18.3 General expressions for velocity $\vec{v}$ and acceleration $\vec{a}$ of a moving point

Now that we have some comfort with moving frames we can develop formulas that are not so strongly attached to base vectors. That is, we take account that the base vectors rotate with the frame, but develop formulas that don't use the base vectors explicitly. Thus the formulas we develop here work equally for any frame that is glued to the rotating frame of choice, independent of its orientation.

Absolute velocity of a point moving relative to a moving frame

Imagine that you know the absolute velocity $\vec{v}_{0'/\mathcal{F}}$ of some point O' on an object $\mathcal{B}$, say the center of a car tire and the angular velocity of the tire, $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$. Finally, imagine you also know the relative velocity of point P, $\vec{v}_{\mathrm{P}/\mathcal{B}}$, say of a bug crawling on the tire.

If the frame $\mathcal{B}$ is translating or rotating, the velocity of particle P relative to the frame $\vec{v}_{\mathrm{P}/\mathcal{B}}$ is not the absolute velocity (the velocity relative to a Newtonian frame). The absolute velocity in this case is $\vec{v}_{\mathrm{P}/\mathcal{F}}$, or more simply $\vec{v}_{\mathrm{P}}$, or more simply still, just $\vec{v}$. We want to know the relationship between the relative velocity $\vec{v}_{\mathrm{P}/\mathcal{B}}$ and the absolute velocity $\vec{v}_{\mathrm{P}/\mathcal{F}}$ (otherwise known as just $\vec{v}$).

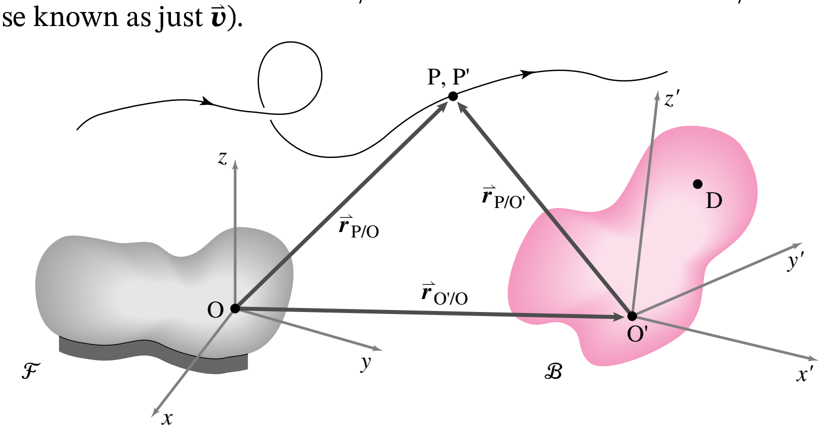


Figure 18.32: The position of point P relative to the origin O' of moving frame $\mathcal{B}$ is $\vec{r}_{\mathrm{P/O'}}$. The position of the origin of the frame $\mathcal{B}$ relative to the origin O of the fixed frame $\mathcal{F}$ is $\vec{r}_{\mathrm{O'/O}}$. The position of point P relative to O is the sum of $\vec{r}_{\mathrm{P/O'}}$ and $\vec{r}_{\mathrm{O'/O}}$. The motion of point P relative to the fixed frame may be complicated.

Let's start by looking at the position. The position of a point P that is moving is:

$$\vec{r}_{\mathrm{P/O}} = \vec{r}_{\mathrm{O'/O}} + \vec{r}_{\mathrm{P/O'}}$$

where O' is the origin of a coordinate system which is glued to the rigid object, as shown in fig. 18.32.

To find the absolute velocity of point P we will use the $\dot{\vec{Q}}$ formula, equation 18.19, for computing the rate of change of a vector. The velocity of P is the rate of change of its position. Here, we use $\vec{Q} = \vec{r}_{P/O'}$

$$
\begin{aligned}
\vec{v}_{P/\mathcal{F}} &= {}^{\mathcal{F}}\dot{\vec{r}}_{P/O} \\
&= {}^{\mathcal{F}}\dot{\vec{r}}_{O'/O} + {}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} \\
&= \vec{v}_{O'/\mathcal{F}} + \overbrace{\underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}}^{{}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}}
\end{aligned}
$$

The $\dot{\vec{Q}}$ formula 18.19 was used in the calculation to compute ${}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}$

$$
{}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} = {}^{\mathcal{B}}\dot{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}. \tag{18.27}
$$

Thus, the 'three term velocity formula'.

$$
\overbrace{\vec{v}_{P}}^{\boxed{\text{Absolute velocity of point P}}} = \underbrace{\vec{v}_{O'}}_{\boxed{\text{1. Velocity of reference point } O' \text{ on object } \mathcal{B}}} + \overbrace{\vec{v}_{P/\mathcal{B}}}^{\boxed{\text{2. Velocity of point P relative to moving frame } \mathcal{B}}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}_{\boxed{\text{3. Velocity relative to point } O' \text{ of the point } P' \text{ glued to object } \mathcal{B} \text{ which is instantaneously coincident with point P}}} \tag{18.28}
$$

Another way to write the formula for absolute velocity is as

$$
\vec{v}_{P} = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}
$$

where P' is a point glued to $\mathcal{B}$ which is instantaneously coincident with P, so the absolute velocity of P' is

$$
\vec{v}_{P'} = \vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}. \tag{18.29}
$$

Reconsider the bug crawling on the tire, object $\mathcal{B}$, in fig. 18.33. To find the absolute velocity of the bug, we need to be concerned with how the bug moves relative to the tire *and* how the tire moves relative to the ground.

Example: Absolute velocity of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

Referring to equation 18.27 on page 2, the absolute velocity of the bug is

$$\begin{aligned}\vec{v}_{P/\mathcal{F}} &= \vec{v}_{O'/\mathcal{F}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} \\ &= R\dot{\theta}\hat{\imath} \quad + \quad \dot{\ell}\,\hat{\jmath}' \quad + \quad (-\dot{\theta}\hat{k}') \times (s\hat{\imath}' + \ell\,\hat{\jmath}') \\ &= R\dot{\theta}\hat{\imath} \quad + \quad \ell\dot{\theta}\hat{\imath}' \quad + \quad (\dot{\ell} - s\dot{\theta})\hat{\jmath}'.\end{aligned}$$

At the instant of interest, the direction of the bug's absolute velocity depends upon the relative magnitudes of $\dot{\ell}$ and $\dot{\theta}$ as well as the orientation of $\hat{\imath}'$ and $\hat{\jmath}'$.

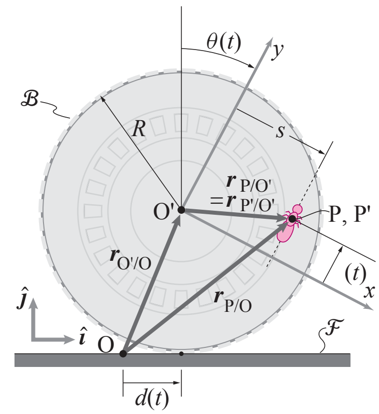


Figure 18.33

As we noted earlier, another way to write the formula for absolute velocity is

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where, in the example above, $\vec{v}_{P'} = R\dot{\theta}\hat{\imath} + \dot{\theta}(\ell\hat{\imath}' - s\,\hat{\jmath}')$ and $\vec{v}_{P/\mathcal{B}} = \dot{\ell}\,\hat{\jmath}'$. At the instant of concern, we can think of the absolute velocity of the bug as the velocity of the mark labeled P' under the bug *plus* the velocity of the bug relative to the tire.

Acceleration

We would like to find acceleration of a point using information about its motion relative to a moving frame. The result, the 'five term acceleration formula' is the most complicated formula in this book. (For reference, it is in Table II, 5c).

Acceleration relative to an object or frame

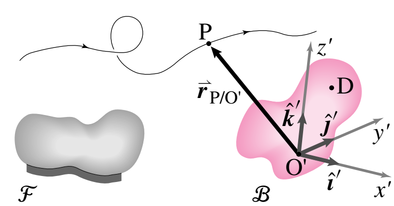


Figure 18.34: An object or reference frame $\mathcal{B}$ and a point P move around. We are interested in the acceleration of point P.

The acceleration of a point relative to an object or frame is the acceleration you would calculate if you were looking at the particle while you translated and rotated with the frame and took no account of the outside world. That is, if the position of a particle P relative to the origin O' of a coordinate system in a moving frame $\mathcal{B}$ is given by:

$$\vec{r}_{P/O'} = \underbrace{r_{Px'/O'}}_{x'}\hat{\imath}' + \underbrace{r_{Py'/O'}}_{y'}\hat{\jmath}' + \underbrace{r_{Pz'/O'}}_{z'}\hat{k}',$$

then the acceleration of the particle P relative to the frame is:

$$\vec{a}_{P/\mathcal{B}} = \underbrace{\ddot{r}_{Px'/O'}}_{\ddot{x}'}\hat{\imath}' + \underbrace{\ddot{r}_{Py'/O'}}_{\ddot{y}'}\hat{\jmath}' + \underbrace{\ddot{r}_{Pz'/O'}}_{\ddot{z}'}\hat{k}'.$$

That is, the acceleration relative to the frame takes no account of (a) the motion of the frame or of (b) the rotation of the base vectors with the frame to which they are fixed.

Reconsider the bug labeled point P crawling on the tire, object $\mathcal{B}$, in fig. 18.35.

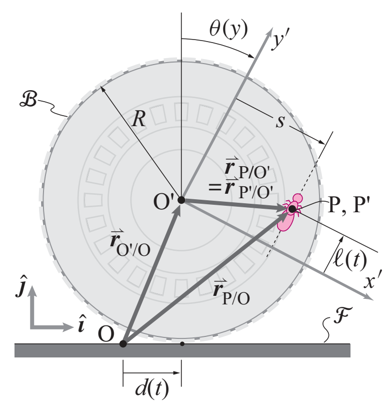


Figure 18.35: A bug crawling on a tire at point P. Point P' is on the tire and instantaneously corresponds to point P.

Example: Acceleration relative to a frame (2-D): Bug crawling on a tire, again
If we are sitting on the tire, all that we see is the bug crawling in a straight line at non-constant rate relative to us. Thus, its acceleration relative to the tire is

$$\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} = \ddot{\ell}\,\hat{\boldsymbol{\jmath}}'.$$

So, at the instant of interest, the bug has an acceleration relative to the tire frame parallel to the y'-axis.

Absolute acceleration of a point P′ glued to a moving frame

Imagine that you know the absolute acceleration of some point O' at the center of a frame $\mathcal{B}$, say the center of a car tire. Imagine you also know the angular velocity of the tire, $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$, and the angular acceleration, $\vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}}$. Then, you can find the absolute acceleration of a piece of gum labeled point D stuck to the sidewall (see fig. 18.37). If we start with the equation 18.29 for the absolute velocity of a point glued to a moving frame on page 2 and differentiate with respect to time, we get the absolute acceleration of a point D fixed in a moving frame $\mathcal{B}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{D}} &= \frac{d}{dt}[\vec{\boldsymbol{v}}_{\mathrm{O}'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'}] \\
&= \vec{\boldsymbol{a}}_{\mathrm{O}'/\mathrm{O}} + [\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'})] \\
&= \vec{\boldsymbol{a}}_{\mathrm{O}'/\mathrm{O}} + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{O}'}) \qquad (18.30)
\end{aligned}$$

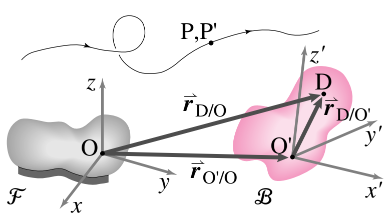


Figure 18.36: Tracking a moving particle from two different reference frames.

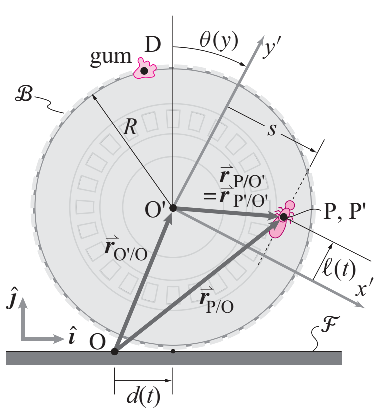


Figure 18.37: Gum is glued to a rolling tire at D. The point P′ glued to the tire is just where the point P happens to be passing at the instant of interest.

