18.1 Polar coordinates and path coordinates

As you learned in Chapter 15, when a particle moves in a plane while going in circles around the origin its position, velocity, and acceleration can be described like this:

$$\begin{aligned}
\vec{\boldsymbol{r}} &= R\hat{\boldsymbol{e}}_R \\
\vec{\boldsymbol{v}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta = v_\theta\hat{\boldsymbol{e}}_\theta \\
\vec{\boldsymbol{a}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta = -\frac{v_\theta^2}{R}\hat{\boldsymbol{e}}_R + \dot{v}_\theta\hat{\boldsymbol{e}}_\theta.
\end{aligned}$$

These three equations say that the position is the distance from the origin times a unit vector towards the point; that the velocity is tangent to the circle of motion; and that the acceleration has a centripetal component proportional to the speed squared, and a tangential component, tangent to the circle of motion with magnitude equal to the rate of change of speed.

We will now generalize these results two different ways.

- First we will use polar base vectors for non-constant R.
- Next we will use path base vectors to show that, in a sense to be explained, the 2nd and 3rd formulas above (for $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$) apply to any wild motion in 2D or 3D.

As mentioned, in principle these new methods are not needed. We could just use one fixed coordinate system with base vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ and write the velocity and acceleration of a point at position $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ as

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \quad \text{and} \quad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}$$

as in Chapters 9-12. But, as for the circular motion of Chapter 15, rotating base vectors are helpful for simplifying some kinematics and mechanics problems.

Polar coordinates for general (non-circular) motion

The extension of *polar coordinates* to 3 dimensions as *cylindrical coordinates* is shown in fig. 18.1.

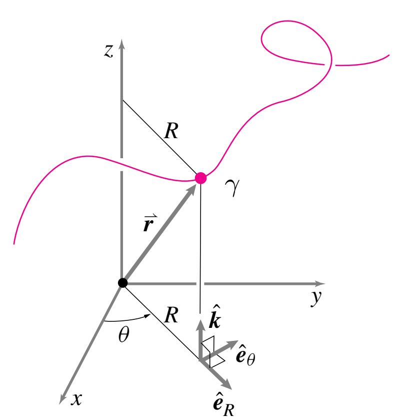


Figure 18.1: Polar coordinates.

Rather than identifying the location of a point by its x, y and z coordinates, a point is located by its cylindrical coordinates

R, the distance to the point from the z axis,

θ, the angle that the most direct line from the z axis to the point makes with the positive x direction,

z, the conventional z coordinate of the particle,

and base vectors:

$\hat{\boldsymbol{e}}_R$, a unit vector normal to the z axis that points from the z axis to the particle
(in 2-D $\hat{\boldsymbol{e}}_R = \vec{\boldsymbol{r}}/r$,
in 3-D $\hat{\boldsymbol{e}}_R = (\vec{\boldsymbol{r}} - (\vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}})/|\vec{\boldsymbol{r}} - (\vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}}|$
$=$ a unit vector in the direction of the shadow of $\vec{\boldsymbol{r}}$ in the xy plane)

$\hat{\boldsymbol{e}}_\theta$, a vector in the xy plane normal to $\hat{\boldsymbol{e}}_R$ (Formally $\hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_R$),

$\hat{\boldsymbol{k}}$, the conventional $\hat{\boldsymbol{k}}$ base vector.

The position vector of a particle is

$$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}.$$

The z component of position, velocity and acceleration is the same in cylindrical coordinates as in Cartesian coordinates. If only two-dimensional problems are being considered then

$$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R.$$

As the particle moves, the values of its coordinates R, θ and z change as do the base vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$.

Example: Oblong path
A particle that moves on the oblong path $R = A + B\cos(2\theta)$ with $A > B$ is shown in fig. 18.2. The position vector is

$$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R = (A + B\cos(2\theta))\hat{\boldsymbol{e}}_R.$$

Note that, unlike for circular motion, $\hat{\boldsymbol{e}}_\theta$ is *not* tangent to the particle's path in general. The base vector $\hat{\boldsymbol{e}}_\theta$ is only tangent to the path at those points where the path is closest to, or furthest from, the origin (which, in this example, are also the points where the path crosses the x and y axes).

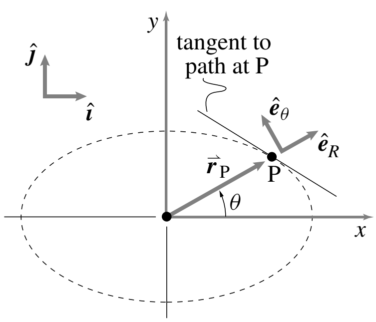


Figure 18.2: Particle P on an oblong path. Note that the velocity is not parallel to $\hat{\boldsymbol{e}}_\theta$ at most points on the path.

