18.2 Rotating reference frames and their time-varying base vectors

In this section you will learn about rotating reference frames, how to take the derivative of a vector 'in' a rotating frame, and how to use that derivative to find the derivative in a Newtonian or fixed frame. We start by showing the alternative, just using one frame with one set of fixed base vectors.

The fixed base-vector method

To motivate the sections that follow we first show the "fixed base vector" method. Consider the task of determining the acceleration of a bug walking at constant speed as it walks on a straight line marked on the surface of a tire rolling at constant rate. Artificial as this problem seems, it is similar to the sort of calculation needed in the kinematics of mechanisms. For now, imagine you really care how strong the bug's legs need to be to hold on (unreasonably neglecting air friction). So knowing the bug's acceleration determines the net force on it by $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Now we try to find $\vec{\boldsymbol{a}}$ by taking two time derivatives of position.

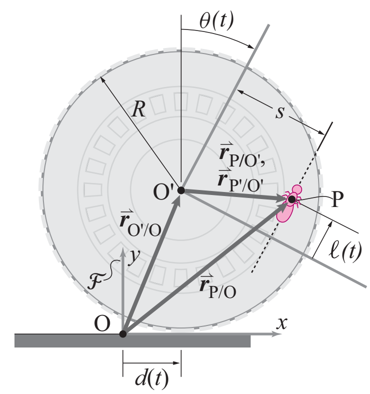


Figure 18.17: A bug walks at constant speed on a straight line (dotted) marked on a tire that is rotating clockwise (θ relative to a vertical line). The distance of the line from the center of the tire is s. The distance the bug has walked from the closest point to the tire center is ℓ. The fixed base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are aligned with the fixed x and y axis. The line segment connecting the tire center with the closest point on the line is in the direction $\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$. The direction along the line on which the bug walks is $\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

If we want to avoid using rotating base vectors we have to write an expression for the position of the bug in terms of x and y components. Choosing a suitable origin of the coordinate system we have

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{P/0} &= \vec{\boldsymbol{r}}_{0'/0} + \vec{\boldsymbol{r}}_{P/0'} \\
&= \underbrace{(R\theta\hat{\boldsymbol{\imath}} + R\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{0'/0}} + \underbrace{(s(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\,\hat{\boldsymbol{\jmath}}) + \ell(\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}))}_{\vec{\boldsymbol{r}}_{P/0'}} \\
&= \left(R\theta + s\cos\theta + \ell\,\sin\theta\right)\hat{\boldsymbol{\imath}} + \left(R - s\sin\theta + \ell\,\cos\theta\right)\hat{\boldsymbol{\jmath}}. \qquad (18.15)
\end{aligned}
$$

To find the velocity we take the time derivative, taking account that both θ and ℓ are functions of t. Thus, for example looking at the term $\ell\cos\theta$ both the product rule and chain rule need be applied. Proceeding we get

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{P/0} \;=\; & \left(R\dot\theta - s\dot\theta\sin\theta + \dot\ell\,\sin\theta + \ell\dot\theta\cos\theta\right)\hat{\boldsymbol{\imath}} \\
& + \left(-s\dot\theta\cos\theta + \dot\ell\,\cos\theta - \ell\dot\theta\sin\theta\right)\hat{\boldsymbol{\jmath}}. \qquad (18.16)
\end{aligned}
$$

To get the acceleration of the bug we differentiate one more time. This time we use the product rule and chain rule again, but get to use the simplification for this problem that the rolling and bug walking are at constant rate so $\ddot\theta = 0$ and $\ddot\ell = 0$:

$$
\begin{aligned}
\vec{\boldsymbol{a}}_{P/0} \;=\; & \left(-s\dot\theta^2\cos\theta + \dot\ell\dot\theta\cos\theta + \dot\ell\dot\theta\cos\theta - \ell\dot\theta^2\sin\theta\right)\hat{\boldsymbol{\imath}} \\
& + \left(\;s\dot\theta^2\,\sin\theta - \dot\ell\dot\theta\,\sin\theta - \dot\ell\dot\theta\,\sin\theta - \ell\dot\theta^2\cos\theta\right)\hat{\boldsymbol{\jmath}} \\
=\; & \left(-\dot\theta^2(s\cos\theta + \ell\,\sin\theta) + 2\dot\ell\dot\theta\cos\theta\right)\hat{\boldsymbol{\imath}} \\
& + \left(\;\dot\theta^2(s\,\sin\theta - \ell\,\cos\theta) - 2\dot\ell\dot\theta\,\sin\theta\right)\hat{\boldsymbol{\jmath}} \qquad (18.17)
\end{aligned}
$$

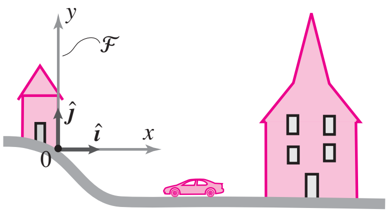


Figure 18.18: A fixed reference frame $\mathcal{F}$ is defined by an origin 0 and coordinate axes xyz or base vectors $\hat{\imath}\hat{\jmath}\hat{k}$. Once the xy (or $\hat{\imath}\,\hat{\jmath}$) directions are chosen the z (or $\hat{k}$) direction is implicitly defined by the right hand rule.

①**A fine point for experts: reference frame *vs* coordinate system.** There is a semantic debate about the degree to which the phrases "coordinate system" and " reference frame" are synonymous. For simplicity we take them to mean the same thing. An alternative definition distinguishes reference frames from coordinate systems. It turns out that coordinate systems which are rotated, but not rotating, with respect to each other both calculate the same time-derivative of a given vector. Because these coordinate systems are equivalent in this regard they are sometimes called the same reference frame. That is, some people consider one reference frame to be the set of all coordinate systems that are glued to each other, no matter what their position or orientation. In this way of thinking, a frame is made manifest by the use of one of its coordinate systems, but no particular coordinate system is unique to the frame.