Example: Absolute acceleration of a point glued to a moving frame (2-D): Bug crawling on a tire, again
Here, the acceleration of point P′ glued to the tire, relative to the tire is zero, $\vec{\boldsymbol{a}}_{\mathrm{P}'/\mathcal{B}} = 0$ (see fig. 18.37). The angular velocity of the wheel with respect to the ground is $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} = -\dot{\theta}\hat{\boldsymbol{k}} = -\dot{\theta}\hat{\boldsymbol{k}}'$. The angular speed is increasing at a rate $\ddot{\theta}$. Thus, $\vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}} = -\ddot{\theta}\hat{\boldsymbol{k}} = -\ddot{\theta}\hat{\boldsymbol{k}}'$. The position of P' relative to O' is $\vec{\boldsymbol{r}}_{\mathrm{P}'/\mathrm{O}'} = s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}'$.
Using equation 18.30 on page 4, we get the absolute acceleration of point P' to be

$$\vec{\boldsymbol{a}}_{\mathrm{P}'/\mathcal{F}} = \vec{\boldsymbol{a}}_{\mathrm{O}'/\mathcal{F}} + \underbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{\mathrm{P}'/\mathrm{O}'})}_{\text{centripetal term}} + \underbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{\mathrm{P}'/\mathrm{O}'}}_{\text{tangential term}}$$

$$= \underbrace{R\ddot{\theta}\hat{\boldsymbol{\imath}}}_{\text{acceleration of origin of moving frame}} - \overbrace{\dot{\theta}^2(s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}')}^{\text{centripetal term}} + \overbrace{R\ddot{\theta}(s\,\hat{\boldsymbol{\jmath}}' - \ell\,\hat{\boldsymbol{\imath}}')}^{\text{tangential term}}$$

In this example, the absolute acceleration P' is due to:

1. the increase in the translational speed of the tire relative to the ground (acceleration of origin of moving frame),
2. its going in circles at a non-constant rate about point O' relative to the ground (‘tangential term’), and

3. 'centripetal term' towards the origin of the moving frame. (In three-dimensional problems, this term is directed towards an axis through $\vec{\omega}$ that goes through O').

Absolute acceleration of a point moving relative to a moving frame

If we start with the equation for absolute velocity 18.27 on page 2 and differentiate with respect to time we get the absolute acceleration of a point P using a moving frame $\mathcal{B}$. To do this calculation we need to use the product rule of differentiation. Refer to the $\dot{\vec{Q}}$ formula, eqn. (18.24) on page 953. Here is the calculation:

$$
\begin{aligned}
\vec{a}_{\mathrm{P}} &= \frac{d}{dt}[\vec{v}_{\mathrm{O'/O}} + \vec{v}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}] \\
&= \vec{a}_{\mathrm{O'/O}} + (\vec{a}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}) \\
&\quad + \ [\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}})] \\
&= \underbrace{\vec{a}_{\mathrm{O'/O}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}) + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}}_{\vec{a}_{\mathrm{P'}}} + \vec{a}_{\mathrm{P}/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}.
\end{aligned}
$$

The collection of terms $\vec{a}_{\mathrm{P'}}$ is the acceleration of a point P' which is glued to body $\mathcal{B}$ and is instantaneously coincident with P. It is the same as $\vec{a}_{\mathrm{D}}$ using $D = P'$ in equation 18.30. To repeat, the result is

The 'five term' acceleration formula

1. acceleration of reference point O' on object $\mathcal{B}$

3. tangential acceleration: acceleration relative to O' of point glued to object, coincident with P, due to non-constant $\vec{\omega}$

$$
\underbrace{\vec{a}_{\mathrm{P}}}_{\text{The absolute acceleration}} = \overbrace{\vec{a}_{\mathrm{O'/O}}}^{1} + \underbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}})}_{2} + \overbrace{\vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}}^{3} + \underbrace{\vec{a}_{\mathrm{P}/\mathcal{B}}}_{4} + \underbrace{2\,\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}}_{\text{5. Coriolis accel.}} \tag{18.31}
$$

2. centripetal acceleration: acceleration relative to O' of point glued to object, coincident with P, and going in circles around O' at constant rate

4. acceleration of P relative to moving frame $\mathcal{B}$

$$
= \underbrace{\vec{a}_{\mathrm{P'}}}_{\text{The first 3 terms above}} + \vec{a}_{\mathrm{P}/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}} \tag{18.32}
$$

This 'five-term-acceleration' formula is both famous and infamous. It's famous because it is given a lot of emphasis by some instructors①, and

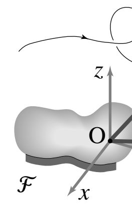


Figure 18.3
which insta
point P' whi

(1) **True story.** Once a teacher wrote eqn. (18.31) in big letters on a fresh clean black board, including the explanations in boxes. Then he talked it through, telling the class that this was what they had been building up to. That after all the vector notation, the more involved kinematic development and the messy notation that this was finally the climax. There was still silence in the room as, so the teacher was hoping, the students could feel satisfied with the accomplishment. After time for a slow breath a student asked "Was it as good for you as it was for us?"

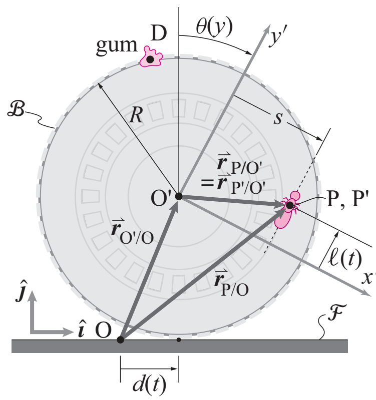


Figure 18.39

infamous because it takes some getting used to. Eqn. 18.32 is the 'three term acceleration formula'. It combines the first three terms in the 5-term formula and interprets them as the acceleration of the point P′ on $\mathcal{B}$ that instantaneously coincides with P. The best way to get used to the five term acceleration formula is to find situations where some of the terms drop out.

Reconsider the bug labeled point P crawling on the tire, object $\mathcal{B}$, in fig. 18.37. To find the absolute acceleration of the bug we need to think about how the bug moves relative to the tire *and* how the tire moves relative to the ground.

Example: Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

From the previous bug examples on page 3 and page 4 we know that

$$\vec{v}_{P/\mathcal{B}} = \dot{\ell}\,\hat{\jmath}', \quad \text{and } \vec{a}_{P/\mathcal{B}} = \ddot{\ell}\,\hat{\jmath}'.$$

Referring to the five term acceleration formula, equation 18.31 on page 5, the absolute acceleration of the bug is

$$\begin{aligned}
\vec{a}_{P/\mathcal{F}} \quad &= \quad \vec{a}_{O'/O} \\
&\quad + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}) \\
&\quad + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P/O'} \\
&\quad + \vec{a}_{P/\mathcal{B}} \\
&\quad + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} \\
&= \underbrace{R\ddot{\theta}\hat{\imath} - \dot{\theta}^2(s\hat{\imath}' + \ell\,\hat{\jmath}') + \ddot{\theta}(\ell\hat{\imath}' - s\,\hat{\jmath}')}_{\vec{a}_{P'/\mathcal{F}}} + \ddot{\ell}\,\hat{\jmath}' + 2\dot{\theta}\dot{\ell}\,\hat{\imath}' \\
&= R\ddot{\theta}\hat{\imath} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\theta}\dot{\ell})\hat{\imath}' + (-s\ddot{\theta} + \ddot{\ell} - \ell\dot{\theta}^2)\hat{\jmath}'.
\end{aligned}$$

So, at the instant of interest, the bug's absolute acceleration is due to:

1. the translational acceleration of the tire, $\vec{a}_{O'/O} = R\ddot{\theta}\hat{\imath}$
2. the centripetal acceleration of going in circles of radius $\sqrt{s^2 + \ell^2}$ about the center of the tire as it rolls, $-\dot{\theta}^2(s\hat{\imath}' + \ell\,\hat{\jmath}')$, pointing at the center of the tire,
3. the tangential acceleration of going in circles about the center of the tire as the tire rolls at non-constant rate, $\ddot{\theta}(\ell\hat{\imath}' - s\,\hat{\jmath}')$,
4. the acceleration of the bug relative to the tire as it crawls on the line, $\vec{a}_{P/\mathcal{B}} = \ddot{\ell}\,\hat{\jmath}'$, and
5. the Coriolis acceleration caused, in part, by the change in direction, relative to the ground, of the velocity of the bug relative to the tire, $2\dot{\theta}\dot{\ell}\,\hat{\imath}'$.

Items 1, 2 and 3 sum to be the acceleration of point P' *on* the tire but instantaneously coinciding with moving point P.

Motion relative to a point versus motion relative to a frame

We can now give a different interpretation of the expressions we have been using $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$. Rather than thinking of $\vec{v}_{B/A}$ as the difference between $\vec{v}_B$ and $\vec{v}_A$ we can think of $\vec{v}_{B/A}$ as the $\vec{v}_{B/\mathcal{A}}$ where $\mathcal{A}$ is a frame with origin that moves with point A and which has no rotation rate relative to $\mathcal{F}$. That is

$$\vec{\boldsymbol{v}}_{\mathrm{B/A}} \quad \text{means} \quad \vec{\boldsymbol{v}}_{\mathrm{B}/\mathcal{A}}$$

Similarly,

$$\vec{\boldsymbol{a}}_{\mathrm{B/A}} \quad \text{means} \quad \vec{\boldsymbol{a}}_{\mathrm{B}/\mathcal{A}}.$$

18.2 Relation between moving frame formulae and polar coordinate formulae

A similarity exists between the polar coordinate velocity formula

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

and the second two terms in the 'three-term' velocity formula

$$\vec{v}_P = \vec{v}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}}.$$

In fact, we have tried to build your understanding of moving frames by means of that connection.

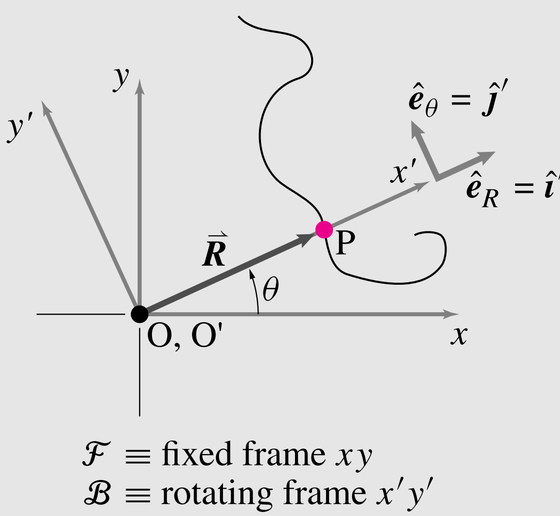
Similarly, the polar coordinate formula for acceleration

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$$

is somehow closely linked to the last four terms of the '5-term' acceleration formula

$$\begin{aligned}\vec{a}_P &= \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) \\ &\quad + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}.\end{aligned}$$

Let's make these connections explicit. Imagine a particle P moving around on the xy-plane.



Let's create a moving frame $\mathcal{B}$ with rotating coordinate system $x'y'$ attached to it whose origin O' is coincident with origin O of a coordinate system xy attached to a fixed frame $\mathcal{F}$. Let this frame rotate in exactly such a way so that the particle is always on the x'-axis. So, in this frame, $\vec{r}_{P/O'} = R\hat{\imath}'$, $\vec{v}_{P/\mathcal{B}} = \dot{R}\hat{\imath}'$, and $\vec{a}_{P/\mathcal{B}} = \ddot{R}\hat{\imath}'$. Also, the frame motion is characterized by $\vec{v}_{O'/O} = \vec{0}, \vec{a}_{O'/O} = \vec{0}$, $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{k}$, and $\dot{\vec{\omega}}_{\mathcal{B}} = \ddot{\theta}\hat{k}$. So, if we plug in the three-term velocity formula, we get

$$\begin{aligned}\vec{v}_P &= \vec{v}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}} \\ &= \vec{0} + (\dot{\theta}\hat{k}) \times (R\hat{\imath}') + \dot{R}\hat{\imath}' \\ &= R\dot{\theta}\hat{j}' + \dot{R}\hat{\imath}' \\ &= R\dot{\theta}\hat{e}_\theta + \dot{R}\hat{e}_r \qquad (\text{because } \hat{\imath}' = \hat{e}_r \text{ and } \hat{j}' = \hat{e}_\theta)\end{aligned}$$

which is the polar coordinate velocity formula.

Similarly, if we plug into the five-term acceleration formula, we get

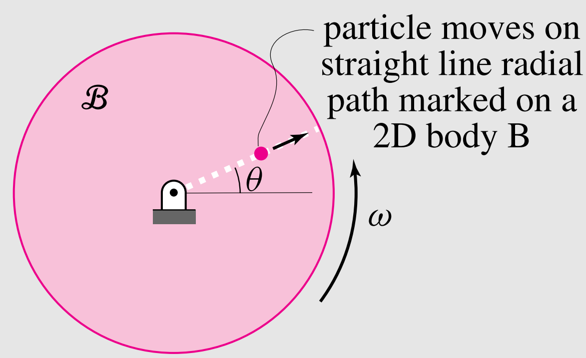
$$\begin{aligned}\vec{a}_P &= \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) \\ &\quad + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} \\ &= \vec{0} + \ddot{R}\hat{\imath}' + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times R\hat{\imath}') \\ &\quad + (\ddot{\theta}\hat{k}) \times (R\hat{\imath}') + 2(\dot{\theta}\hat{k}) \times \dot{R}\hat{\imath}' \\ &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\end{aligned}$$

Again, we recover the appropriate polar coordinate formula.

We have just shown how the polar coordinate formulae are special cases of the relative motion formulae.

Warning!

In problems where we want the rotating frame to be a rotating object on which a particle moves, the polar coordinate formulae only correspond term by term with the relative motion formulae if the particle path is a straight radial line fixed on a 2D object, as in the example of a bug walking on a straight line scribed on the surface of a rotating CD or a bead sliding in a tube rotating about an axis perpendicular to the tube.



Because we are sometimes interested in more general relative motions, the polar coordinate formulae do not always apply and we must make use of the more general relative motion formulae.