Velocity in polar coordinates

The velocity and acceleration are found by differentiating the position $\vec{\boldsymbol{r}}$, taking account that the base vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ also change with time just as they did for circular motion:

$$\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta \quad \text{and} \quad \dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R.$$

We find the velocity by taking the time derivative of the position, using the product rule of differentiation:

$$
\begin{aligned}
\vec{\boldsymbol{v}} = \frac{d}{dt}\vec{\boldsymbol{r}} &= \frac{d}{dt}[R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}] \\
&= \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) + \frac{d}{dt}(z\hat{\boldsymbol{k}}) \\
&= (\dot{R}\hat{\boldsymbol{e}}_R + R\underbrace{\dot{\hat{\boldsymbol{e}}}_R}_{\dot{\theta}\hat{\boldsymbol{e}}_\theta}) + (\dot{z}\hat{\boldsymbol{k}}) \\
&= \underbrace{\dot{R}}_{v_R}\hat{\boldsymbol{e}}_R + \underbrace{R\dot{\theta}}_{v_\theta}\hat{\boldsymbol{e}}_\theta + \underbrace{\dot{z}}_{v_z}\hat{\boldsymbol{k}}. \qquad (18.1)
\end{aligned}
$$

This formula is intuitive. The velocity is the sum of three vectors: one, $\dot{R}\hat{\boldsymbol{e}}_R$, due to moving towards or away from the z axis; one, $\dot{\theta}R\hat{\boldsymbol{e}}_\theta$, having to do with the angle being swept; and in 3-D, one, $\dot{z}\hat{\boldsymbol{k}}$, for motion perpendicular to the xy plane. In 2-D this is shown in fig. 18.3.

But, as has been emphasized before, this isn't a new vector $\vec{\boldsymbol{v}}$ but just a new way of representing the same vector:

$$
\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{v}} \\
v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}} &= v_R\hat{\boldsymbol{e}}_R + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{k}}.
\end{aligned}
$$

The vector $\vec{\boldsymbol{v}}$ can be represented in different base vector systems, in this case cartesian and polar.

Note that eqn. (18.1) adds two terms to the circular motion case from Chapter 15: one for variable R and one for variable z.

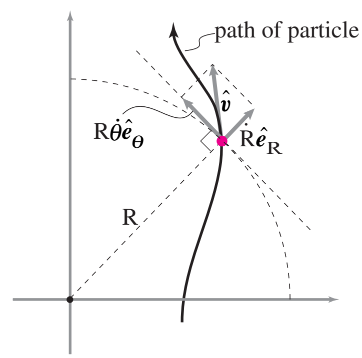


Figure 18.3: Velocity in polar coordinates for general planar motion. The velocity has a radial $\hat{\boldsymbol{e}}_R$ component $\dot{R}$ and a circumferential $\hat{\boldsymbol{e}}_\theta$ component $R\dot{\theta}$.

Acceleration in polar coordinates

To find the acceleration, we differentiate once again. The resulting formula has new terms generated by the product rule of differentiation.

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= \frac{d}{dt}\vec{\boldsymbol{v}} \\
&= \frac{d}{dt}(\dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}) \\
&= (\ddot{R}\hat{\boldsymbol{e}}_R + \dot{R}\underbrace{\dot{\hat{\boldsymbol{e}}}_R}_{\dot{\theta}\hat{\boldsymbol{e}}_\theta}) + (\dot{R}\dot{\theta}\hat{\boldsymbol{e}}_\theta + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta + R\dot{\theta}\underbrace{\dot{\hat{\boldsymbol{e}}}_\theta}_{-\dot{\theta}\hat{\boldsymbol{e}}_R}) + (\ddot{z}\hat{\boldsymbol{k}}) \\
&= \underbrace{(\ddot{R} - \dot{\theta}^2 R)}_{a_R}\hat{\boldsymbol{e}}_R + \underbrace{(2\dot{R}\dot{\theta} + R\ddot{\theta})}_{a_\theta}\hat{\boldsymbol{e}}_\theta + \underbrace{\ddot{z}}_{a_z}\hat{\boldsymbol{k}}. \qquad (18.2)
\end{aligned}
$$

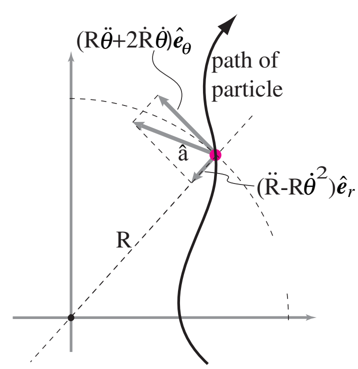


Figure 18.4: Acceleration in polar coordinates for general planar motion. The acceleration has a radial $\hat{\boldsymbol{e}}_R$ component $\ddot{R} - R\dot{\theta}^2$ and a circumferential $\hat{\boldsymbol{e}}_\theta$ component $R\ddot{\theta} + 2\dot{R}\dot{\theta}$.