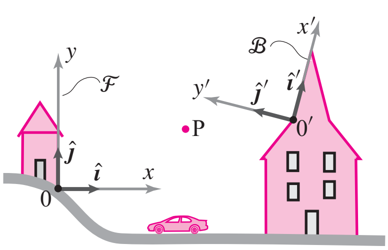


Figure 18.19: A second reference frame $\mathcal{B}$ is defined by an origin 0' and coordinate axes $x'y'z'$ or base vectors $\hat{\imath}'\,\hat{\jmath}'\hat{k}'$. Once the $x'y'$ (or $\hat{\imath}'\,\hat{\jmath}'$) directions are chosen the z' (or $\hat{k}'$) direction is implicitly defined by the right hand rule.

which is a bit of a mess. We could regroup the terms, but there would still be 6 of them.

The moving-reference-frame methods that follow don't change this answer. But they give a somewhat simpler derivation. And they also group the terms in a physically meaningful way. One would be hard pressed to make sense of all the terms in eqn. (18.17). With the time-varying base vector methods below we can interpret the terms.

Reference frames

A *reference frame* is a coordinate system①. It has an origin and a set of preferred mutually orthogonal directions represented by base vectors. You can think of a reference frame as a giant piece of graph paper, or in 3-D as a giant jungle gym, that permeates space. It has the look of a wire *frame*. Because we will use various frames, we name them. We always have one frame that we think of as *fixed* for the purposes of Newtonian mechanics. We call this frame $\mathcal{F}$ (or sometimes $\mathcal{N}$). Most often we choose a frame that is 'glued' to the ground with an origin at a convenient point and with at least one base vector lined up with something convenient (e.g., up, sideways, along a slope, along the edge of an important part, *etc.*). $\mathcal{F}$ is a frame in which the mechanics laws we use are accurate. We define it by its origin and the direction of its coordinate axes, thus we would write

$$\mathcal{F} \text{ is } 0xyz \quad \text{or} \quad \mathcal{F} \text{ is } O\hat{\imath}\hat{\jmath}\hat{k}.$$

where we would generally have a picture showing the position of the origin and the orientation of the coordinate axes (see fig. 18.18).

When we write casually 'position $\vec{r}$' of a point we mean $\vec{r}_{P/0}$. When we write 'velocity $\vec{v}$' we mean $\frac{d}{dt}\vec{r}$ as calculated in $\mathcal{F}$. That is, if $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then we define the derivative of $\vec{r}$ with respect to t in $\mathcal{F}$ as

$$\frac{{}^{\mathcal{F}}d\vec{r}}{dt} = {}^{\mathcal{F}}\dot{\vec{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}.$$

The script $\mathcal{F}$ shows explicitly that when we take the time derivative of the vector we take the time derivative of its components, using the components associated with $\mathcal{F}$ and holding constant the base vectors associated with $\mathcal{F}$. That is

$\dfrac{{}^{\mathcal{F}}d\vec{r}_{P/0}}{dt} = {}^{\mathcal{F}}\dot{\vec{r}}_{P/0}$ are just fancy ways of writing what we have been calling $\vec{v}$.

The elaborate notation just makes explicit how $\vec{v}$ is defined. The only need for this elaborate notation is if there is ambiguity. There is only ambiguity if more than one reference frame is used in a given problem.

Using more than one reference frame

Let's add a second reference frame called $\mathcal{B}$ glued to and oriented with the roof of the building. We will always use script capital letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ or $\mathcal{N}$) to name reference frames. We define $\mathcal{B}$ by writing

$$\mathcal{B} \text{ is } 0'x'y'z' \quad \text{or} \quad \mathcal{B} \text{ is } O'\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'\hat{\boldsymbol{k}}'$$

and by drawing a picture (see fig. 18.19). This new frame, as we have drawn it, is also a good Newtonian or fixed frame. So we could write all positions using the $\mathcal{B}$ coordinates and base vectors and then proceed with all of our mechanics equations. The only confusion being that gravity doesn't point in the $-\hat{\boldsymbol{\jmath}}'$ direction, but in some crooked direction relative to $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ (which we would have to work out from the angle of the roof). Although one hardly notices when using just a single fixed frame, we actually use frames for three somewhat distinct purposes:

I. To define a vector. For example if we were tracking the motion of a cannon ball at P we could define its position vector $\vec{\boldsymbol{r}}$ as $\vec{\boldsymbol{r}}_{\mathrm{P}/0}$, using frame $\mathcal{F}$ to define $\vec{\boldsymbol{r}}$. Or we could define $\vec{\boldsymbol{r}}$ as $\vec{\boldsymbol{r}}_{\mathrm{P}/\mathrm{O}'}$ using frame $\mathcal{B}$ to define $\vec{\boldsymbol{r}}$.

II. To assign coordinate values to a given vector. For example, the vector $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ could be written as

$$\begin{aligned} \vec{\boldsymbol{r}}_{\mathrm{B/A}} &= \vec{\boldsymbol{r}}_{\mathrm{B/A}}, \\ 4\,\mathrm{m}\hat{\boldsymbol{\imath}} + 5\,\mathrm{m}\hat{\boldsymbol{\jmath}} &= 6\,\mathrm{m}\hat{\boldsymbol{\imath}}' - 2.24\,\mathrm{m}\hat{\boldsymbol{\jmath}}'. \end{aligned}$$