SAMPLE 18.8 A ‘T’ shaped tube is welded to a massless rigid arm OAB which rotates about O at a constant rate $\omega = 5\,\mathrm{rad/s}$. See also Sample 18.26. At the instant shown a particle P is falling down in the vertical section of the tube with speed $v_{/\mathrm{tube}} = 4\,\mathrm{ft/s}$. Find the absolute velocity of the particle. Take $\ell = 2\,\mathrm{ft}$ in the figure.

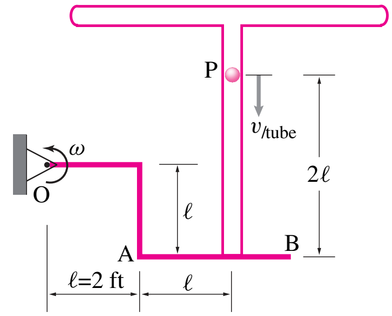
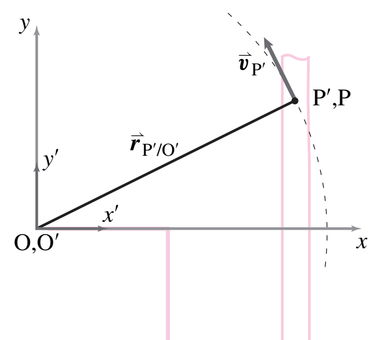


Figure 18.40

Solution Let us attach a frame $\mathcal{B}$ to arm OAB. Thus $\mathcal{B}$ rotates with OAB with angular velocity $\vec{\omega}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}}$ where $\omega = 5\,\mathrm{rad/s}$. To do calculations in $\mathcal{B}$ we attach a coordinate system $x'y'z'$ to $\mathcal{B}$ at point O. At the instant of interest the rotating coordinate system $x'y'z'$ coincides with the fixed coordinate system xyz. (Since the entire motion is in the xy-plane, the z-axis is not shown in the figure). Let P' be a point coincident with P but fixed in $\mathcal{B}$. Now,

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\mathrm{rel}}$$



where

$$\begin{aligned}
\vec{\boldsymbol{v}}_{P'} &= \underbrace{\vec{\boldsymbol{v}}_{O'}}_{\vec{\mathbf{0}}} + \vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} \\
&= \omega\hat{\boldsymbol{k}} \times (2\ell\hat{\boldsymbol{\imath}} + \ell\,\hat{\boldsymbol{\jmath}}) \\
&= 2\omega\ell\,\hat{\boldsymbol{\jmath}} - \omega\ell\hat{\boldsymbol{\imath}},
\end{aligned}$$

and

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{rel}} &= \text{Velocity relative to the frame } \mathcal{B} \\
&= -v_{/\mathrm{tube}}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\vec{\boldsymbol{v}}_P &= \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{rel} \\
&= -\omega\ell\hat{\boldsymbol{\imath}} + (2\omega\ell - v_{/\mathrm{tube}})\hat{\boldsymbol{\jmath}} \\
&= -5\,\mathrm{rad/s}\cdot 2\,\mathrm{ft}\hat{\boldsymbol{\imath}} + (2\cdot 5\,\mathrm{rad/s}\cdot 2\,\mathrm{ft} - 4\,\mathrm{ft/s})\hat{\boldsymbol{\jmath}} \\
&= -10\,\mathrm{ft/s}\hat{\boldsymbol{\imath}} + 16\,\mathrm{ft/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_P = -10\,\mathrm{ft/s}\hat{\boldsymbol{\imath}} + 16\,\mathrm{ft/s}\hat{\boldsymbol{\jmath}}}$$

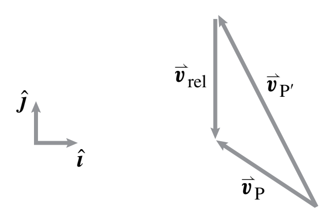


Figure 18.41:

Comments: The kinematics calculation is equivalent to the vector addition shown in Figure 18.41. The velocity of P is the sum of $\vec{\boldsymbol{v}}_{P'}$ and $\vec{\boldsymbol{v}}_{\mathrm{rel}} = \vec{\boldsymbol{v}}_{P/\mathcal{B}}$.

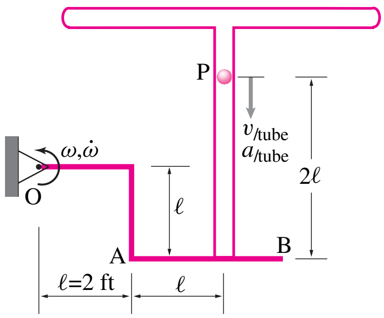


Figure 18.42

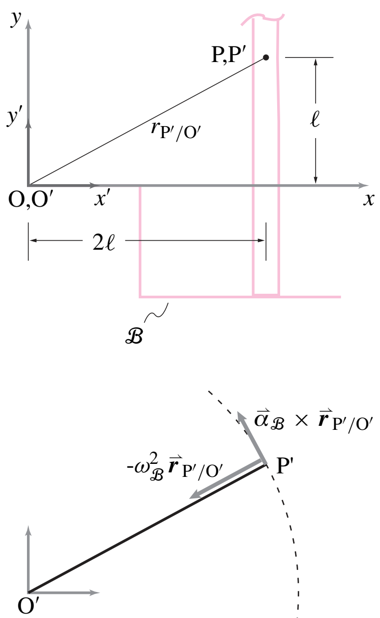


Figure 18.43: Acceleration of point P′. P′ is fixed to the frame $\mathcal{B}$ and at the moment coincides with point P. Therefore, $\vec{\boldsymbol{a}}_{P'} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} - \omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{P'/O'}$.

SAMPLE 18.9 Acceleration of a point in a rotating frame. Consider the rotating tube of Sample 18.8 again. The arm OAB rotates with counterclockwise angular acceleration $\dot\omega = 3\,\text{rad/s}^2$ and, at the instant shown, its angular speed $\omega = 5\,\text{rad/s}$. Also, at the same instant, the particle P falls down with speed $v_{/\text{tube}} = 4\,\text{ft/s}$ and acceleration $a_{/\text{tube}} = 2\,\text{ft/s}^2$. Find the absolute acceleration of the particle at the given instant. Take $\ell = 2\,\text{ft}$ in the figure.

Solution We consider a frame $\mathcal{B}$, with coordinate axes $x'y'z'$, fixed to the arm OAB and thus rotating with $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \dot\omega\hat{\boldsymbol{k}} = 3\,\text{rad/s}^2\hat{\boldsymbol{k}}$. The acceleration of point P is given by

$$\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{\text{cor}} + \vec{\boldsymbol{a}}_{\text{rel}}$$

where

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= \text{acceleration of a point P}' \text{ that is fixed in } \mathcal{B} \\
& \text{and at the moment coincides with P,} \\
\vec{\boldsymbol{a}}_{\text{cor}} &= \text{Coriolis acceleration, and} \\
\vec{\boldsymbol{a}}_{\text{rel}} &= \text{acceleration of P relative to frame } \mathcal{B}.
\end{aligned}$$

Now we calculate each of these terms separately. For calculating $\vec{\boldsymbol{a}}_{P'}$, imagine a rigid rod from point O to point P, rotating with the frame $\mathcal{B}$. Mark the far end of the rod as P′ (same as point P). The acceleration of this end of the rod is $\vec{\boldsymbol{a}}_{P'}$. To find the relative terms $\vec{\boldsymbol{v}}_{\text{rel}}$ and $\vec{\boldsymbol{a}}_{\text{rel}}$, freeze the motion of the frame $\mathcal{B}$ at the given moment and watch the motion of point P. The non-intuitive term $\vec{\boldsymbol{a}}_{\text{cor}}$ has no such simple physical interpretation but has a simple formula. Thus,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= \overbrace{\vec{\boldsymbol{a}}_{O'}}^{\vec{0}} + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'})}^{-\omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{P'/O'}} \\
&= \dot\omega\hat{\boldsymbol{k}} \times (2\ell\hat{\boldsymbol{\imath}} + \ell\hat{\boldsymbol{\jmath}}) - \omega^2(2\ell\hat{\boldsymbol{\imath}} + \ell\hat{\boldsymbol{\jmath}}) \\
&= 2\dot\omega\ell\hat{\boldsymbol{\jmath}} - \dot\omega\ell\hat{\boldsymbol{\imath}} - 2\omega^2\ell\hat{\boldsymbol{\imath}} - \omega^2\ell\hat{\boldsymbol{\jmath}} \\
&= -\ell(\dot\omega + 2\omega^2)\hat{\boldsymbol{\imath}} + \ell(2\dot\omega - \omega^2)\hat{\boldsymbol{\jmath}} \\
&= -(106\hat{\boldsymbol{\imath}} + 38\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{cor}} &= 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\text{rel}} \\
&= 2\omega\hat{\boldsymbol{k}} \times v_{/\text{tube}}(-\hat{\boldsymbol{\jmath}}) \\
&= 2\omega v_{/\text{tube}}\hat{\boldsymbol{\imath}} = 2 \cdot 5\,\text{rad/s} \cdot 4\,\text{ft/s} \\
&= 40\,\text{ft/s}^2\hat{\boldsymbol{\imath}},
\end{aligned}$$

$$\vec{\boldsymbol{a}}_{\text{rel}} = a_{/\text{tube}}(-\hat{\boldsymbol{\jmath}}) = -2\,\text{ft/s}^2\hat{\boldsymbol{\jmath}}.$$

Adding the three terms together, we get

$$\begin{aligned}
\vec{\boldsymbol{a}}_P &= -106\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 38\,\text{ft/s}^2\hat{\boldsymbol{\jmath}} + 40\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 2\,\text{ft/s}^2\hat{\boldsymbol{\jmath}} \\
&= -66\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 40\,\text{ft/s}^2\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_P = -(66\hat{\boldsymbol{\imath}} + 40\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2}$$

Note that the single term $\vec{\boldsymbol{a}}_{P'}$ encompasses three terms of the five term acceleration formula.

SAMPLE 18.10 A small collar P is pinned to a rigid rod AB at length $\ell = 1\,\text{m}$ along the rod. The collar is free to slide in a straight track on a disk of radius $r = 400\,\text{mm}$. The disk rotates about its center O at a constant $\omega = 2\,\text{rad/s}$. At the instant shown, when $\theta = 45^\circ$ and the collar is at a distance $\frac{3}{4}r$ in the track from the center O, find

1. the angular velocity of the rod AB and
2. the velocity of point P relative to the disk.

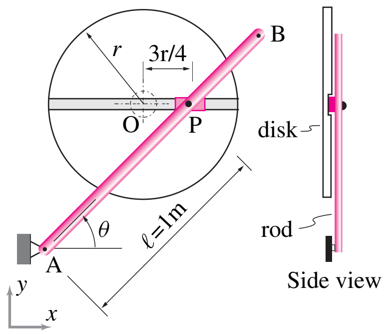
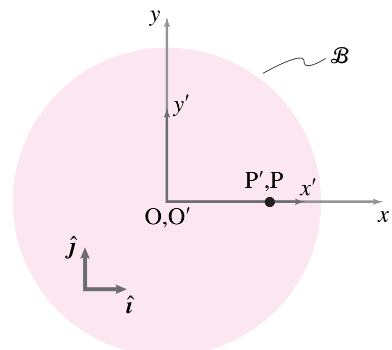


Figure 18.44:



xy is the fixed frame. $x'y'$ rotates with the disk with ω_B. P′ is fixed in the rotating frame.

