SAMPLE 18.5 Acceleration of a point moving in a rotating frame. It is given that the arm OAB rotates with counterclockwise angular acceleration $\dot{\omega} = 3\,\text{rad/s}^2$ and at the instant shown the angular speed $\omega = 5\,\text{rad/s}$. Also, at the same instant, the particle P is sliding (roughly) down with speed $v_{/\text{tube}} = 4\,\text{ft/s}$ and acceleration $a_{/\text{tube}} = 2\,\text{ft/s}^2$. Find the absolute acceleration of the particle at the given instant. Take $\ell = 2\,\text{ft}$ in the figure.

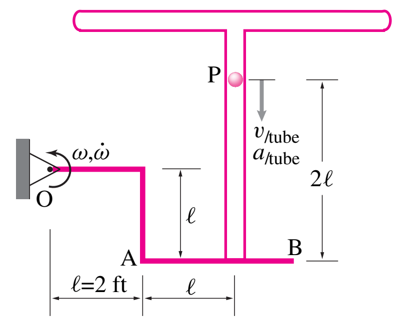

Figure 18.26

Solution Let us attach a body frame $\mathcal{B}$ to the rigid arm OAB. For calculations we fix a coordinate system $x'y'z'$ in this frame such that the origin O' of the coordinate system coincides with O, and at the given instant, the axes are aligned with the inertial coordinate axes xyz. Since $x'y'z'$ is fixed in the frame $\mathcal{B}$ and $\mathcal{B}$ rotates with the rigid arm with $\dot{\vec{\omega}}_{\mathcal{B}} = 3\,\text{rad/s}^2\hat{\boldsymbol{k}}$ and $\vec{\omega}_{\mathcal{B}} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$, the basis vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$ and $\hat{\boldsymbol{k}}'$ rotate with the same $\dot{\vec{\omega}}_{\mathcal{B}}$ and $\vec{\omega}_{\mathcal{B}}$.

In the rotating (primed) coordinate system,

$$
\begin{aligned}
\vec{\boldsymbol{r}}_P &= x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' \\
\vec{\boldsymbol{v}}_P &= \frac{d}{dt}(\vec{\boldsymbol{r}}_P) = \frac{d}{dt}(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') \\
&= \dot{x}'\hat{\boldsymbol{\imath}}' + x'\dot{\hat{\boldsymbol{\imath}}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + y'\dot{\hat{\boldsymbol{\jmath}}}'.
\end{aligned}
$$

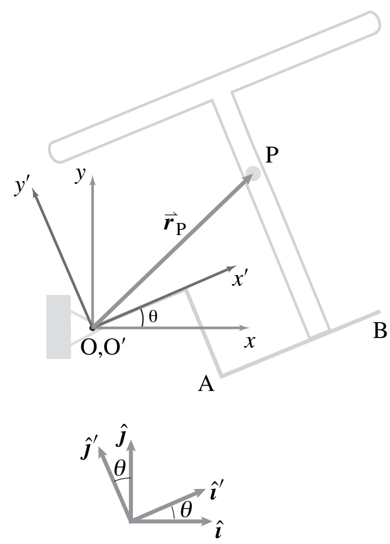

Figure 18.27

Now, we use the $\dot{\vec{\boldsymbol{Q}}}$ formula to evaluate $\dot{\hat{\boldsymbol{\imath}}}'$ and $\dot{\hat{\boldsymbol{\jmath}}}'$, *i.e.,*

$$
\begin{aligned}
\dot{\hat{\boldsymbol{\imath}}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}' = \omega\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\imath}}' = \omega\hat{\boldsymbol{\jmath}}' \\
\dot{\hat{\boldsymbol{\jmath}}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}' = \omega\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\jmath}}' = -\omega\hat{\boldsymbol{\imath}}'.
\end{aligned}
$$

Also, note that x' is constant since in frame $\mathcal{B}$, the motion of the particle is always along the tube, *i.e.,* along the negative y' axis (see Fig. 18.27). Thus, $x' = 2\ell$, $\dot{x}' = 0$, $y' = \ell$, and $\dot{y}' = -v_{/\text{tube}}$. Substituting these quantities in $\vec{\boldsymbol{v}}_P$, we get:

$$
\begin{aligned}
\vec{\boldsymbol{v}}_P &= x'\omega\hat{\boldsymbol{\jmath}}' - v_{/\text{tube}}\hat{\boldsymbol{\jmath}}' + y'(-\omega\hat{\boldsymbol{\imath}}') \\
&= (x'\omega - v_{/\text{tube}})\hat{\boldsymbol{\jmath}}' - \omega y'\hat{\boldsymbol{\imath}}'.
\end{aligned} \tag{18.26}
$$