Alternatively, if we just want to look at the components of a given vector we use $[]_{\mathcal{F}}$ to indicate the components of the vector in the square brackets, [ ], using the base vectors of $\mathcal{F}$. Thus

$$[\vec{\boldsymbol{r}}_{\mathrm{B/A}}]_{\mathcal{F}} = [4\,\mathrm{m}, \quad 5\,\mathrm{m}]' \quad \text{and} \quad [\vec{\boldsymbol{r}}_{\mathrm{B/A}}]_{\mathcal{B}} = [6\,\mathrm{m}, \quad -2.24\,\mathrm{m}]'$$

where we have used $[]'$ to put the components in their standard column form (although this is a picky detail). Note that although $\vec{\boldsymbol{r}}_{\mathrm{B/A}} = \vec{\boldsymbol{r}}_{\mathrm{B/A}}$ that

$$\begin{aligned} [\vec{\boldsymbol{r}}_{\mathrm{B/A}}]_{\mathcal{F}} &\neq [\vec{\boldsymbol{r}}_{\mathrm{B/A}}]_{\mathcal{B}} \\ \text{because} \quad \begin{bmatrix} 4\,\mathrm{m} \\ 5\,\mathrm{m} \end{bmatrix} &\neq \begin{bmatrix} 6\,\mathrm{m} \\ -2.24\,\mathrm{m} \end{bmatrix} \end{aligned}$$

That is, the components of a vector are different when expressed in different frames. Two different lists of numbers represent the same vector. The magnitude, however, is independent of frame ($4^2 + 5^2 = 6^2 + 2.24^2$, to two decimal places).

III. To find the rate of change of a given vector. The position of P relative to A changes with time. We can calculate this rate of change two different ways. First using frame $\mathcal{F}$

$$\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{v}}_{\mathrm{P/A}}}{dt} = \overset{\mathcal{F}.}{\vec{\boldsymbol{r}}}_{\mathrm{P/A}} = \left(\frac{d}{dt}x_{\mathrm{P/A}}\right)\hat{\boldsymbol{\imath}} + \left(\frac{d}{dt}y_{\mathrm{P/A}}\right)\hat{\boldsymbol{\jmath}}$$

$$\text{or more informally as} \quad \overset{\mathcal{F}.}{\vec{\boldsymbol{r}}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$$

if we are clear in our minds that x and y are the coordinates of P relative to A. But we can also calculate the rate of change of the same vector $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ using frame $\mathcal{B}$ as

$$\frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \overset{\mathcal{B}}{\dot{\vec{\boldsymbol{r}}}}_{\mathrm{P/A}} = \left(\frac{d}{dt}x'_{\mathrm{P/A}}\right)\hat{\boldsymbol{\imath}}' + \left(\frac{d}{dt}y'_{\mathrm{P/A}}\right)\hat{\boldsymbol{\jmath}}'$$

$$\text{or more informally as} \quad \overset{\mathcal{B}}{\dot{\vec{\boldsymbol{r}}}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'.$$

For the two frames $\mathcal{F}$ and $\mathcal{B}$ $\overset{\mathcal{B}}{\dot{\vec{\boldsymbol{r}}}} = \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{r}}}}$ because the two frames are not rotating relative to each other. Specifically, for $\mathcal{F}$ and $\mathcal{B}$ the formula for finding x and y from x' and y' does not involve time. Similarly, the formulas for finding $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ from $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ do not involve time. For frames that are *rotated* with respect to each other but not *rotating*, the two time derivatives of a given vector are related the same way the vector itself is related to itself in the two frames. The vectors are the same but their coordinates are different. That is, for rotated but not relatively rotating frames

$$\vec{\boldsymbol{r}}_{\mathrm{P/A}} = \vec{\boldsymbol{r}}_{\mathrm{P/A}} \quad \text{and} \quad \frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt}$$

$$\text{but} \quad [\vec{\boldsymbol{r}}_{\mathrm{P/A}}]_{\mathcal{F}} \neq [\vec{\boldsymbol{r}}_{\mathrm{P/A}}]_{\mathcal{B}} \quad \text{and} \quad \left[\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt}\right]_{\mathcal{F}} \neq \left[\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt}\right]_{\mathcal{B}}.$$

Going back and forth between these three uses of frames with ease is one of the advanced skills of a person who can analyze the dynamics of complex systems (And being confused about the distinctions is an almost universal part of learning advanced dynamics).

Example: Two fixed frames $\mathcal{F}$ and $\mathcal{B}$
Consider $\mathcal{F}$ and $\mathcal{B}$ both to be fixed to the ground. Let's look at $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ where P is moving up at constant rate (see fig. 18.20). First look at the position using both frames:

$$\begin{aligned} \vec{\boldsymbol{r}}_{\mathrm{P/A}} &= \vec{\boldsymbol{r}}_{\mathrm{P/A}} \quad \text{and} \\ d\hat{\boldsymbol{\imath}} + ct\,\hat{\boldsymbol{\jmath}} &= \left(\sqrt{2}ct/2\right)\hat{\boldsymbol{\imath}}' + \left(\sqrt{2}ct/2\right)\hat{\boldsymbol{\jmath}}' \quad \text{but} \\ [\vec{\boldsymbol{r}}_{\mathrm{P/A}}]_{\mathcal{F}} = [d, \quad ct]' &\neq [\vec{\boldsymbol{r}}_{\mathrm{P/A}}]_{\mathcal{B}} = [\sqrt{2}ct/2, \quad \sqrt{2}ct/2]'. \end{aligned}$$

Now look at the rate of change of position using both frames. First $\mathcal{F}$:

$$\begin{aligned} \frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \overset{\mathcal{F}}{\dot{\vec{\boldsymbol{r}}}}_{\mathrm{P/A}} &= \left(\frac{d}{dt}x_{\mathrm{P/A}}\right)\hat{\boldsymbol{\imath}} + \left(\frac{d}{dt}y_{\mathrm{P/A}}\right)\hat{\boldsymbol{\jmath}} \\ &= c\,\hat{\boldsymbol{\jmath}} \end{aligned}$$

Then the rate of change of $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ as calculated in $\mathcal{B}$:

$$\begin{aligned} \frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \overset{\mathcal{B}}{\dot{\vec{\boldsymbol{r}}}}_{\mathrm{P/A}} &= \left(\frac{d}{dt}x'_{\mathrm{P/A}}\right)\hat{\boldsymbol{\imath}}' + \left(\frac{d}{dt}y'_{\mathrm{P/A}}\right)\hat{\boldsymbol{\jmath}}' \\ &= (\sqrt{2}c/2)\hat{\boldsymbol{\imath}}' + (\sqrt{2}c/2)\hat{\boldsymbol{\jmath}}' = c\,\hat{\boldsymbol{\jmath}} \end{aligned}$$

You can quickly verify that $\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt}$ by noting that $\hat{\boldsymbol{\imath}}' = \sqrt{2}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})/2$ and $\hat{\boldsymbol{\jmath}}' = \sqrt{2}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})/2$.

