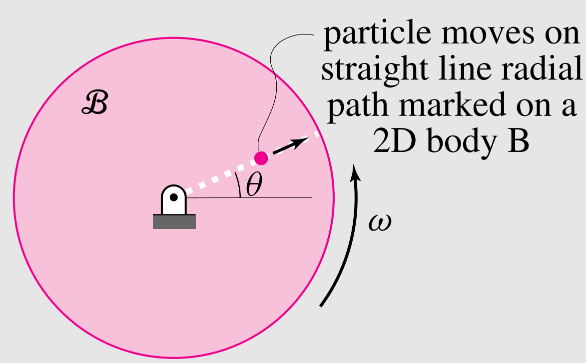
$$\begin{aligned}\vec{\boldsymbol{a}}_{\mathrm{P}} &= \vec{\boldsymbol{a}}_{\mathrm{O'/O}} + \vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O'}}) \\ &\quad + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O'}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} \\ &= \vec{\boldsymbol{0}} + \ddot{R}\hat{\boldsymbol{\imath}}' + \dot{\theta}\hat{\boldsymbol{k}} \times (\dot{\theta}\hat{\boldsymbol{k}} \times R\hat{\boldsymbol{\imath}}') \\ &\quad + (\ddot{\theta}\hat{\boldsymbol{k}}) \times (R\hat{\boldsymbol{\imath}}') + 2(\dot{\theta}\hat{\boldsymbol{k}}) \times \dot{R}\hat{\boldsymbol{\imath}}' \\ &= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta\end{aligned}$$

Again, we recover the appropriate polar coordinate formula.

We have just shown how the polar coordinate formulae are special cases of the relative motion formulae.

Warning!

In problems where we want the rotating frame to be a rotating object on which a particle moves, the polar coordinate formulae only correspond term by term with the relative motion formulae if the particle path is a straight radial line fixed on a $2D$ object, as in the example of a bug walking on a straight line scribed on the surface of a rotating CD or a bead sliding in a tube rotating about an axis perpendicular to the tube.



Because we are sometimes interested in more general relative motions, the polar coordinate formulae do not always apply and we must make use of the more general relative motion formulae.