Now substituting $x' = 2\ell = 4\,\text{ft}$, $\omega = 5\,\text{rad/s}$, $y' = \ell = 2\,\text{ft}$, $v_{/\text{tube}} = 4\,\text{ft/s}$ and noting that $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ at the given instant, we get:

$$
\vec{\boldsymbol{v}}_P = [(20-4)\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{\imath}}]\,\text{ft/s} = (-10\hat{\boldsymbol{\imath}} + 16\hat{\boldsymbol{\jmath}})\,\text{ft/s}.
$$

We can find $\vec{\boldsymbol{a}}_P$ by differentiating Eq. (18.26) and noting again that $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ at the given instant(1):

$$
\begin{aligned}
\vec{\boldsymbol{a}}_P &= \frac{d}{dt}(\vec{\boldsymbol{v}}_P) = \frac{d}{dt}[(x'\omega - v_{/\text{tube}})\hat{\boldsymbol{\jmath}}' - \omega y'\hat{\boldsymbol{\imath}}'] \\
&= (\underbrace{\dot{x}'}_{0}\omega + x'\dot{\omega} - \overbrace{\dot{v}_{/\text{tube}}}^{a_{/\text{tube}}})\hat{\boldsymbol{\jmath}}' + (x'\omega - v_{/\text{tube}})\dot{\hat{\boldsymbol{\jmath}}}' - (\dot{\omega}y' + \omega\overbrace{\dot{y}'}^{-v_{/\text{tube}}})\hat{\boldsymbol{\imath}}' - \omega y'\dot{\hat{\boldsymbol{\imath}}}' \\
&= (x'\dot{\omega} - a_{/\text{tube}})\hat{\boldsymbol{\jmath}}' + (x'\omega - v_{/\text{tube}})(-\omega\hat{\boldsymbol{\imath}}') - (\dot{\omega}y' - \omega v_{/\text{tube}})\hat{\boldsymbol{\imath}}' - \omega y'(\omega\hat{\boldsymbol{\jmath}}') \\
&= -a_{/\text{tube}}\hat{\boldsymbol{\jmath}}' + 2\omega v_{/\text{tube}}\hat{\boldsymbol{\imath}}' - \omega^2(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') + \dot{\omega}(x'\hat{\boldsymbol{\jmath}}' - y'\hat{\boldsymbol{\imath}}') \\
&= -2\,\text{ft/s}^2\hat{\boldsymbol{\jmath}} + 40\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 25(4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2 + 3(4\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{\imath}})\,\text{ft/s}^2 \\
&= -(66\hat{\boldsymbol{\imath}} + 40\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{a}}_P = -(66\hat{\boldsymbol{\imath}} + 40\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2}$$

(1) We will revisit this problem in Sample 18.9. Here we use the $\dot{\vec{\boldsymbol{Q}}}$ formula on various base vectors. In Sample 18.9 we will use the relative velocity and acceleration formulae.

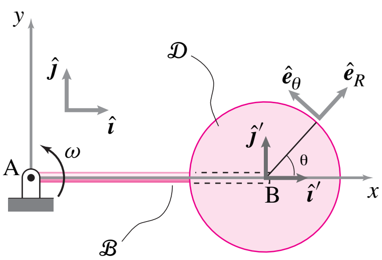

Figure 18.28

SAMPLE 18.6 Rate of change of unit vectors. A circular disk $\mathcal{D}$ is welded to a rigid rod AB. The rod rotates about point A with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$. A frame $\mathcal{B}$ is attached to the disk and therefore rotates with the same $\vec{\omega}$. Two coordinate systems, $(\hat{\imath}', \hat{\jmath}')$ and $(\hat{e}_R, \hat{e}_\theta)$ are fixed in frame $\mathcal{B}$ as shown in the figure.