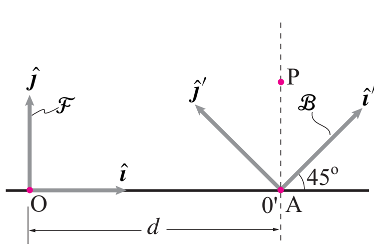


Figure 18.20: A particle P is moving up at speed c. Two fixed frames $\mathcal{F}$ and $\mathcal{B}$ are used to keep track of it.

So long as $\mathcal{B}$ is not rotating with respect to $\mathcal{F}$ then the rate of change of a given vector is the same in both reference frames.

Translating and rotating reference frames

Now look at a third reference frame $\mathcal{C}$ that is glued to the roof of the car as it starts up hill (see fig. 18.21). We define $\mathcal{C}$ by the origin of its coordinate system $0''$ and its time-varying base vectors $\hat{\imath}''$ and $\hat{\jmath}''$. The issues with **defining** a vector with $\mathcal{C}$ and with **writing components** using $\mathcal{C}$ are the same as for $\mathcal{B}$. However taking the **time derivative** of a given vector in $\mathcal{C}$ is different than taking the time derivative in $\mathcal{B}$ or $\mathcal{F}$ because $\mathcal{C}$ is rotating relative to them.

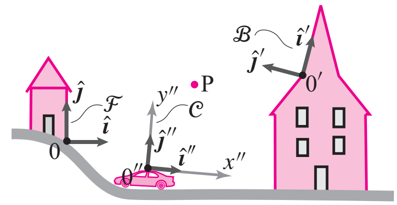


Figure 18.21: A third reference frame $\mathcal{C}$ is defined by an origin $0''$ and coordinate axes $x''y''z''$ or base vectors $\hat{\imath}''\hat{\jmath}''\hat{k}''$. In this case $\mathcal{C}$ is moving and rotating with respect to $\mathcal{F}$ and $\mathcal{B}$. Once the $x''y''$ (or $\hat{\imath}''\hat{\jmath}''$) directions are chosen the z'' (or $\hat{k}''$) direction is implicitly defined by the right hand rule.

Rate of change of a vector relative to a rotating frame: the $\overset{\mathcal{C}}{\dot{\vec{Q}}}$ formula

Because dynamics involves the time derivatives of so many different vectors (e.g. $\vec{r}$, $\vec{v}$, $\vec{L}$, $\vec{H}_{/\mathrm{C}}$, and $\vec{\omega}$) it is easier to think about the derivative of some arbitrary or general vector, call it $\vec{Q}$, and then apply what we learn to these other vectors.

Recalling our three uses of frames:

I. To define a vector.

II. To express the coordinates of a given vector.

III. To take the time derivative of a vector.

we see that items [III.] and [I.] can be combined. That is, once a vector $\vec{Q}$ is defined clearly by some means then we can define a new vector as the derivative of that vector in, say, moving frame $\mathcal{C}$. Once this new vector ${}^{\mathcal{C}}\dot{\vec{Q}}$ is defined it can be expressed in terms of the coordinates of any convenient frame.

Example: Derivative in a moving frame of a constant vector
Consider as $\vec{Q}$ the relative position vector $\vec{r}_{\mathrm{P/A}}$ of the points A and P that do not move in the fixed frame $\mathcal{F}$. That is, the points A and P don't move in the ordinary sense of the words (see fig. 18.22). Now also look at the frame $\mathcal{B}$ that is rotating with respect to $\mathcal{F}$ at the rate $\dot{\theta}$. We have

$$\begin{aligned} \vec{r}_{\mathrm{P/A}} &= \vec{r}_{\mathrm{P/A}} \\ \ell\,\hat{\jmath} &= \ell\sin\theta\,\hat{\imath}' + \ell\cos\theta\,\hat{\jmath}' \end{aligned}$$

So we can now calculate the derivative in each frame by holding the correspond-

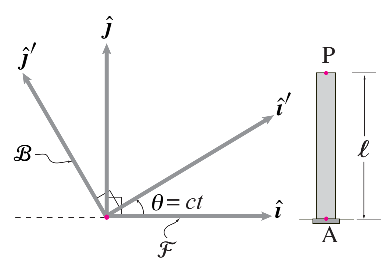


Figure 18.22: A fixed frame $\mathcal{F}$ and a rotating frame $\mathcal{B}$ both keep track of the position vector $\vec{r}_{\mathrm{P/A}}$.

ing base vectors as constant. So

$$\begin{aligned} \frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} &= \dot{\ell}\,\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}} \\ \text{and}\quad \frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} &= \ell\dot{\theta}\cos\theta\,\hat{\boldsymbol{\imath}}' - \ell\dot{\theta}\sin\theta\,\hat{\boldsymbol{\jmath}}' \\ &= \ell\dot{\theta}\,\underbrace{(\cos\theta\,\hat{\boldsymbol{\imath}}' - \sin\theta\,\hat{\boldsymbol{\jmath}}')}_{\hat{\boldsymbol{\imath}}} \\ &= \ell\dot{\theta}\,\hat{\boldsymbol{\imath}} \end{aligned}$$

That is, the stationary vector $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ and the rotating frame $\mathcal{B}$ define a new vector, the derivative of $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ in $\mathcal{B}$. This is also $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} - \vec{\boldsymbol{v}}_{\mathrm{A}/\mathcal{B}}$, the difference between the velocity of P and the velocity of A in the frame $\mathcal{B}$. This new vector can be expressed in any coordinate system of choice for example the $\hat{\boldsymbol{\imath}}\,\hat{\boldsymbol{\jmath}}$ system. So we wrote above

$$\frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \ell\dot{\theta}\,\hat{\boldsymbol{\imath}}$$

which looks mixed up but isn't. The frame $\mathcal{B}$ is used to help define a vector which is then expressed in the coordinates of $\mathcal{F}$.