The acceleration for an arbitrary planar path is shown in fig. 18.4. Four of the five terms comprising the polar coordinate formula for acceleration are easy to understand.

$\ddot{R}$ is just the acceleration due to the distance from the origin changing with time.

$\dot{\theta}^2 R$ is the familiar *centripetal* acceleration.

$R\ddot{\theta}$ is the acceleration due to rotation proceeding at a faster and faster rate. And

$\ddot{z}$ is the same as for Cartesian coordinates.

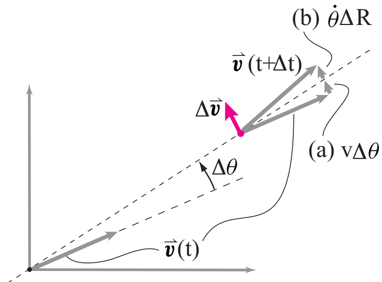


Figure 18.5: The simplest case for understanding of the *Coriolis* term in the acceleration. A particle moves at constant speed $\dot{R}$ out on a line which is rotating at constant rate $\dot{\theta}$. The figure shows a velocity vector as an arrow when the particle is at the origin and a time Δt later. The change in the velocity vector comes from two effects: (a) the radial line has rotated, and (b) a circumferential component is added to the velocity when the particle is away from the origin. For small $\Delta\theta$ these two changes to the velocity are both approximately perpendicular to $\vec{\boldsymbol{v}}$ and parallel to each other.

The Coriolis term. The difficult term in the polar coordinate expression for the acceleration is the

$2\dot{\theta}\dot{R}$ term, called the *Coriolis* acceleration, after the civil engineer Gustave-Gaspard Coriolis who first wrote about it in 1835 (in a slightly more general context).

The presence of the '2' in this term is due to the two effects from which it derives: 1 from the change of the $\dot{R}\hat{\boldsymbol{e}}_R$ term in the velocity and 1 from the change of the $R\dot{\theta}\hat{\boldsymbol{e}}_\theta$ term in the velocity $(1 + 1 = 2)$.

The Coriolis acceleration occurs even if both $\dot{\theta}$ and $\dot{R}$ are constant. That is, a particle that moves at constant speed ($\dot{R} =$ constant) on a straight line that is itself rotating at constant rate ($\dot{\theta}$ = constant) does not have a straight-line path, and thus has some acceleration. The Coriolis term catches this acceleration. Here is a situation in which the Coriolis term is the only non-zero term in the general polar-coordinate acceleration formula (eqn. (18.2)).

Example: The simplest Coriolis example

Consider, like above, a particle moving at constant speed $\dot{R}$ along a line which is itself rotating at constant $\dot{\theta}$ about the origin. Let's look at a particle as it passes through the origin at time t and a small amount of time Δt later (see fig. 18.5). At time $t + \Delta t$ the direction of the scribed line has changed by an angle $\Delta\theta = \dot{\theta}\Delta t$. So that, even if at that later time $\dot{\theta} = 0$ the direction of $\vec{\boldsymbol{v}}$ has changed so $\vec{\boldsymbol{v}}$ has changed by an amount $v\Delta\theta$. But the rotation of the line does continue, so the velocity includes a part in the $\hat{\boldsymbol{e}}_\theta$ direction with magnitude $\dot{\theta}\Delta R$. That is $\vec{\boldsymbol{v}}$ is changed by both $\Delta\theta v$ and by $\dot{\theta}\Delta R$ so

$$
\begin{aligned}
\Delta\vec{\boldsymbol{v}} &\approx \left(v\Delta\theta + \dot{\theta}\Delta R\right)\hat{\boldsymbol{e}}_\theta \\
&\approx \left(\dot{R}(\dot{\theta}\Delta t) + \dot{\theta}(\dot{R}\Delta t)\right)\hat{\boldsymbol{e}}_\theta \\
&\approx \left(\dot{R}\dot{\theta} + \dot{\theta}\dot{R}\right)\hat{\boldsymbol{e}}_\theta\Delta t \\
\Rightarrow \frac{\Delta\vec{\boldsymbol{v}}}{\Delta t} &\approx \left(\dot{R}\dot{\theta} + \dot{\theta}\dot{R}\right)\hat{\boldsymbol{e}}_\theta \\
\Rightarrow \vec{\boldsymbol{a}} &\approx \left(\dot{R}\dot{\theta} + \dot{\theta}\dot{R}\right)\hat{\boldsymbol{e}}_\theta = 2\dot{R}\dot{\theta}\hat{\boldsymbol{e}}_\theta
\end{aligned}
$$

as predicted by the general polar coordinate acceleration formula.