Solution We will think of P in two ways: one as attached to the rod and the other as sliding in the slot. First, let us attach a frame $\mathcal{B}$ to the disk. Thus $\mathcal{B}$ rotates with the disk with angular velocity $\vec{\omega}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}} = 2\,\text{rad/s}\hat{\boldsymbol{k}}$. We attach a coordinate system $x'y'z'$ to $\mathcal{B}$ at point O. At the instant of interest, the rotating coordinate system $x'y'z'$ coincides with the fixed coordinate system xyz. Now let us consider point P' which is fixed on the disk (and hence in $\mathcal{B}$) and coincides with point P at the instant of interest. We can write the velocity of P as:

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}}$$

where

$$\begin{aligned}
\vec{\boldsymbol{v}}_{P'} &= \overbrace{\vec{\boldsymbol{v}}_{O'}}^{\vec{\mathbf{0}}} + \vec{\omega}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} = \omega\hat{\boldsymbol{k}} \times \left(\frac{3}{4}r\hat{\boldsymbol{\imath}}'\right) = \frac{3}{4}\omega r\,\hat{\boldsymbol{\jmath}}' = \frac{3}{4}\omega r\,\hat{\boldsymbol{\jmath}}, \\
\vec{\boldsymbol{v}}_{\text{rel}} &\equiv \vec{\boldsymbol{v}}_{P/\mathcal{B}} = v_{\text{rel}}\hat{\boldsymbol{\imath}}' = v_{\text{rel}}\hat{\boldsymbol{\imath}}.
\end{aligned}$$

In the last expression, $\vec{\boldsymbol{v}}_{\text{rel}} = v_{\text{rel}}\hat{\boldsymbol{\imath}}$, we do not know the magnitude of $\vec{\boldsymbol{v}}_{\text{rel}}$ and hence have left it as an unknown v_{rel}, but its direction is known because $\vec{\boldsymbol{v}}_{\text{rel}}$ has to be along the track and the track at the given instant is along the x-axis. Thus,

$$\vec{\boldsymbol{v}}_P = \frac{3}{4}\omega r\,\hat{\boldsymbol{\jmath}} + v_{\text{rel}}\hat{\boldsymbol{\imath}}. \tag{18.33}$$

Now let us consider the motion of rod AB. Let $\vec{\boldsymbol{\Omega}} = \Omega\hat{\boldsymbol{k}}$ be the angular velocity of AB at the instant of interest where Ω is unknown. Since P is pinned to the rod, it executes circular motion about A with radius $AP = \ell$. Therefore,

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{r}}_{P/A} = \Omega\hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) = \Omega\ell(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}). \tag{18.34}$$

But, and this trivial formula is the key, $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_P$. Therefore, from Eqn. (18.33) and (18.34),

$$\frac{3}{4}\omega r\,\hat{\boldsymbol{\jmath}} + v_{\text{rel}}\hat{\boldsymbol{\imath}} = \Omega\ell(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}). \tag{18.35}$$

Taking dot product of both sides of the above equation with $\hat{\boldsymbol{\jmath}}$ we get

$$\begin{aligned}
\frac{3}{4}\omega r &= \Omega\ell\cos\theta \\
\Rightarrow \quad \Omega &= \frac{3\omega r}{4\ell\cos\theta} = \frac{3\cdot 2\,\text{rad/s}\cdot 0.4\,\text{m}}{4\cdot 1\,\text{m}\cdot\frac{1}{\sqrt{2}}} = 0.85\,\text{rad/s}.
\end{aligned}$$

Again taking the dot product of both sides of Eqn. (18.35) with $\hat{\boldsymbol{\imath}}$ we get

$$v_{\text{rel}} = -\Omega\ell\sin\theta = -0.85\,\text{rad/s}\cdot 1\,\text{m}\cdot\frac{1}{\sqrt{2}} = -0.6\,\text{m/s}.$$

$$(i)\ \vec{\boldsymbol{\Omega}} = 0.85\,\text{rad/s}\hat{\boldsymbol{k}}, \quad (ii)\ \vec{\boldsymbol{v}}_{\text{rel}} = -0.6\,\text{m/s}\hat{\boldsymbol{\imath}}$$

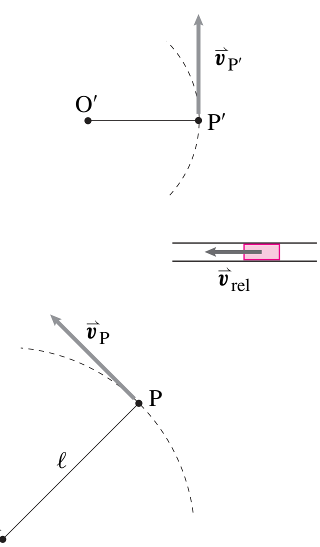


Figure 18.45:

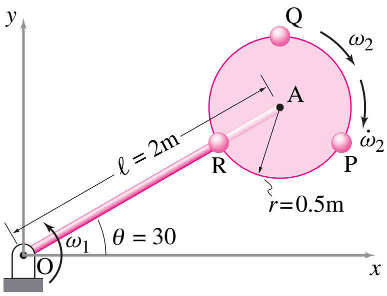


Figure 18.46

SAMPLE 18.11 Spinning wheel on a rotating rod in 2-D . A rigid body OA is attached to a wheel that is massless except for three point masses P, Q, and R, placed symmetrically on the wheel. Each of the three masses is $m = 0.5\,\text{kg}$. The rod OA rotates about point O in the counterclockwise direction at a constant rate $\omega_1 = 3\,\text{rad/s}$. The wheel rotates with respect to the arm about point A with angular acceleration $\dot{\omega}_2 = 1\,\text{rad/s}^2$ and at the instant shown it has angular speed $\omega_2 = 5\,\text{rad/s}$. Note that both ω_2 and $\dot{\omega}_2$ are given with respect to the arm.

Using a rotating frame $\mathcal{B}$ attached to the rod and a coordinate system attached to the frame with origin at O, find

1. the velocity of the mass P and
2. the acceleration of the mass P.

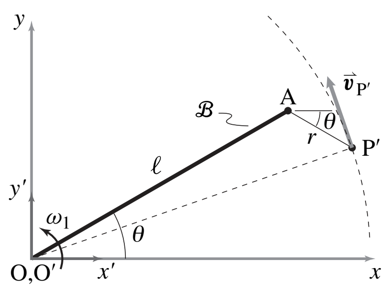


Figure 18.47: P′ goes in circles around point O with radius OP′. Therefore, $\vec{v}_{P'}$ is tangential to the circular path at P′.

Solution Frame $\mathcal{B}$ is attached to the rod. We choose a coordinate system $x'y'z'$ in frame $\mathcal{B}$ with its origin at O and, at the instant, aligned with the fixed coordinate system xyz. We consider a point P′ momentarily coincident with point P but fixed in frame $\mathcal{B}$. Since P′ is fixed in $\mathcal{B}$, it rotates with $\mathcal{B}$ with $\vec{\omega}_{\mathcal{B}} = \omega_1 \hat{\boldsymbol{k}}$. To visualize the motion of P′ imagine a rigid rod from O to P′ (see Fig. 18.47). Now we can calculate the velocity and acceleration of point P′ as follows.

1. **Velocity of point P:**

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}.$$

Now we calculate the two terms separately:

$$\begin{aligned}
\vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} \\
&= \omega_1 \hat{\boldsymbol{k}} \times (\vec{r}_{A/O'} + \vec{r}_{P'/A}) \\
&= \omega_1 \hat{\boldsymbol{k}} \times [\underbrace{\ell(\cos\theta \hat{\boldsymbol{\imath}} + \sin\theta \hat{\boldsymbol{\jmath}})}_{\vec{r}_{A/O'}} + \underbrace{r(\cos\theta \hat{\boldsymbol{\imath}} - \sin\theta \hat{\boldsymbol{\jmath}})}_{\vec{r}_{P'/A}}] \\
&= \omega_1(\ell + r)\cos\theta \hat{\boldsymbol{\jmath}} - \omega_1(\ell - r)\sin\theta \hat{\boldsymbol{\imath}} \\
&= 3\,\text{rad/s} \cdot 2.5\,\text{m} \cdot \cos 30^\circ \hat{\boldsymbol{\jmath}} - 3\,\text{rad/s} \cdot 1.5\,\text{m} \cdot \sin 30^\circ \hat{\boldsymbol{\imath}} \\
&= (6.50\hat{\boldsymbol{\jmath}} - 2.25\hat{\boldsymbol{\imath}})\,\text{m/s}.
\end{aligned}$$

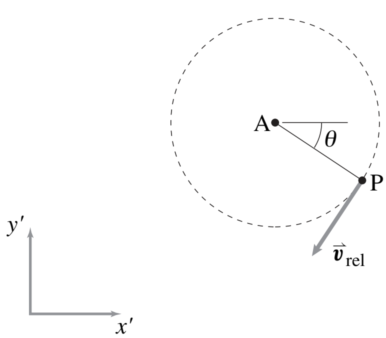


Figure 18.48: Velocity of point P with respect to the frame $\mathcal{B}$.

Since the wheel rotates with angular speed ω_2 with respect to the rod, an observer sitting in frame $\mathcal{B}$ would see a circular motion of point P about point A. Therefore,

$$\begin{aligned}
\vec{v}_{\text{rel}} &= \vec{\omega}_{\text{wheel}/\mathcal{B}} \times \vec{r}_{P/A} \\
&= -\omega_2 \hat{\boldsymbol{k}}' \times r(\cos\theta \hat{\boldsymbol{\imath}}' - \sin\theta \hat{\boldsymbol{\jmath}}') \\
&= -\omega_2 r(\cos\theta \hat{\boldsymbol{\jmath}}' + \sin\theta \hat{\boldsymbol{\imath}}') \\
&= -(2.16\hat{\boldsymbol{\jmath}}' + 1.25\hat{\boldsymbol{\imath}}')\,\text{m/s}.
\end{aligned}$$

But at the instant of interest, $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}$. So,

$$\vec{v}_{\text{rel}} = -(2.16\hat{\boldsymbol{\jmath}} + 1.25\hat{\boldsymbol{\imath}})\ \text{m/s}$$

Therefore,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = 4.33\,\text{m/s}\hat{\boldsymbol{\jmath}} - 3.50\,\text{m/s}\hat{\boldsymbol{\imath}}.$$

$$\vec{v}_P = (-3.50\hat{\boldsymbol{\imath}} + 4.33\hat{\boldsymbol{\jmath}})\,\text{m/s}$$

2. **Acceleration of point P:** We can similarly find the acceleration of point P:

$$\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{\text{cor}} + \vec{\boldsymbol{a}}_{\text{rel}}$$

where

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= \text{acceleration of point P}' \\
&= \underbrace{\vec{\boldsymbol{a}}_{O'}}_{\vec{\mathbf{0}}} + \underbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{B}}}_{\vec{\mathbf{0}}} \times \vec{\boldsymbol{r}}_{P'/O'} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'}) \\
&= -\omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{P'/O'} \\
&= -\omega_1^2[(\ell + r)\cos\theta \hat{\boldsymbol{\imath}} + (\ell - r)\sin\theta \hat{\boldsymbol{\jmath}}] \\
&= -9(\text{ rad/s})^2[2.5\,\text{m}\cdot\cos 30^\circ \hat{\boldsymbol{\imath}} + 1.5\,\text{m}\cdot\sin 30^\circ \hat{\boldsymbol{\jmath}}] \\
&= -(19.48\hat{\boldsymbol{\imath}} + 6.75\hat{\boldsymbol{\jmath}})\,\text{m/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{cor}} &= \text{Coriolis acceleration} \\
&= 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\text{rel}} \\
&= 2\omega_1 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{\text{rel}} \qquad \text{(see part (a) above for } \vec{\boldsymbol{v}}_{\text{rel}}\text{).} \\
&= (6\,\text{rad/s})\hat{\boldsymbol{k}} \times (-2.16\hat{\boldsymbol{\jmath}} - 1.25\hat{\boldsymbol{\imath}})\,\text{m/s} \\
&= (12.99\hat{\boldsymbol{\imath}} - 7.50\hat{\boldsymbol{\jmath}})\,\text{m/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{rel}} &= \text{acceleration of P relative to frame } \mathcal{B} \\
&= \vec{\boldsymbol{a}}_{P/\mathcal{B}} = \dot{\vec{\boldsymbol{\omega}}}_2 \times \vec{\boldsymbol{r}}_{P/A} - \omega_2^2 \vec{\boldsymbol{r}}_{P/A} \\
&= -\dot{\omega}_2 \hat{\boldsymbol{k}}' \times r(\cos\theta\hat{\boldsymbol{\imath}}' - \sin\theta\hat{\boldsymbol{\jmath}}') - \omega_2^2 r(\cos\theta\hat{\boldsymbol{\imath}}' - \sin\theta\hat{\boldsymbol{\jmath}}') \\
&= -r[(\dot{\omega}_2\sin\theta + \omega_2^2\cos\theta)\hat{\boldsymbol{\imath}}' + (\dot{\omega}_2\cos\theta - \omega_2^2\sin\theta)\hat{\boldsymbol{\jmath}}'] \\
&= -0.5\,\text{m}[(1\,\text{rad/s}^2\cdot\sin 30^\circ + 25(\text{ rad/s})^2\cdot\cos 30^\circ)\hat{\boldsymbol{\imath}}' \\
&\qquad +(1\,\text{rad/s}^2\cdot\cos 30^\circ - 25(\text{ rad/s})^2\cdot\sin 30^\circ)\hat{\boldsymbol{\jmath}}'] \\
&= (-11.08\hat{\boldsymbol{\imath}}' + 5.82\hat{\boldsymbol{\jmath}}')\,\text{m/s}^2 \\
&= (-11.08\hat{\boldsymbol{\imath}} + 5.82\hat{\boldsymbol{\jmath}})\,\text{m/s}^2.
\end{aligned}$$

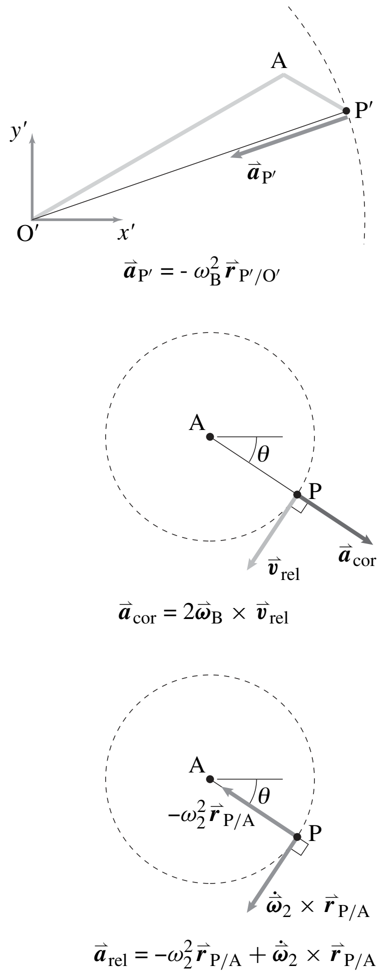


Figure 18.49

The term $\vec{\boldsymbol{a}}_{P'}$ encompasses three terms of the five term acceleration formula. The last line in the calculation of $\vec{\boldsymbol{a}}_{\text{rel}}$ follows from the fact that at the instant of interest $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$.