Using the moving-frame derivative to calculate the fixed-frame derivative

Given a new vector ${}^{\mathcal{C}}\dot{\vec{\boldsymbol{Q}}}$, the derivative of $\vec{\boldsymbol{Q}}$ as calculated in a rotating frame $\mathcal{C}$, one calculation of common use is the determination of the derivative ${}^{\mathcal{F}}\dot{\vec{\boldsymbol{Q}}}$ of the same vector in the fixed frame $\mathcal{F}$.

First think of a line segment that is marked between two points that are glued to a moving frame $\mathcal{C}$. We know (at least in 2-D and for fixed axis rotation) that

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}.$$

Likewise for any vector which is fixed in $\mathcal{C}$. It is especially useful to apply this formula to unit base vectors, so

[$\vec{\boldsymbol{Q}}(t)$ is an arbitrary vector not attached to $\mathcal{C}$ or $\mathcal{F}$.]

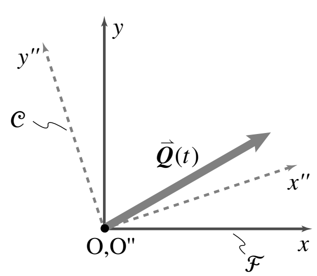


Figure 18.23: A fixed frame $\mathcal{F}$ defined by $Oxyz$ and a moving frame $\mathcal{C}$ defined by $O'x'y'z'$. Also shown is an arbitrary vector $\vec{\boldsymbol{Q}}$ which changes relative to both frames.

$$\begin{aligned} \dot{\hat{\boldsymbol{\imath}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}'', \\ \dot{\hat{\boldsymbol{\jmath}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}'', \quad \text{and} \\ \dot{\hat{\boldsymbol{k}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{k}}''. \end{aligned} \tag{18.18}$$

In some minds, Eqns. 18.18 are the core of rigid body kinematics. Box 18.1 on 10 shows how these relations give 'the Q dot' formula: For any time dependent vector $\vec{\boldsymbol{Q}}$

$${}^{\mathcal{F}}\dot{\vec{\boldsymbol{Q}}} = {}^{\mathcal{C}}\dot{\vec{\boldsymbol{Q}}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/F} \times \vec{\boldsymbol{Q}}. \tag{18.19}$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}.$$

where $\dot{\vec{Q}}_{rel}$ is the time derivative of $\vec{Q}$ relative to the moving frame of interest (in this case $\mathcal{C}$). The 'Q dot' formula says that

The derivative of a vector with respect to a Newtonian frame $\mathcal{F}$ (or 'absolute derivative') can be calculated as the derivative ${}^{\mathcal{C}}\dot{\vec{Q}}$ of the vector with respect to a moving frame $\mathcal{C}$, plus a term $\vec{\omega}_{\mathcal{C}} \times \vec{Q}$ that corrects for the rotation of frame $\mathcal{C}$ relative to frame $\mathcal{F}$.

Note that if $\vec{Q}$ is a constant in the frame $\mathcal{C}$, like the relative position vector of two points glued to $\mathcal{C}$, then ${}^{\mathcal{C}}\dot{\vec{Q}} = \vec{Q}_{rel} = \vec{0}$ and the $\dot{\vec{Q}}$ formula reduces to

$${}^{\mathcal{F}}\dot{\vec{Q}} = \vec{\omega} \times \vec{Q}.$$

The $\dot{\vec{Q}}$ formula 18.19 is useful for the derivation of a variety of formulas and is also useful in the solution of problems.

While we have shown how to use this formula to calculate the rate of change of a vector with respect to a Newtonian frame, the formula can be used to calculate its rate of change with respect to a non-Newtonian frame. Letting $\mathcal{A}$ and $\mathcal{B}$ be two possibly non-Newtonian frames, the $\dot{\vec{Q}}$ formula for the rate of change of $\vec{Q}$ with respect to frame $\mathcal{A}$ is

$${}^{\mathcal{A}}\dot{\vec{Q}} = {}^{\mathcal{B}}\dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{Q}. \tag{18.20}$$

Both $\mathcal{A}$ and $\mathcal{B}$ could be non-Fixed (non-Newtonian).

Summary of the $\dot{\vec{Q}}$ formula

For a vector $\vec{Q}$ fixed in $\mathcal{B}$,

$$\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}} \times \vec{Q} \qquad \text{or} \qquad {}^{\mathcal{F}}\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

For any time dependent vector $\vec{Q}$,

$$\dot{\vec{Q}} = {}^{\mathcal{B}}\dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}} \times \vec{Q} \qquad \text{or} \qquad {}^{\mathcal{F}}\dot{\vec{Q}} = {}^{\mathcal{B}}\dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

Some examples of applying the $\dot{\vec{Q}}$ formula are:

- Absolute velocity of a point P relative to O':

$$\dot{\vec{r}}_{P/O'} = \overset{\mathcal{B}}{\dot{\vec{r}}}_{P/O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}$$

- Rate of change of a rotating unit vector which is fixed in $\mathcal{B}$:

$$\dot{\hat{\boldsymbol{\imath}}}' = \underbrace{\dot{\hat{\boldsymbol{\imath}}}'_{rel}}_{\vec{\mathbf{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}'$$

The varying base-vectors method of computing velocity and acceleration

One way to calculate velocity, acceleration is to express the position of a particle in terms of a combination of base vectors, some of which change in time. Velocity and acceleration are then determined by directly differentiating the expression for position, taking account that the base vectors themselves are changing. This method is sometimes convenient for bodies connected in series, one body to the next, etc. The overall approach is as follows:

1) Glue a coordinate system to every moving body. If needed, also create moving frames that move independently of any particular body.
2) Call the basis vectors associated with these frames $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ for the fixed frame $\mathcal{F}$; $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, $\hat{\boldsymbol{k}}'$ for the moving frame $\mathcal{B}$; and $\hat{\boldsymbol{\imath}}''$, $\hat{\boldsymbol{\jmath}}''$, $\hat{\boldsymbol{k}}''$ for the moving frame $\mathcal{C}$, etc.
3) Evaluate all of the relative angular velocities; $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}}$, etc. in terms of the scalar angular rates $\dot{\theta}$, $\dot{\phi}$, etc. and the base vectors glued to the frames.
4) Express all of the absolute angular velocities in terms of the relative angular velocities.
5) Differentiate to get the angular accelerations using, for example,

$$\begin{aligned} \dot{\hat{\boldsymbol{\imath}}}' &= \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}' \text{ or} \\ \dot{\hat{\boldsymbol{\jmath}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}'' \end{aligned}$$

6) Write the position of all points of interest in terms of the various base vectors.
7) Differentiate the position to get the velocities (again using $\dot{\hat{\boldsymbol{\jmath}}}'' = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}''$, etc.)
8) Differentiate again to get acceleration.

First, reconsider the bug crawling on the tire in fig. 18.24.

Example: Absolute velocity of a point moving relative to a moving frame: Bug crawling on a tire

We write the position of the bug in terms of the various basis vectors as

$$\begin{aligned} \vec{\boldsymbol{r}}_{P/O} &= \vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'} \\ &= \overbrace{d\hat{\boldsymbol{\imath}} + R\hat{\boldsymbol{\jmath}}}^{\vec{\boldsymbol{r}}_{O'/O}} + \overbrace{s\hat{\boldsymbol{\imath}}' + \ell\,\hat{\boldsymbol{\jmath}}')}^{\vec{\boldsymbol{r}}_{P/O'}}. \end{aligned}$$

To get the absolute velocity $\frac{^{\mathcal{F}}d}{dt}(\vec{\boldsymbol{r}}_{P/O})$ of the bug at the instant shown, we differentiate the position of the bug once, using the product rule and the rates of change of the rotating basis vectors with respect to the fixed frame, to get

$$\begin{aligned}
\dot{\vec{\boldsymbol{r}}}_{P/O} = \vec{\boldsymbol{v}}_P &= \dot{d}\hat{\boldsymbol{\imath}} + \underbrace{\dot{R}}_{0}\hat{\boldsymbol{\jmath}} + \underbrace{\dot{s}}_{0}\hat{\boldsymbol{\imath}}' + s\dot{\hat{\boldsymbol{\imath}}}' + \dot{\ell}\hat{\boldsymbol{\jmath}}' + \ell\dot{\hat{\boldsymbol{\jmath}}}' \\
&= \dot{d}\hat{\boldsymbol{\imath}} + s(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}') + \dot{\ell}\hat{\boldsymbol{\jmath}}' + \ell(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}') \\
&= \dot{d}\hat{\boldsymbol{\imath}} + s(-\dot{\theta}\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\imath}}') + \dot{\ell}\hat{\boldsymbol{\jmath}}' + \ell(-\dot{\theta}\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\jmath}}') \\
&= \dot{d}\hat{\boldsymbol{\imath}} + \ell\dot{\theta}\hat{\boldsymbol{\imath}}' + (\dot{\ell} - s\dot{\theta})\hat{\boldsymbol{\jmath}}'. \qquad (18.21)
\end{aligned}$$

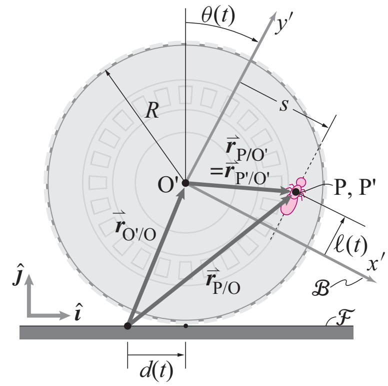


Figure 18.24: The xyz coordinate system is attached to the fixed frame with basis vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ and the $x'y'z'$ coordinate system is attached to the rolling tire with basis vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, $\hat{\boldsymbol{k}}'$ .The absolute angular velocity of the tire is $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} = -\dot{\theta}\hat{\boldsymbol{k}}' = -\dot{\theta}\hat{\boldsymbol{k}}$.

Example: Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

Differentiating equation 18.21 from the example above again, we get the absolute acceleration $\frac{^{\mathcal{F}}d^2}{dt^2}(\vec{\boldsymbol{r}}_{P/O}) = \frac{^{\mathcal{F}}d}{dt}(\vec{\boldsymbol{v}}_P)$ of the bug at the instant shown,

$$\begin{aligned}
\ddot{\vec{\boldsymbol{r}}}_{P/O} = \dot{\vec{\boldsymbol{v}}}_P = \vec{\boldsymbol{a}}_P &= \ddot{d}\hat{\boldsymbol{\imath}} + (\ell\ddot{\theta} + \dot{\ell}\dot{\theta})\hat{\boldsymbol{\imath}}' + \ell\dot{\theta}(-\dot{\theta}\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\imath}}') \\
&\quad + (\ddot{\ell} - s\ddot{\theta} - \underbrace{\dot{s}}_{0}\dot{\theta})\hat{\boldsymbol{\jmath}}' + (\dot{\ell} - s\dot{\theta})(-\dot{\theta}\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\jmath}}') \\
&= \ddot{d}\hat{\boldsymbol{\imath}} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\ell}\dot{\theta})\hat{\boldsymbol{\imath}}' + (\ddot{\ell} - \ell\dot{\theta}^2 - s\ddot{\theta})\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