1. Find the rate of change of unit vectors $\hat{e}_R$, $\hat{e}_\theta$, $\hat{\imath}'$ and $\hat{\jmath}'$ using the $\dot{\vec{Q}}$ formula.
2. Express the $\hat{e}_R$ and $\hat{e}_\theta$ vectors in terms of $\hat{\imath}'$ and $\hat{\jmath}'$ and verify the results obtained above for $\dot{\hat{e}}_R$ and $\dot{\hat{e}}_\theta$ by direct differentiation.

Solution Since the disk is welded to the rod and frame $\mathcal{B}$ is fixed in the disk, the frame rotates with $\vec{\omega}_\mathcal{B} = \omega\hat{\mathbf{k}}$.

1. To find the rate of change of the unit vectors using the $\dot{\vec{Q}}$ formula, we substitute the desired unit vector in place of $\vec{Q}$ in the formula (eqn. (18.19)). For example,

$$\dot{\hat{e}}_R = {}^{\mathcal{B}}\dot{\hat{e}}_R + \vec{\omega}_\mathcal{B} \times \hat{e}_R.$$

It should be clear that ${}^{\mathcal{B}}\dot{\hat{e}}_R = 0$, since $\hat{e}_R$ does not change with respect to an observer sitting in frame $\mathcal{B}$. Therefore,

$$\dot{\hat{e}}_R = \vec{\omega}_\mathcal{B} \times \hat{e}_R = \omega\hat{\mathbf{k}} \times \hat{e}_R = \omega\hat{e}_\theta.$$

Similarly,

$$\begin{aligned}
\dot{\hat{e}}_\theta &= \underbrace{{}^{\mathcal{B}}\dot{\hat{e}}_\theta}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{e}_\theta = \omega\overbrace{\hat{\mathbf{k}} \times \hat{e}_\theta}^{-\hat{e}_R} = -\omega\hat{e}_R.\\
\dot{\hat{\imath}}' &= \underbrace{{}^{\mathcal{B}}\dot{\hat{\imath}}'}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{\imath}' = \omega\hat{\mathbf{k}} \times \hat{\imath}' = \omega\hat{\jmath}'.\\
\dot{\hat{\jmath}}' &= \underbrace{{}^{\mathcal{B}}\dot{\hat{\jmath}}'}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{\jmath}' = \omega\hat{\mathbf{k}} \times \hat{\jmath}' = -\omega\hat{\imath}'.
\end{aligned}$$

$$\boxed{\dot{\hat{e}}_R = \omega\hat{e}_\theta, \quad \dot{\hat{e}}_\theta = -\omega\hat{e}_R, \quad \dot{\hat{\imath}}' = \omega\hat{\jmath}', \text{ and } \dot{\hat{\jmath}}' = -\omega\hat{\imath}'}$$

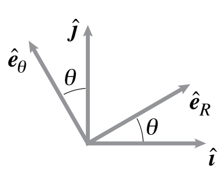

Figure 18.29

2. Since $\hat{e}_R = \cos\theta\hat{\imath}' + \sin\theta\hat{\jmath}'$ and $\hat{e}_\theta = -\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}'$, we get their rates of change by direct differentiation as

$$\begin{aligned}
\dot{\hat{e}}_R &= \cos\theta\dot{\hat{\imath}}' + \sin\theta\dot{\hat{\jmath}}'\\
&= \cos\theta(\omega\hat{\jmath}') + \sin\theta(-\omega\hat{\imath}')\\
&= \omega(-\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}') = \omega\hat{e}_\theta,\\
\dot{\hat{e}}_\theta &= -\sin\theta\dot{\hat{\imath}}' + \cos\theta\dot{\hat{\jmath}}'\\
&= -\sin\theta(\omega\hat{\jmath}') + \cos\theta(-\omega\hat{\imath}')\\
&= -\omega(\cos\theta\hat{\imath}' + \sin\theta\hat{\jmath}') = -\omega\hat{e}_R.
\end{aligned}$$

Here we have used the fact that θ, the angle between the unit vectors $\hat{e}_R$ and $\hat{\imath}'$, remains constant during the motion. The results obtained are the same as in part (a).

SAMPLE 18.7 **Rate of change of a position vector.** A rigid rod OAB rotates counterclockwise about point O with constant angular speed $\omega = 5\,\mathrm{rad/s}$. A collar C slides out on the bent arm AB with constant speed $v = 0.5\,\mathrm{m/s}$ with respect to the arm. Find the velocity of the collar using the $\dot{\vec{Q}}$ formula.