Now adding the three parts of $\vec{\boldsymbol{a}}_P$ we get

$$\begin{aligned}
\vec{\boldsymbol{a}}_P &= \vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{\text{cor}} + \vec{\boldsymbol{a}}_{\text{rel}} \\
&= -17.57\,\text{m/s}^2\hat{\boldsymbol{\imath}} - 8.43\,\text{m/s}^2\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_P = -(17.57\hat{\boldsymbol{\imath}} + 8.43\hat{\boldsymbol{\jmath}})\,\text{m/s}^2}$$

18.3 General expressions for velocity and acceleration

18.3.1 A bug walks on a straight line engraved on a rotating turntable (the bug's path in the room is not a straight line). The line passes through the center of the turntable. The bug's speed on this line is 1 in/s (the bug's absolute speed is *not* 1 in/s). The turntable rotates at a constant rate of 2 revolutions every π s in a positive sense about the z-axis. Its surface is always in the xy-plane. At time $t = 0$, the engraved line is aligned with the x-axis, the bug is at the origin and headed towards the positive x direction.

a) What is the x component of the bug's velocity at $t = \pi$ s?

b) What is the y component of the bug's velocity at $t = \pi$ seconds?

c) What is the y component of the bug's acceleration at $t = \pi$?

d) What is the radius of curvature of the bug's path at $t = 0$?

18.3.2 Actual path of bug trying to walk a straight line. A straight line is inscribed on a horizontal turntable. The line goes through the center. Let ϕ be the angle of rotation of the turntable which spins at constant rate $\dot{\phi}_0$. A bug starts on the outside edge of the turntable of radius R and walks towards the center, passes through it, and continues to the opposite edge of the turntable. The bug walks at a constant speed v_A, as measured by how far her feet move per step, on the line inscribed on the table. Ignore gravity.

a) **Picture.** Make an accurate drawing of the bug's path as seen in the room (which is not rotating with the turntable). In order to make this plot, you will need to assume values of v_A and $\dot{\phi}_0$ and initial values of R and ϕ. You will need to write a parametric equation for the path in terms of variables that you can plot (probably x and y coordinates). You will also need to pick a range of times. Your plot should include the instant at which the bug walks through the origin. Make sure your x and y-axes are drawn to the same scale. A computer plot would be nice.

b) Calculate the radius of curvature of the bug's path as it goes through the origin.

c) Accurately draw (say, on the computer) the osculating circle when the bug is at the origin on the picture you drew for (a) above.

d) **Force.** What is the force on the bug's feet from the turntable when she starts her trip? Draw this force as an arrow on your picture of the bug's path.

e) **Force.** What is the force on the bug's feet when she is in the middle of the turntable? Draw this force as an arrow on your picture of the bug's path.

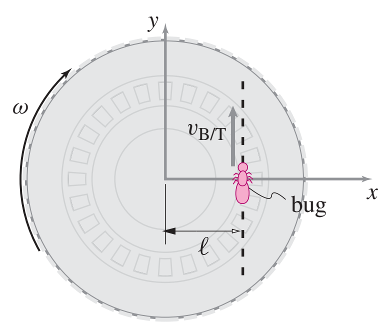
18.3.3 A small bug is crawling on a straight line scratched on an old record. The scratch is a distance $\ell = 6$ cm from the center of the turntable. The turntable is turning clockwise at a constant angular rate $\omega = 2$ rad/s. The bug is walking, relative to the turntable, at a constant rate $v_{B/T} = 12$ cm/s, straight along the scratch in the y-direction. At the instant of interest, everything is aligned as shown in the figure. The bug has a mass $m_B = 1$ gram.

a) What is the bug's velocity?

b) What is the bug's acceleration?

c) What is the sum of all forces acting on the bug?

d) Sketch the path of the bug in the neighborhood of its location at the time of interest (indicate the direction the bug is moving on this path).

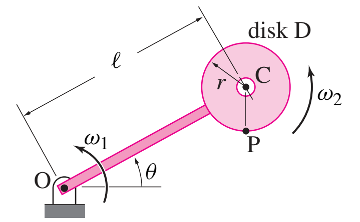


Problem 18.3.3

18.3.4 Arm OC rotates with constant rate ω_1. Disc D of radius r rotates about point C at constant rate ω_2 measured with respect to the arm OC. What are the absolute velocity and acceleration of point P on the disc, $\vec{v}_P$ and $\vec{a}_P$? (To do this problem will require defining a moving frame of reference. More than one choice is possible.) *

a) Pick a suitable moving frame and do the problem.

b) Pick another suitable moving frame and redo the problem. Make sure the answers are the same.

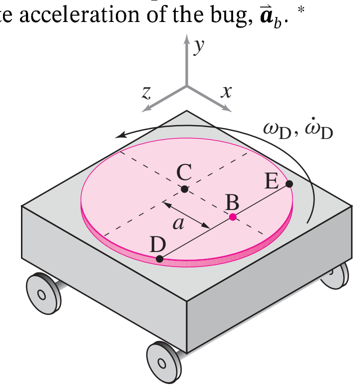


Problem 18.3.4

18.3.5 For the configuration in problem 18.3.4 what is the absolute angular velocity of the disk, D? *

18.3.6 For the configuration in problem 18.3.4, taking ω_2 to be the angular velocity of disk $\mathcal{D}$ relative to the rod, what is the absolute angular acceleration of the disk, D? What is the absolute angular acceleration of the disk if ω_1 and ω_2 are not constant?

18.3.7 A turntable oscillates with displacement $\vec{x}_C(t) = A\sin(st)\hat{\imath}$. The disc of the turntable rotates with angular speed and acceleration ω_D and $\dot{\omega}_D$. A small bug walks along line DE with velocity v_b relative to the turntable. At the instant shown, the turntable is at its maximum amplitude $x = A$, the line DE is currently aligned with the z-axis, and the bug is passing through point B on line DE. Point B is a distance a from the center of the turntable, point C. Find the absolute acceleration of the bug, $\vec{a}_b$. *



Problem 18.3.7

18.3.8 A small 0.1 kg toy train engine is going *clockwise* at a constant rate (relative to the track) of 2 m/s on a circular track of radius 1 m. The track itself is on a turntable $\mathcal{B}$ that is rotating *counterclockwise* at a constant rate of 1 rad/s. The dimensions are as shown. At the instant of interest the train is pointing due south ($-\mathbf{j}$) and is *at* the center O of the turntable.

a) What is the velocity of the train relative to the turntable $\vec{\boldsymbol{v}}_{/\mathcal{B}}$?

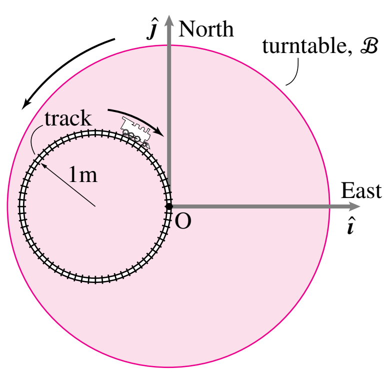
b) What is the absolute velocity of the train $\vec{\boldsymbol{v}}$?

c) What is the acceleration of the train relative to the turntable $\vec{\boldsymbol{a}}_{/\mathcal{B}}$?

d) What is the absolute acceleration of the train $\vec{\boldsymbol{a}}$?

e) What is the total force acting on the train?

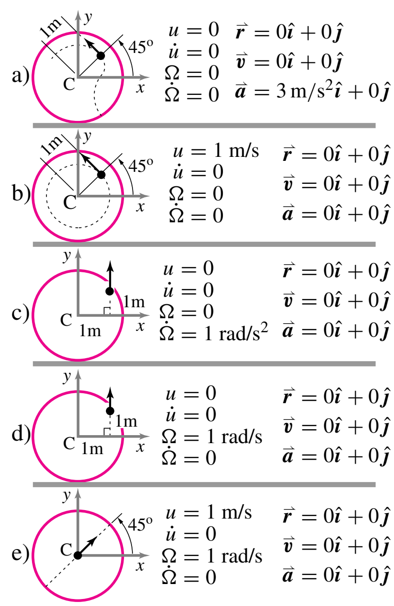
f) Sketch the path of the train for one revolution of the turntable (surprise)?



Problem 18.3.8

18.3.9 A giant bug walks on a horizontal disk. An xyz frame is attached to the disk. The disk is rotating about the z axis (out of the paper) and simultaneously translating with respect to an inertial frame XYZ plane. In each of the cases shown in the figure, determine the total force acting on the bug. In each case, the dotted line is scratched on the disk and is the path the bug follows walking in the direction of the arrow. The location of the bug is marked with a dot. At the instants shown, the xyz coordinate system shown is aligned with the inertial XYZ frame (which is not shown in the pictures). In this problem:

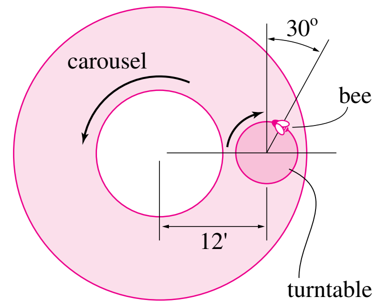
$\vec{\boldsymbol{r}}$ = the position of the center of the disk,
$\vec{\boldsymbol{v}}$ = the velocity of the center of the disk,
$\vec{\boldsymbol{a}}$ = the acceleration of the center of the disk,
Ω the angular velocity of the disk (i.e., $\vec{\boldsymbol{\Omega}} = \Omega \boldsymbol{k}$),
$\dot{\Omega}$ = the angular acceleration of the disk,
u = the speed of the bug traversing the dotted line (arc length on the disk per unit time),
$\dot{u} = du/dt$, and
m = 1 kg = mass of the bug.



Problem 18.3.9

18.3.10 Repeat questions (a)-(f) for the toy train in problem 18.3.8 going *counterclockwise* at constant rate (relative to the track) of 2 m/s on the circular track.

18.3.11 A nostalgic honeybee alights on a phonograph turntable that is being carried by a carnival goer who is riding on a carousel. The situation is sketched below. The carousel has angular velocity of 5 rpm, which is increasing (accelerating) at 10 rev/min^2; the phonograph rotates at a constant 33 1/3 rpm. The honeybee is at the outer edge of the phonograph record in the position shown in the figure; the radius of the record is 7 inches. Calculate the magnitude of the acceleration of the honeybee.



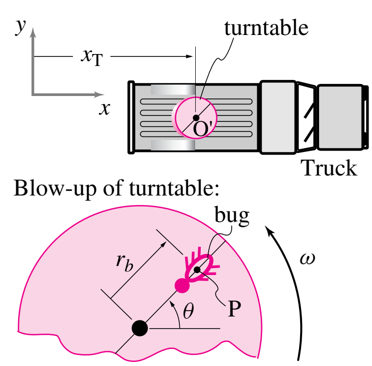
Problem 18.3.11

18.3.12 Consider a turntable on the back of a pick-up truck. A bug walks on a line on the turntable. The line may or may not be drawn through the center of the turntable. The truck may or may not be going at constant rate. The turntable may or may not be spinning, and, if it spins, it may or may not go at a constant rate. The bug may be anywhere on the line and may or may not be walking at a constant rate.

a) **Draw a picture of the situation.** Clearly define all variables you are going to use. What is the moving frame, and what is the reference point on that frame?

b) **One term at a time.** For each term in the five term acceleration formula, find a situation where all but one term is zero. Use the turntable as the moving frame, the bug as the particle of interest. *

c) **Two terms at a time.** (harder) How many situations can you find where a pair of terms is not zero but all other terms are zero? (Don't try to do all 10 cases unless you really think this infamous formula is fun. Try at least one or two.) *



Problem 18.3.12: **Picking apart the five term acceleration formula.**

18.3.2 Actual path of bug trying to walk a straight line. A straight line is inscribed on a horizontal turntable. The line goes through the center. Let ϕ be the angle of rotation of the turntable which spins at constant rate $\dot{\phi}_0$. A bug starts on the outside edge of the turntable of radius R and walks towards the center, passes through it, and continues to the opposite edge of the turntable. The bug walks at a constant speed v_A, as measured by how far her feet move per step, on the line inscribed on the table. Ignore gravity.

a) **Picture.** Make an accurate drawing of the bug's path as seen in the room (which is not rotating with the turntable). In order to make this plot, you will need to assume values of v_A and $\dot{\phi}_0$ and initial values of R and ϕ. You will need to write a parametric equation for the path in terms of variables that you can plot (probably x and y coordinates). You will also need to pick a range of times. Your plot should include the instant at which the bug walks through the origin. Make sure your x and y-axes are drawn to the same scale. A computer plot would be nice.

b) Calculate the radius of curvature of the bug's path as it goes through the origin.

c) Accurately draw (say, on the computer) the osculating circle when the bug is at the origin on the picture you drew for (a) above.

d) **Force.** What is the force on the bug's feet from the turntable when she starts her trip? Draw this force as an arrow on your picture of the bug's path.

e) **Force.** What is the force on the bug's feet when she is in the middle of the turntable? Draw this force as an arrow on your picture of the bug's path.