Summary of the varying base-vector method

In the varying base vector method, we calculate the velocity of a point by looking at the position as the sum of two position vectors, one of which is expressed in the moving base vectors. We then differentiate the position, taking account that the base vectors of the moving frame change with time. In general

$$\begin{aligned}
\vec{\boldsymbol{v}}_P &= \frac{d}{dt}\vec{\boldsymbol{r}}_P \\
&= \frac{d}{dt}\left[\vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'}\right] \\
&= \frac{d}{dt}\left[(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) + (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' + z'\hat{\boldsymbol{k}}')\right] \\
&= (\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}) + (\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + \dot{z}'\hat{\boldsymbol{k}}') + \\
&\quad \left[x'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}') + z'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{k}}')\right]
\end{aligned}$$

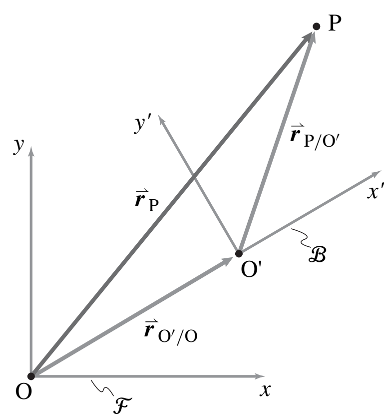


Figure 18.25: The position of a point P relative to the origin of a fixed frame $\mathcal{F}$ is represented as the sum of two vectors: the position of the new origin relative to the old, and the position of P relative to the new origin. (Here we use "new" as an informal synonym for "moving frame".)

We could calculate $\vec{\boldsymbol{a}}_P$ similarly using a combination of the product rule of differentiation and the facts that $\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}'$, $\dot{\hat{\boldsymbol{\jmath}}}' = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}'$, and $\dot{\hat{\boldsymbol{k}}}' = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{k}}'$, and would get a formula with 15 non-zero terms.

18.1 The $\dot{\vec{Q}}$ formula

We think about some vector $\vec{Q}$ as a quantity that could be represented by an arrow. We can write $\vec{Q}$ using the coordinates of the fixed Newtonian frame with base vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$: $\vec{Q} = Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k}$. Similarly we could write $\vec{Q}$ in terms of the coordinates of some moving and rotating frame $\mathcal{B}$ with base vectors $\hat{\imath}', \hat{\jmath}', \hat{k}'$: $\vec{Q} = Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'$. Now of course

$$\begin{aligned} \vec{Q} &= \vec{Q} \\ \text{so} \quad Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k} &= Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'. \end{aligned}$$

Similarly, $\dot{\vec{Q}} = \dot{\vec{Q}}$ so long as what we mean by $\dot{\vec{Q}}$ is its derivative in a fixed frame. That is, we use $\dot{\vec{Q}}$ as an informal notation for ${}^{\mathcal{F}}\dot{\vec{Q}} = \frac{{}^{\mathcal{F}}d\vec{Q}}{dt}$. We can calculate $\dot{\vec{Q}}$ the same way we have from the start of the book, namely,

$$\dot{\vec{Q}} = \dot{Q}_x\hat{\imath} + \dot{Q}_y\hat{\jmath} + \dot{Q}_z\hat{k}.$$

We didn't have to use the product rule of differentiation because the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, associated with a fixed frame, are constant in time.

What if we wanted to use the coordinate information that was given to us by a person who was moving and rotating with the moving frame $\mathcal{B}$? Now we calculate $\dot{\vec{Q}}$ taking account that the $\mathcal{B}$ base vectors change in time.

$$\begin{aligned} \dot{\vec{Q}} &= \frac{d}{dt}[Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'] \\ &= \underbrace{[\dot{Q}_{x'}\hat{\imath}' + \dot{Q}_{y'}\hat{\jmath}' + \dot{Q}_{z'}\hat{k}']}_{{}^{\mathcal{B}}\dot{\vec{Q}}} \\ &\quad + \underbrace{[Q_{x'}\dot{\hat{\imath}}' + Q_{y'}\dot{\hat{\jmath}}' + Q_{z'}\dot{\hat{k}}']}_{?}. \end{aligned} \tag{18.22}$$

The first term in the product rule is just the derivative of $\vec{Q}$ in the moving frame ${}^{\mathcal{B}}\dot{\vec{Q}}$. That is, ${}^{\mathcal{B}}\dot{\vec{Q}}$ is calculated by differentiating the components in $\mathcal{B}$ holding the base vectors in $\mathcal{B}$ fixed. The second term depends on evaluating $\dot{\hat{\imath}}'$, $\dot{\hat{\jmath}}'$, and $\dot{\hat{k}}'$. We know (at least for 2-D and for fixed-axis rotation in 3-D) that

$$\begin{aligned} \dot{\hat{\imath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\imath}', \\ \dot{\hat{\jmath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\jmath}', \quad \text{and} \\ \dot{\hat{k}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{k}'. \end{aligned} \tag{18.23}$$

Eqns. 18.23 are the core of rigid body kinematics.

Now we can go back to the second group of terms in Eqn. 18.22.

$$\begin{aligned} ? &= [Q_{x'}\dot{\hat{\imath}}' + Q_{y'}\dot{\hat{\jmath}}' + Q_{z'}\dot{\hat{k}}'] \\ &= [Q_{x'}\vec{\omega}_{\mathcal{B}} \times \hat{\imath}' + Q_{y'}\vec{\omega}_{\mathcal{B}} \times \hat{\jmath}' + Q_{z'}\vec{\omega}_{\mathcal{B}} \times \hat{k}'] \\ &= \vec{\omega}_{\mathcal{B}} \times [Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'] \\ &= \vec{\omega}_{\mathcal{B}} \times \vec{Q} \end{aligned}$$

Going back to Eqn. 18.22 we get the desired result:

$${}^{\mathcal{F}}\dot{\vec{Q}} \overset{\mathcal{B}}{=} \dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}. \tag{18.24}$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}. \tag{18.25}$$

Geometric 'derivation' of the $\dot{\vec{Q}}$ formula

Here is a geometrical 'derivation' of the $\dot{\vec{Q}}$ formula in two dimensions. Referring to the figure at right, we look at a vector $\vec{Q}$ at two successive times. We then look at how $\vec{Q}$ seems to change in a frame that rotates slightly as $\vec{Q}$ changes. The picture shows how to account for the difference between the change of $\vec{Q}$ as perceived by the two different frames.