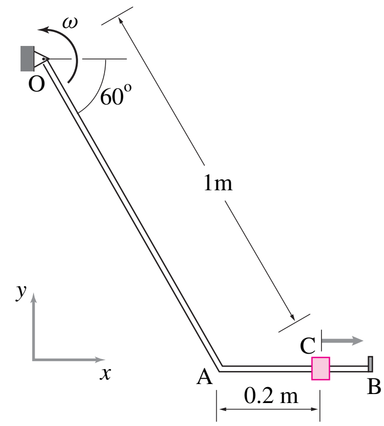

Figure 18.30

Solution Let $\vec{r}_C$ be the position vector of the collar. Then the velocity of the collar is $\dot{\vec{r}}_C$. Let the rod OAB be the rotating frame $\mathcal{B}$. Now we can find $\dot{\vec{r}}_C$ using the $\dot{\vec{Q}}$ formula:

$$\dot{\vec{r}}_C = {}^{\mathcal{B}}\dot{\vec{r}}_C + \vec{\omega}_{\mathcal{B}} \times \vec{r}_C$$

To compute $\dot{\vec{r}}_C$, let us first find ${}^{\mathcal{B}}\dot{\vec{r}}_C$, the rate of change of $\vec{r}_C$ as seen in frame $\mathcal{B}$ (this term represents the velocity of the collar you see if you sit on the rod and watch the collar; also called $\vec{v}_{rel}$).

$$\begin{aligned} \vec{r}_C &= \vec{r}_A + \vec{r}_{C/A} \\ {}^{\mathcal{B}}\dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_A + {}^{\mathcal{B}}\dot{\vec{r}}_{C/A} \end{aligned}$$

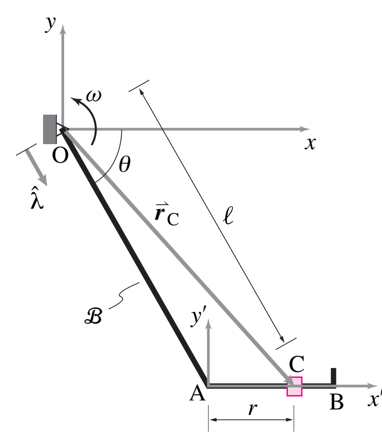

Figure 18.31

Note that the vector $\vec{r}_A = \ell\hat{\boldsymbol{\lambda}}$ does not change in frame $\mathcal{B}$ since both its magnitude, ℓ, and direction, $\hat{\boldsymbol{\lambda}}$, remain fixed in $\mathcal{B}$. Therefore,

$${}^{\mathcal{B}}\dot{\vec{r}}_A = 0$$

$$\text{Now} \quad \vec{r}_{C/A} = r\hat{\imath}'. \quad \Rightarrow \quad {}^{\mathcal{B}}\dot{\vec{r}}_{C/A} = \dot{r}\hat{\imath}' = 0.5\,\mathrm{m/s}\hat{\imath}'$$

because $\hat{\imath}'$ does not change in $\mathcal{B}$ and $\dot{r}$ = speed of the collar with respect to the arm (see Figure 18.31). Thus,

$${}^{\mathcal{B}}\dot{\vec{r}}_C = 0.5\,\mathrm{m/s}\hat{\imath}'.$$

Hence,

$$\begin{aligned} \dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega\hat{\boldsymbol{k}} \times (\ell\hat{\boldsymbol{\lambda}} + r\hat{\imath}') \\ &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega\ell(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}) + \omega r(\hat{\boldsymbol{k}} \times \hat{\imath}') \\ &= \dot{r}\hat{\imath}' + \omega\ell(\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) + \omega r\hat{\jmath}' \\ &= 0.5\,\mathrm{m/s}\hat{\imath}' + 5\,\mathrm{m/s}(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}) + 1\,\mathrm{m/s}\hat{\jmath}' \\ &= 4.83\,\mathrm{m/s}\hat{\imath} + 3.5\,\mathrm{m/s}\hat{\jmath} \end{aligned}$$

where we have used the fact that at the given instant, $\hat{\imath}' = \hat{\imath}$ and $\hat{\jmath}' = \hat{\jmath}$.

$$\vec{v}_C = 4.83\,\mathrm{m/s}\hat{\imath} + 3.5\,\mathrm{m/s}\hat{\jmath}$$