HW # 12 P66 18.3.2

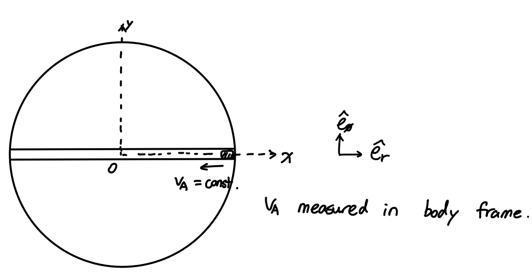
Based on 2019 Old Solution
Modified by Teaya Yang
Corrected May 17, 2021

$\hat{e}_\phi$ $\hat{e}_r$

$= \dot{\phi}_0$

r

ϕ



a) Draw accurate path of the bug relative to the room.

Assume values of V_A and $\dot{\phi}$ and initial values of R and ϕ

The parametric equation for the path in terms of t is:

$$\begin{cases} x(t) = r\cos\phi \\ y(t) = r\sin\phi \end{cases} \quad \text{where} \quad \begin{cases} r(t) = R - V_A t \\ \phi(t) = \dot{\phi}_0 t \end{cases}$$

Write in terms of x-y coordinates

$$\begin{cases} x(t) = (R - V_A t)\cos(\dot{\phi}_0 t) \\ y(t) = (R - V_A t)\sin(\dot{\phi}_0 t) \end{cases}$$ pick

Pick $R = 1\,m$, $V_A = 0.2\,m/s$ $\dot{\phi}_0 = 1\,rad/s$

$\boxed{\text{Code and plots see attached}}$

b) Find the radius of curvature of the path at origin.

The velocity of the bug is :

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\left(r(t)\,\hat{e}_r\right) = \dot{r}(t)\,\hat{e}_r + r(t)\,\dot{\hat{e}}_r$$

$$= -V_A\hat{e}_r + (R - V_A t)\,\dot{\phi}_0\,\hat{e}_\phi \times \hat{e}_r$$

$$= -V_A\hat{e}_r + (R - V_A t)\,\dot{\phi}_0\,\hat{e}_\phi$$

($\hat{e}_r$, $\hat{e}_\phi$ are polar coordinate unit vectors)

The acceleration is

$$\vec{a} = -V_A\dot{\phi}_0\,\hat{e}_\phi - V_A\dot{\phi}_0\,\hat{e}_\phi + (R - V_A t)\,\ddot{\phi}^{\,0}\,\hat{e}_\phi - (R - V_A t)\,\dot{\phi}_0^2\,\hat{e}_r$$

when the bug goes through the center, $r = R - V_A t = 0$

$\Rightarrow \quad t = \frac{R}{v_A}$

At this instant,

$$\vec{v} = -v_A \hat{e}_r \ , \quad \vec{a} = -2 v_A \dot{\phi}_0 \hat{e}_\phi$$

compare this to the path coordinates expression

$$\vec{v} = v \hat{e}_t \ , \quad \vec{a} = \dot{v} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n$$

where ρ is the radius of curvature

we get when the bug is at the origin

$$\begin{cases} v = |\vec{v}| = v_A \\ \hat{e}_t = -\hat{e}_r \\ \dot{v} = 0 \qquad \text{(since } \vec{a} \perp \vec{v} \text{ at this instant)} \\ \frac{v^2}{\rho} = 2 v_A \dot{\phi}_0 \\ \hat{e}_n = -\hat{e}_\phi \end{cases}$$

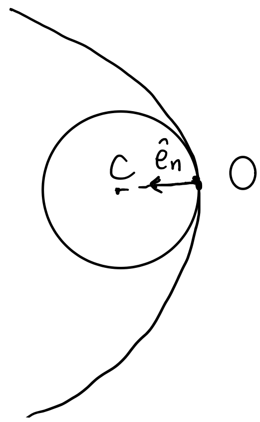
$$\Rightarrow \boxed{\rho = \frac{v^2}{2 v_A \dot{\phi}_0} = \frac{v_A^2}{2 v_A \dot{\phi}_0} = \frac{v_A}{2 \dot{\phi}_0}}$$

c) Coding to draw the osulating circle when the bug is at the origin on picture drawn in a)

$\Rightarrow$ Find the center & radius of the circle

get in part b)

Now find the center of the osculating circle



C is the center of osculating circle

O is the origin

Recall Thus from part b) $\overline{r_{O/C}} = \rho\hat{e}_n = \rho\hat{e}_\phi$

$t = \frac{R}{V_A}$ since $\hat{e}_n = -\hat{e}_\phi$ $\Rightarrow$ $\phi = \dot{\phi}_0 t = \frac{R\dot{\phi}_0}{V_A}$

$\rho = \frac{V_A}{2\dot{\phi}_0}$ $\Rightarrow$ $(x_0 - x_c)\hat{i} + (y_0 - y_c)\hat{j} = \rho\hat{e}_\phi$ ①

where $x_0 = y_0 = 0$

When the ~~$\hat{e}_n$~~ bug ~~is~~ $\hat{e}_\phi$ at origin, $t = \frac{R}{V_A}$ (from above)

So $\vec{r}_{C/O} = x_c\hat{i} + y_c\hat{j} = -\rho\hat{e}_\phi$

Thus we have $-\rho(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -\sin(\dot{\phi}_0\frac{R}{V_A})\hat{i} + \cos(\dot{\phi}_0\frac{R}{V_A})\hat{j}$ ②

① ② $\Rightarrow$ $\begin{cases} x_c = \frac{V_A}{2\dot{\phi}_0}\sin\left(\frac{R\dot{\phi}_0}{V_A}\right) \\ y_c = -\frac{V_A}{2\dot{\phi}_0}\cos\left(\frac{R\dot{\phi}_0}{V_A}\right) \end{cases}$ $\Rightarrow$ $x_c = \frac{V_A}{2\dot{\phi}_0}\sin(\dot{\phi}_0\frac{R}{V_A})$, $y_c = \frac{-V_A}{2\dot{\phi}_0}\cos(\dot{\phi}_0\frac{R}{V_A})$

with the position of C, we can draw the circle see Matlab code attached

d) Find the force on the bug's feet from the turntable at the starting point

Draw the force as an arrow on the path

We know from b) that

$$\vec{a} = -2V_A \dot{\phi}_0 \hat{e}_\phi - R\dot{\phi}_0^2 \hat{e}_r$$

since when $t=0$ $\hat{e}_r = \hat{i}$, $\hat{e}_\phi = \hat{j}$ then

$$\vec{a}(0) = -2V_A \dot{\phi}_0 \hat{j} - R\dot{\phi}_0^2 \hat{i}$$

LMB : $\vec{F} = m\vec{a}$

$$\Rightarrow \boxed{\vec{F} = -m(R\dot{\phi}_0^2 \hat{i} + 2V_A \dot{\phi}_0 \hat{j})}$$

is the force acting on the bug at the beginning.

e) Do the same as d) except at origin

$t = \frac{R}{V_A}$ $\phi = \dot{\phi}_0 \frac{R}{V_A}$

$$a(t=\tfrac{R}{V_A}) = -2V_A \dot{\phi}_0 \hat{e}_\phi \quad (\text{from } b)$$

$$= -2V_A \dot{\phi}_0 \left(-\sin(\dot{\phi}_0 \tfrac{R}{V_A})\hat{i} + \cos(\dot{\phi}_0 \tfrac{R}{V_A})\hat{j}\right)$$

$$= 2V_A \dot{\phi}_0 \sin(\dot{\phi}_0 \tfrac{R}{V_A})\hat{i} - 2V_A \dot{\phi}_0 \cos(\dot{\phi}_0 \tfrac{R}{V_A})\hat{j}$$

LMB: $\vec{F} = m\vec{a}$

$$\boxed{\vec{F} = 2mV_A \dot{\phi}_0 \left(\sin(\dot{\phi}_0 \tfrac{R}{V_A})\hat{i} - \cos(\dot{\phi}_0 \tfrac{R}{V_A})\hat{j}\right)}$$

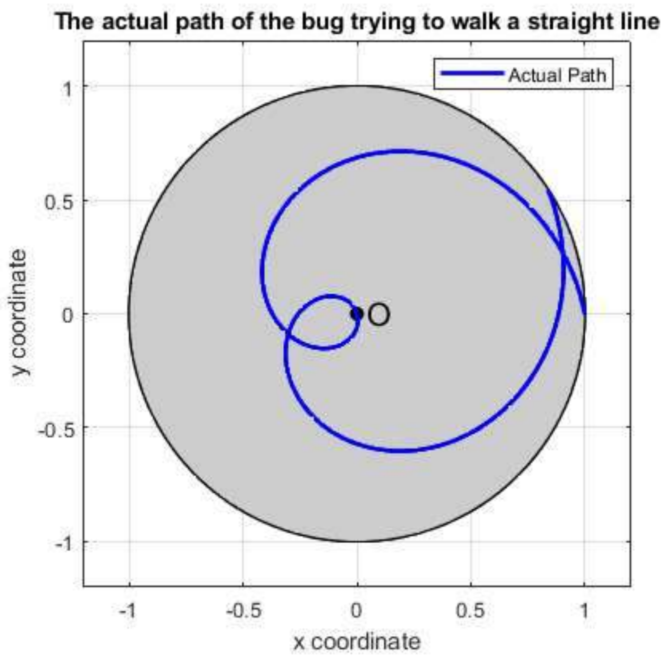
Contents


```
% MAE2030 HW12 P66 8.3.2
% 2019 Solution code, modified by Teaya Yang
% Actual path of bug trying to walk a straight line

% Parameters
R=1; % radius of the turntable
vA=0.2; % velocity of the bug on the turntable
phidot=1; % constant angular velocity of the turntable
```

Part a) Plotting the actual path

Draw the turntable

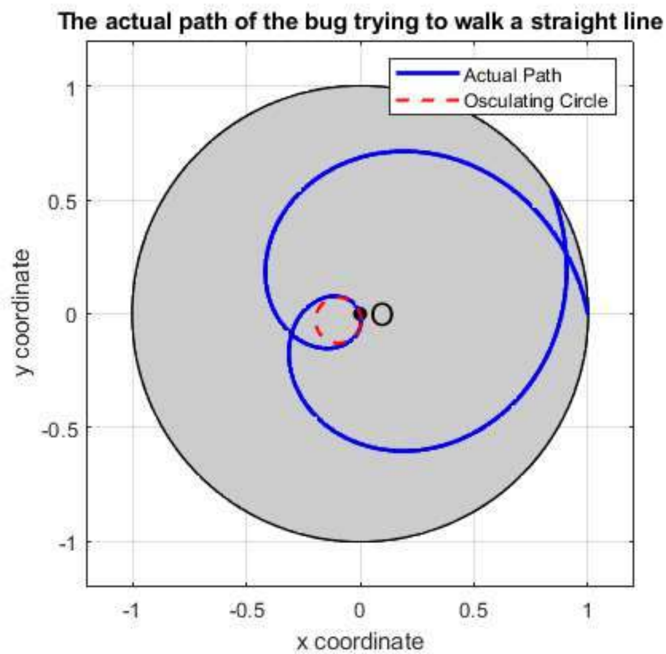
```
figure(1)
box on
hold on
axis equal
grid on
axis([-R R -R R]*1.2)
angle = 0:pi/80:2*pi;
circlex = R*cos(angle);
circley = R*sin(angle);
p(1) = plot(circlex,circley,'k-','linewidth',1.5);
p(2) = fill(circlex,circley,[0.8 0.8 .8]);
p(3) = plot(0,0,'ko','MarkerSize',6,'MarkerFaceColor','k');
text(.04,0,'O','fontsize',15);

% Draw the acutual path of the bug
tend = 2*R/vA; % Time when the bug reaches the other end
t = linspace(0,2*R/vA,1000);
x=(R-vA*t).*cos(phidot*t);
y=(R-vA*t).*sin(phidot*t);
p(4) = plot(x,y,'b-','linewidth',2);
title('The actual path of the bug trying to walk a straight line');
xlabel('x coordinate');
ylabel('y coordinate');
legend(p(4),'Actual Path')
```