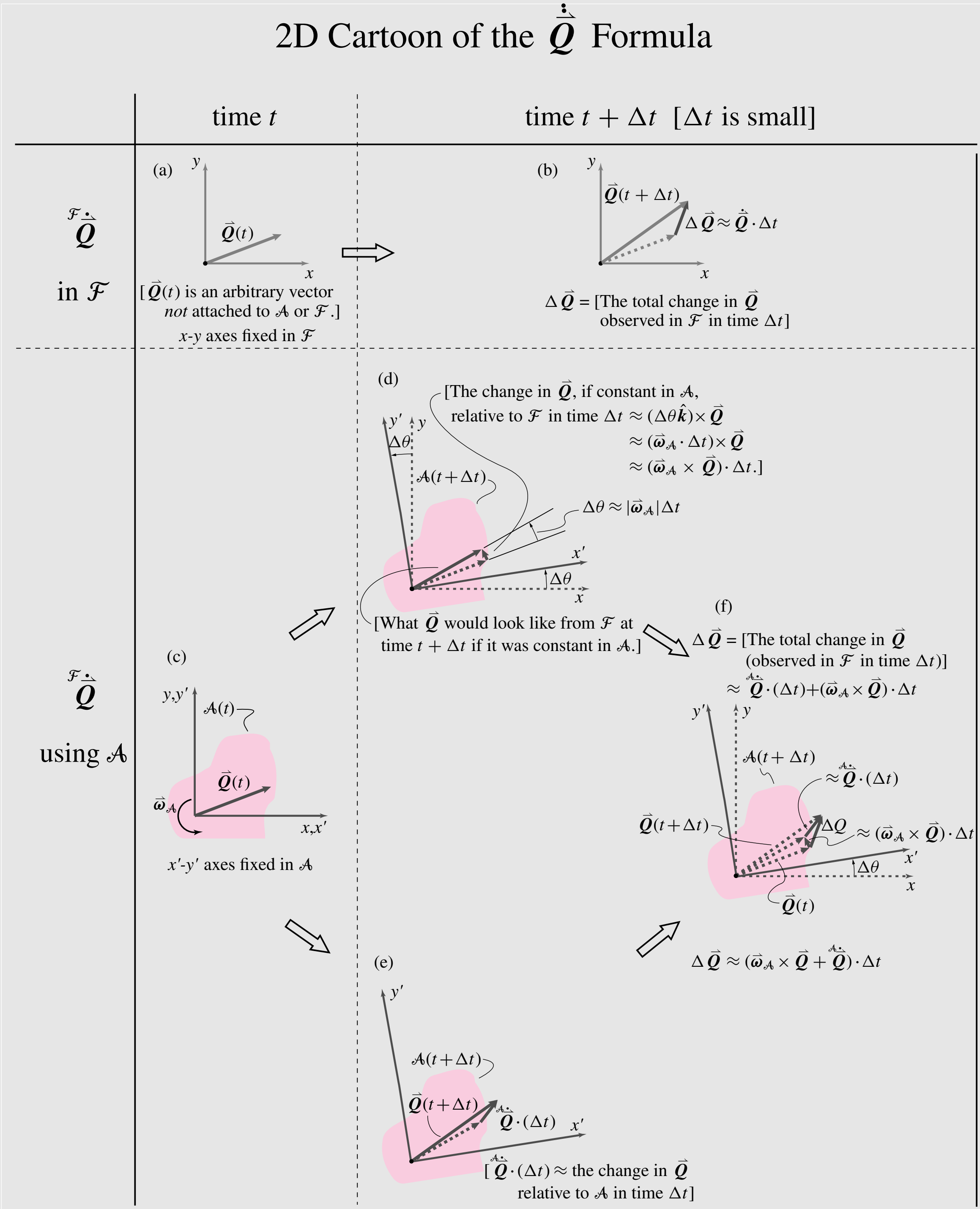
In detail the parts (a) to (e) of the picture show the following.

- Part (a) shows a vector $\vec{Q}$ at time t.
- Part (b) shows $\vec{Q}$ at time $t + \Delta t$ *and* the change in $\vec{Q}$, $\Delta\vec{Q} \approx \dot{\vec{Q}} \cdot \Delta t$.
- Part (c) is like (a) but shows a moving body or frame $\mathcal{A}$.
- Part (d) shows the change in $\vec{Q}$, $(\vec{\omega}_{\mathcal{A}} \times \vec{Q}) \cdot \Delta t$, that would occur if $\vec{Q}$ were fixed (constant) in $\mathcal{A}$.
- Part (e) shows the change in $\vec{Q}$ that would be observed in the moving frame $\mathcal{A}$.
- Part (f) shows the net change in $\vec{Q}$, $\Delta\vec{Q}$, that is the same as that in (b) above; here, it is shown as the sum of the two contributions from (d) and (e).

Thus, using $\mathcal{A}$, $\Delta\vec{Q}$ for small Δt is composed of two parts: (1) the $\Delta\vec{Q}$ observed in $\mathcal{A}(t)$, and (2) the change in $\vec{Q}$ which would occur if $\vec{Q}$ were constant in $\mathcal{A}(t)$ and thus rotating with it. Dividing $\Delta\vec{Q}$ by Δt gives the '$\dot{\vec{Q}}$ formula', $\dot{\vec{Q}} = {}^{\mathcal{A}}\dot{\vec{Q}} + \vec{\omega}_{\mathcal{A}} \times \vec{Q}$.

(continued...)

18.1 The $\dot{\vec{Q}}$ formula (continued)



Two different looks at the change in the vector $\vec{Q}$, $\Delta\vec{Q}$, over a time interval Δt.