Part b) Drawing Osculating circle at origin

Draw osculating circle when bug goes through the center

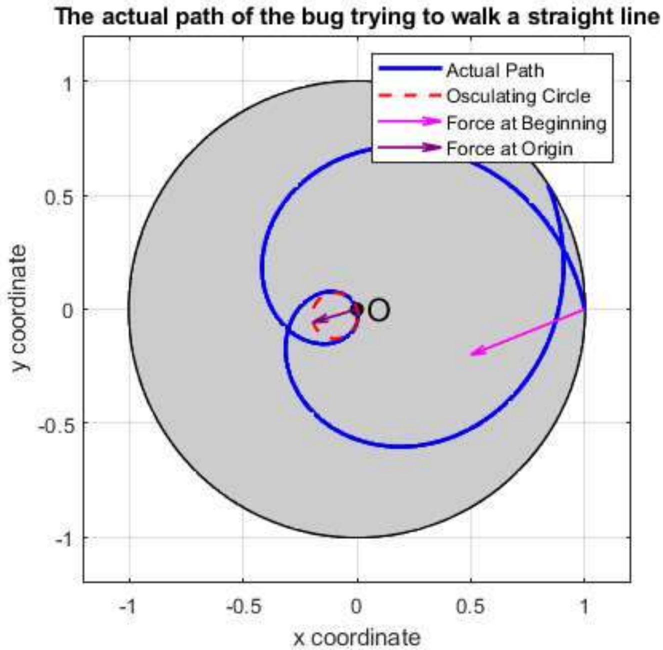
```
rho=vA/(2*phidot); % radius of curvature of the path at the origin
xC= vA*sin(phidot*R/vA)/(2*phidot);
yC= -vA*cos(phidot*R/vA)/(2*phidot); % position of the center
theta=0:0.01:2*pi;
x=xC+rho*cos(theta);
y=yC+rho*sin(theta);
hold on;
p(5) = plot(x,y,'r--','linewidth',1.5);
legend([p(4),p(5)],'Actual Path','Osculating Circle')
```



Part c)d) Drawing force vector at t=0 and at the origin

draw force vector

```
m=1; %mass of the bug
scale=0.5; % scale for graphics
f1x=-m*R*phidot; → f1x = -m*R* phidot ^2;
f1y=-m*2*vA*phidot;
 % draw force at the beginning
p(6) = quiver(R,0,f1x,f1y,scale,'m','linewidth',1.2,'MaxHeadSize',.4);
f2x=2*m*vA*phidot*sin(phidot*R/vA);
f2y=-2*m*vA*phidot*cos(phidot*R/vA);
%draw force at the origin
p(7) = quiver(0,0,f2x,f2y,scale,'color',[.5 0 .5],'linewidth',1.2,'MaxHeadSize',1.2);
legend([p(4),p(5),p(6),p(7)],'Actual Path','Osculating Circle','Force at Beginning','Force at Origin')
```



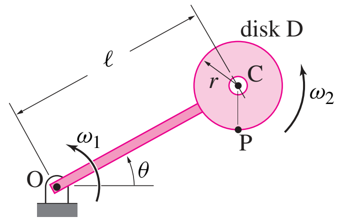
Published with MATLAB® R2020b

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

18.3.4 Arm OC rotates with constant rate ω_1. Disc D of radius r rotates about point C at constant rate ω_2 measured with respect to the arm OC. What are the absolute velocity and acceleration of point P on the disc, $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$? (To do this problem will require defining a moving frame of reference. More than one choice is possible.)

a) Pick a suitable moving frame and do the problem.

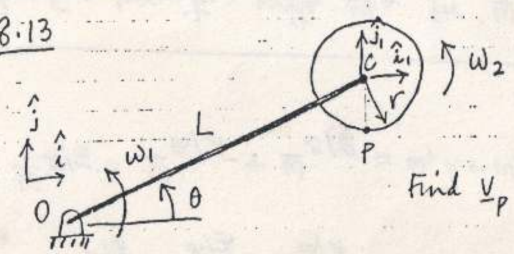
b) Pick another suitable moving frame and redo the problem. Make sure the answers are the same.



Problem 18.4

Answer:
$\vec{\boldsymbol{v}}_P = \omega_1 L(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{\imath}}) + (\omega_1+\omega_2)r\hat{\boldsymbol{\imath}}$,
$\vec{\boldsymbol{a}}_P = -\omega_1^2 L(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) + (\omega_1+\omega_2)^2\hat{\boldsymbol{\jmath}}$.

8·13



Find $\underline{v}_p$ & $\underline{a}_p$

Page 3

Attach a moving frame at point C, with unit vectors $\hat{\imath}_1, \hat{\jmath}_1$, called $\mathcal{B}$

a) First, assume that the moving frame does not rotate (it just translates; unit vectors don't change direction):

$$\underline{\omega}_{disk/\mathcal{B}} = (\omega_1+\omega_2)\hat{k}$$

$$\therefore\ \underline{v}_{P/\mathcal{B}} = (\omega_1+\omega_2)\hat{k}\times \underline{r}_{CP}$$
$$= (\omega_1+\omega_2)r\ \hat{k}\times(-\hat{\jmath}_1)$$
$$= (\omega_1+\omega_2)r\ \hat{\imath}$$

$$\underline{v}_c = \omega_1\hat{k}\times L(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$
$$= \omega_1 L(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$

$$\underline{v}_P = \underline{v}_c + \underline{v}_{P/\mathcal{B}} + \overset{0}{\underline{\omega}_{\mathcal{B}}}\times\underline{r}_{cp}$$

$$\boxed{\underline{v}_P = \omega_1 L(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath}) + (\omega_1+\omega_2)r\,\hat{\imath}}$$
$\omega_1 L\cos\theta\,\hat{\jmath} - \omega_1 L\sin\theta\,\hat{\imath} + (\omega_1+\omega_2)r\,\hat{\imath}$

$$\underline{a}_{P/\mathcal{B}} = (\omega_1+\omega_2)\hat{k}\times[(\omega_1+\omega_2)\hat{k}\times\underline{r}_{cp}]$$
$$= (\omega_1+\omega_2)^2 r\,\hat{\jmath}$$

$$\underline{a}_c = \omega_1\hat{k}\times[\omega_1\hat{k}\times L(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})]$$

$$\underline{a}_P = \underline{a}_c + \underline{a}_{P/\mathcal{B}} + \text{bunch of zeros}$$

$$\boxed{\underline{a}_P = -\omega_1^2 L(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + (\omega_1+\omega_2)^2 r\,\hat{\jmath}}$$

b) Next, consider the situation where moving frame rotates with the rod, i.e., $\underline{\omega}_{\mathcal{B}} = \omega_1\hat{k}$ and $\underline{\omega}_{disk/\mathcal{B}} = \omega_2\hat{k}$

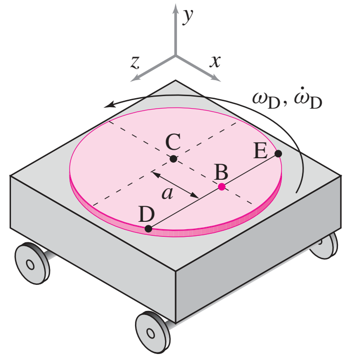
$$\therefore\ \underline{v}_{P/\mathcal{B}} = \omega_2\hat{k}\times\underline{r}_{cp} = \omega_2 r\,\hat{\imath}$$

18.3.5 For the configuration in problem 18.3.4 what is the absolute angular velocity of the disk, D?

Answer:
$\vec{\omega}_D = (\omega_1 + \omega_2)\hat{\boldsymbol{k}}$.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

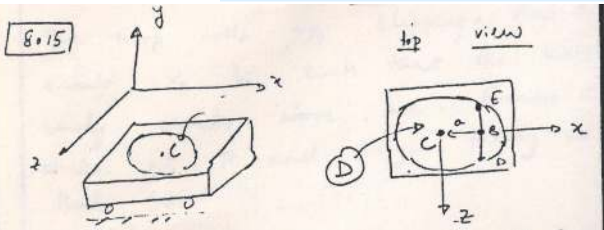
18.3.7 A turntable oscillates with displacement $\vec{\boldsymbol{x}}_C(t) = A\sin(st)\hat{\boldsymbol{\imath}}$. The disc of the turntable rotates with angular speed and acceleration ω_D and $\dot{\omega}_D$. A small bug walks along line DE with velocity v_b relative to the turntable. At the instant shown, the turntable is at its maximum amplitude $x = A$, the line DE is currently aligned with the z-axis, and the bug is passing through point B on line DE. Point B is a distance a from the center of the turntable, point C. Find the absolute acceleration of the bug, $\vec{\boldsymbol{a}}_b$.



Problem 18.7

Answer:
$\vec{\boldsymbol{a}}_b = -(s^2A + a\omega_D^2 + 2\omega_D v_b)\hat{\boldsymbol{\imath}} - a\dot{\omega}_D\hat{\boldsymbol{k}}$.

8.15



Find the acceleration of a bug, at B at the instant shown, but moving with velocity v_b along the line DE.

Solution:

Let B' be a point glued on the turntable but coincident with B at the instant shown.

Then $\boxed{\begin{aligned} \underline{v}_B &= \underline{v}_{B'} + \underline{v}_{rel} \\ \underline{a}_B &= \underline{a}_{B'} + \underline{a}_{rel} + \underline{a}_{cor} \end{aligned}}$

$\underline{a}_{B'} = \underline{a}_C + \dot{\underline{\omega}}_{D/T} \times \underline{r}_{B/C} + \underline{\omega}_{D/T} \times (\underline{\omega}_{D/T} \times \underline{r}_{B'/C})$

$\underline{\omega}_{D/T} = \omega_D \hat{j}$ & $\dot{\underline{\omega}}_{D/T} = \dot{\omega}_D \hat{j}$

And $\underline{r}_{B'/C} = a\hat{\imath}$

$\underline{a}_C = \ddot{x}_C \hat{\imath} = -s^2 A \sin(st)\hat{\imath}$

$= -s^2 A\hat{\imath}$ when the table is at its extreme position.

$\therefore \underline{a}_C = -s^2 A\hat{\imath}$

$\dot{\underline{\omega}}_{D/T} \times \underline{r}_{B'/C} = (a\dot{\omega}_D)(-\hat{k})$

$\underline{\omega}_{D/T} \times (\underline{\omega}_{D/T} \times \underline{r}_{B/C}) = -(a\omega_D^2)\hat{\imath}$

$\underline{a}_{rel} = 0$ $\qquad$ $\underline{a}_{cor} = 2\omega_{D/T} \times v_b(-\hat{k})$

Page 11

$\therefore \boxed{\underline{a}_B = -(s^2A + a\omega_D^2 + 2\omega_D v_b)\hat{\imath} - a\dot{\omega}_D\hat{k}}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

18.3.8 A small 0.1 kg toy train engine is going *clockwise* at a constant rate (relative to the track) of 2 m/s on a circular track of radius 1 m. The track itself is on a turntable $\mathcal{B}$ that is rotating *counter-clockwise* at a constant rate of 1 rad/s. The dimensions are as shown. At the instant of interest the train is pointing due south ($-\mathbf{j}$) and is *at* the center O of the turntable.

a) What is the velocity of the train relative to the turntable $\vec{v}_{/\mathcal{B}}$?

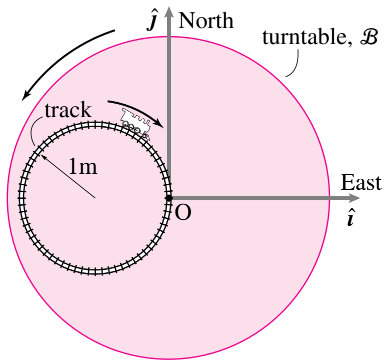
b) What is the absolute velocity of the train $\vec{v}$?

c) What is the acceleration of the train relative to the turntable $\vec{a}_{/\mathcal{B}}$?

d) What is the absolute acceleration of the train $\vec{a}$?

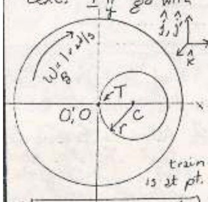
e) What is the total force acting on the train?

f) Sketch the path of the train for one revolution of the turntable (surprise)?



Problem 18.8

8.48 Train. Note that the directions of rotation shown in the book's sketch are not consistent with those shown in the text. I'll go with the ones shown in the sketch:



Attach frame B to the turntable. Its basis vectors $\hat{\imath}', \hat{\jmath}'$ are aligned with $\hat{\imath}$ & $\hat{\jmath}$ at this instant. This frame rotates with $\underset{\sim}{\omega}_B = -1\,\mathrm{rad/s}\,\hat{k}$.

a) $\boxed{\underset{\sim}{V}_{T/B} = -2\,\mathrm{m/s}\,\hat{\jmath}}$

b) $\underset{\sim}{V}_T = \cancel{\underset{\sim}{V}_{O'/O}}^{\,0} + \underset{\sim}{V}_{T/B} + \underset{\sim}{\omega}_B \times \cancel{\underset{\sim}{r}_{T/O'}}^{\,0}$

$\Rightarrow \boxed{\underset{\sim}{V}_T = \underset{\sim}{V}_{T/B} = -2\,\mathrm{m/s}\,\hat{\jmath}}$

continued

8.48 continued p.4

c) $\underset{\sim}{a}_{T/B} = \underset{\sim}{\omega}_{T/B} \times \underset{\sim}{\omega}_{T/B} \times \underset{\sim}{r}_{T/c} + \cancel{\underset{\sim}{\alpha}_{T/B}}^{\,0} \times \underset{\sim}{r}_{T/c}$

$\underset{\sim}{\omega}_{T/B} = \omega_{T/B}\,\hat{k} = \frac{V_{T/B}}{r}\hat{k} = \frac{2\,\mathrm{m/s}}{1\,\mathrm{m}}\hat{k} = 2\,\mathrm{rad/s}\,\hat{k}$

$\Rightarrow \underset{\sim}{a}_{T/B} = 2\,\mathrm{rad/s}\,\hat{k} \times 2\,\mathrm{rad/s}\,\hat{k} \times -1\,\mathrm{m}\,\hat{\imath} =$

$\boxed{\underset{\sim}{a}_{T/B} = 4\,\mathrm{m/s^2}\,\hat{\imath}}$

d) $\underset{\sim}{a}_T = \cancel{\underset{\sim}{a}_{O'/O}}^{\,0} + \underset{\sim}{\omega}_B \times \cancel{\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{T/O'}}^{\,0} + \cancel{\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{T/O'}}^{\,0}$

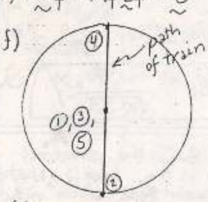
$+ \underset{\sim}{a}_{T/B} + 2\underset{\sim}{\omega}_B \times \underset{\sim}{V}_{rel}$

$\Rightarrow \underset{\sim}{a}_T = 4\,\mathrm{m/s^2}\,\hat{\imath} + 2(-1\,\mathrm{rad/s}\,\hat{k} \times -2\,\mathrm{m/s}\,\hat{\jmath})$

$\underset{\sim}{a}_T = (4\,\mathrm{m/s^2} - 4\,\mathrm{m/s^2})\,\hat{\imath} = \boxed{\underset{\sim}{0} = \underset{\sim}{a}_T}$

e) $\sum \underset{\sim}{F}_T = m_T \underset{\sim}{a}_T = \underset{\sim}{0}$

f)



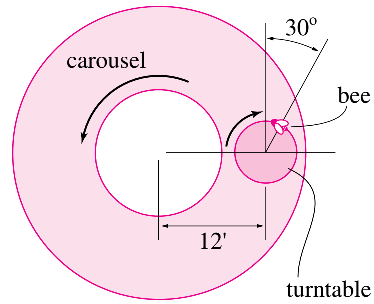
Numbers denote train posn after:

1 : 0 rev (start)
2 : 1/4 rev of turntable
3 : 1/2 rev " "
4 : 3/4 rev " "
5 : 1 rev " "

(This occurs only when $\underset{\sim}{\omega}_{T/B} = -2\omega_B$)

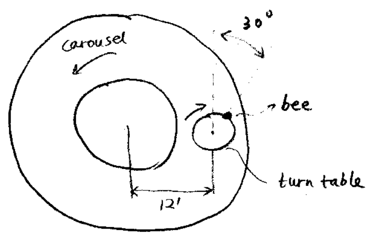
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

18.3.11 A nostalgic honeybee alights on a phonograph turntable that is being carried by a carnival goer who is riding on a carousel. The situation is sketched below. The carousel has angular velocity of 5 rpm, which is increasing (accelerating) at 10 rev/min^2; the phonograph rotates at a constant 33 1/3 rpm. The honeybee is at the outer edge of the phonograph record in the position shown in the figure; the radius of the record is 7 inches. Calculate the magnitude of the acceleration of the honeybee.



Problem 18.11

15.27

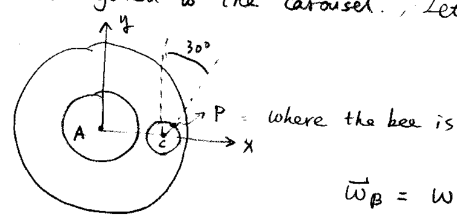


At this instant, the carousel has angular velocity of $\omega_c = 5\,rpm = 10\pi\ rad/min$

angular acceleration $\alpha_c = 10\ r/min^2 = 20\pi\ rad/min^2$

The turntable rotates at $\omega_t = 33\frac{1}{3}\ rpm = 66\frac{2}{3}\pi\ rad/min$, $\alpha_t = 0$

Use the moving frame glued to the carousel. Let's call it $\mathcal{B}$



$\vec{\omega}_{\mathcal{B}} = \omega_c \hat{k} = 10\pi\ rad/min$

$\dot{\vec{\omega}}_{\mathcal{B}} = \alpha_c \hat{k} = 20\pi\ rad/min^2$

$\vec{a}_P = \vec{a}_A + \vec{a}_{P/\mathcal{B}} - \omega_{\mathcal{B}}^2 \vec{r}_{P/A} + \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{P/A} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}$

i) $\vec{a}_A = 0$ since A is fixed

ii) $\vec{a}_{P/\mathcal{B}} = -\omega_t^2 \vec{r}_{P/C} = -(66\frac{2}{3}\pi)^2\, 7\, (\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j})\ in/min^2$

Since P rotates with the turntable at a constant angular velocity.

$\therefore \vec{a}_{P/\mathcal{B}} = (-1.5353\times 10^5\,\hat{i} - 2.6952\times 10^5\,\hat{j})\ in/min^2$

iii) $-\omega_{\mathcal{B}}^2 \vec{r}_{P/A} = -\omega_{\mathcal{B}}^2 (\vec{r}_{P/C} + \vec{r}_{C/A})$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$= -(10\pi)^2\left(\frac{7}{2}\hat{i} + \frac{7\sqrt{3}}{2}\hat{j} + 12\times 12\,\hat{i}\right)$$

$$= -1.4558\times 10^5\,\hat{i} - 5.9831\times 10^3\,\hat{j} \quad \text{in/min}^2.$$

iv) $\dot{\vec{\omega}}_B \times \vec{r}_{P/A}$

$$= 20\pi\,\hat{k} \times \left[\left(\frac{7}{2}+144\right)\hat{i} + \frac{7\sqrt{3}}{2}\hat{j}\right]$$

$$= -380.898\,\hat{i} + 9.2677\times 10^3\,\hat{j} \quad \text{in/min}^2$$

v) $2\vec{\omega}_B \times \vec{v}_{P/B}$

$$\vec{v}_{P/B} = (-\omega_t\hat{k}) \times \vec{r}_{P/C}$$

$$\therefore\ 2\vec{\omega}_B \times \vec{v}_{P/B} = 2\,\omega_C\,\omega_t\,\vec{r}_{P/C}$$

$$= 2\times(10\pi)\times\left(66\tfrac{2}{3}\pi\right)\times\left(\frac{7}{2}\hat{i} + \frac{7\sqrt{3}}{2}\hat{j}\right)$$

$$= 4.6058\times 10^4\,\hat{i} + 7.9775\times 10^4\,\hat{j} \quad \text{in/min}^2$$

Sum all the five terms up.

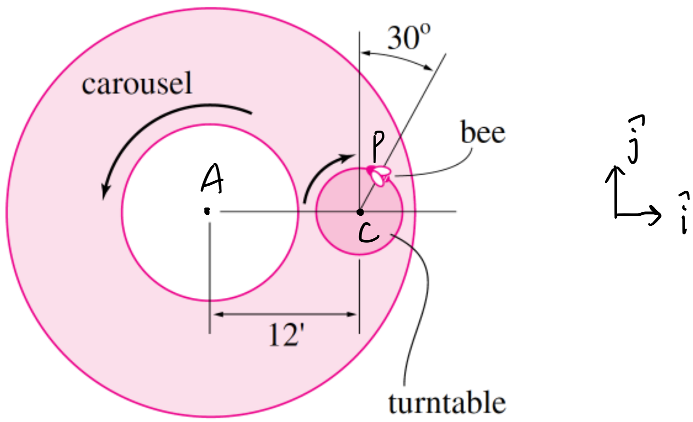
$$\vec{a}_P = \left(-2.5343\times 10^5\,\hat{i} - 1.8646\times 10^5\,\hat{j}\right) \quad \text{in/min}^2$$

$\therefore$ the magnitude of acceleration

$$\boxed{\begin{aligned}|\vec{a}_P| &= 3.1464\times 10^5 \ \text{in/min}^2 \\ &= 87.4 \ \text{in/sec}^2\end{aligned}}$$

17.3.11

Saturday, April 27, 2019 17:16



Carousel : angular velocity $\omega_c = 5 rpm = 10\pi\ rad/min$

angular acceleration $\alpha_c = 10\ r/min^2 = 20\ rad/min^2$

Turntable : const angular velocity $\omega_t = 33\frac{1}{3} rpm = 66\frac{2}{3}\pi\ rad/min$

radius $r_t = 7'$

Find : the magnitude of the acceleration a_b of the bee

Using the rotating frame β which is fixed to the carousel

Thus $\vec{\omega}_\beta = \omega_\beta \hat{k} = \omega_c \hat{k} = 10\pi\ rad/min$

$\dot{\vec{\omega}}_\beta = \alpha_\beta \hat{k} = \alpha_c \hat{k} = 20\pi\ rad/min^2$

$$\vec{a}_P = \vec{a}_A + \vec{a}_{P/\beta} - \omega_\beta^2 \vec{r}_{P/A} + \dot{\vec{\omega}}_\beta \times \vec{r}_{P/A} + 2\vec{\omega}_\beta \times \vec{v}_{P/\beta}$$

i) $\vec{a}_A = 0$ since A is fixed

ii) $\vec{a}_{P/\beta} = -\omega_t^2 \vec{r}_{P/C} = -(66\frac{2}{3}\pi)^2\ 7\ (\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j})\ [in/min^2]$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

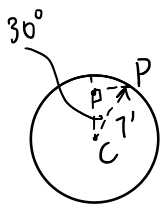
Since P rotates with the turntable at a const angular velocity

$\therefore \overrightarrow{a_{P/\beta}} = (-1.5353 \times 10^5 \hat{i} - 2.6952 \times 10^5 \hat{j})$ [in/min²]

iii) $-\omega_\beta^2 \overrightarrow{r_{P/A}} = -\omega_\beta^2 (\overrightarrow{r_{P/C}} + \overrightarrow{r_{C/A}})$

$= -(10\pi)^2 (\frac{7}{2}\hat{i} + \frac{7\sqrt{3}}{2}\hat{j} + 12\hat{i})$ [in/min²]

$= -1.5298 \times 10^4 \hat{i} - 5.9831 \times 10^3 \hat{j}$ [in/min²]



$|\overrightarrow{r_{P/C}}| = 7'$

$\overrightarrow{r_{P/C}} = \frac{7}{2}\hat{i} + \frac{7\sqrt{3}}{2}\hat{j}$ [in]

$A \xrightarrow[12']{} C$ $\quad \overrightarrow{r_{C/A}} = 12\hat{i}$ [in]

iv) $\dot{\vec{\omega}}_\beta \times \overrightarrow{r_{P/A}}$

$= 20\pi\hat{k} \times [(\frac{7}{2} + 12)\hat{i} + \frac{7\sqrt{3}}{2}\hat{j}]$ [in/min²]

$= -380.898\hat{i} + 973.8937\hat{j}$ [in/min²]

v) $2\vec{\omega}_\beta \times \underbrace{\overrightarrow{v_{P/\beta}}}_{(-\omega_t\hat{k}) \times \overrightarrow{r_{P/C}}}$

$= 2\omega_c\hat{k} \times (-\omega_t\hat{k}) \times \overrightarrow{r_{P/C}}$

$= 2\omega_c\omega_t \overrightarrow{r_{P/C}}$

$= 2 \times 10\pi \times 66\frac{2}{3}\pi \times (\frac{7}{2}\hat{i} + \frac{7\sqrt{3}}{2}\hat{j})$ [in/min²]

$= 4.6058 \times 10^4 \hat{i} + 7.9775 \times 10^4 \hat{j}$ [in/min²]

Sum all the five terms

$\overrightarrow{a_P} = -1.2315 \times 10^5 \hat{i} - 1.9475 \times 10^5 \hat{j}$ [in/min²]

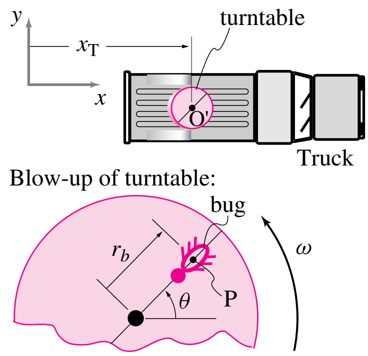
$\therefore |\vec{a}_P| = 2.3042 \times 10^5$ [in/min²] $= 64.0056$ [in/sec²]

18.3.12 Consider a turntable on the back of a pick-up truck. A bug walks on a line on the turntable. The line may or may not be drawn through the center of the turntable. The truck may or may not be going at constant rate. The turntable may or may not be spinning, and, if it spins, it may or may not go at a constant rate. The bug may be anywhere on the line and may or may not be walking at a constant rate.

a) **Draw a picture of the situation.** Clearly define all variables you are going to use. What is the moving frame, and what is the reference point on that frame?

b) **One term at a time.** For each term in the five term acceleration formula, find a situation where all but one term is zero. Use the turntable as the moving frame, the bug as the particle of interest.

c) **Two terms at a time.** (harder) How many situations can you find where a pair of terms is not zero but all other terms are zero? (Don't try to do all 10 cases unless you really think this infamous formula is fun. Try at least one or two.)



Problem 18.12: **Picking apart the five term acceleration formula.**

Answer:
(b) One case: $\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{O'}$, where P refers to the bug; the bug stands still on the turntable, the truck accelerates, and the turntable does not rotate.
(c) One case: $\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{O'} + \vec{\boldsymbol{a}}_{P/\mathcal{D}}$, where $\mathcal{D}$ is the turntable; the bug accelerates along a straight line on the turntable, the truck accelerates, and the turntable does not rotate.