But one need not get confounded by a desire to understand every term intuitively. Equation (18.2) is a way of describing the same acceleration we have described with cartesian coordinates. Namely,

$$
\begin{aligned}
&\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}} \\
\Rightarrow\quad &\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}} = (\underbrace{\ddot{R} - \dot{\theta}^2 R}_{a_R})\hat{\boldsymbol{e}}_R + (\underbrace{2\dot{R}\dot{\theta} + R\ddot{\theta}}_{a_\theta})\hat{\boldsymbol{e}}_\theta + \underbrace{\ \ddot{z}\ }_{a_z}\hat{\boldsymbol{k}}.
\end{aligned}
$$

Example: $R(t)$ **and** $\theta(t)$ **are given functions.**
Say θ and R vary in time according to

$$\theta = at \quad \text{and} \quad R = ct^2$$

with a and c as given constants. Then at any t the position, velocity, and acceleration are (see fig. 18.6)

$$\begin{aligned}
\vec{r} &= R\hat{e}_R = ct^2\hat{e}_R, \\
\vec{v} &= \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta = 2ct\hat{e}_R + act^2\hat{e}_\theta, \quad \text{and} \\
\vec{a} &= (\ddot{R} - \dot{\theta}^2 R)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta \\
&= \left(2c - a^2ct^2\right)\hat{e}_R + (0 + 4act)\,\hat{e}_\theta
\end{aligned}$$

with polar base vectors

$$\begin{aligned}
\hat{e}_R &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} = \cos(at)\hat{\imath} + \sin(at)\hat{\jmath} \quad \text{and} \\
\hat{e}_\theta &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} = -\sin(at)\hat{\imath} + \cos(at)\hat{\jmath}.
\end{aligned}$$

If we substituted these expressions for the polar base vectors into the expressions for $\vec{r}$, $\vec{v}$, and $\vec{a}$ we would get the same cartesian representation (a giant mess that we don't show) that we would get from using $x = R\cos\theta$ and $y = R\sin\theta$ with $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, and $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$. That is $\vec{r} = \vec{r}$, $\vec{v} = \vec{v}$, and $\vec{a} = \vec{a}$ even if the representation is different.

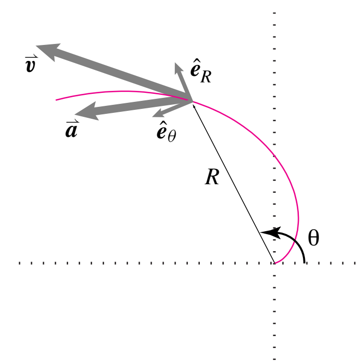


Figure 18.6: A particle moves with $\theta = at$ and $R = ct^2$. In this drawing $a = .5$ and c is anything since no scale is shown.

Path coordinates

Another way still to describe the velocity and acceleration is to use base vectors which are defined by the motion. In particular, the *path* base vectors used are:

1. the unit tangent to the path $\hat{e}_t$, and
2. the unit normal to the path $\hat{e}_n$.

Somewhat surprisingly at first glance, only two base vectors are needed to define the velocity and acceleration, even in three dimensions.

The base vectors can be described geometrically and analytically. Let's start out with a geometric description.

The geometry of the path basis vectors.

As a particle moves through space it traces a path $\vec{r}(t)$. At the instant of interest the path has a unique tangent line. The unit tangent $\hat{e}_t$ is along this line in the direction of motion, as shown in fig. 18.7.

Less clear is that the path has a unique 'kissing' *plane*. One line on this plane is the tangent line. The other line needed to define this plane is determined by the position of the particle just before and just after the time of interest. Just before and just after the time of interest the particle is a little off the tangent line (unless the motion happens to be a straight line and the tangent plane is not uniquely determined). Three points, the position of the particle just before, just at, and just after the instant of interest determine the tangent plane.

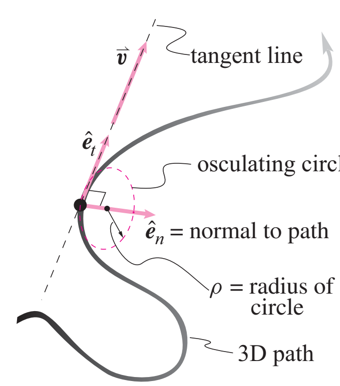


Figure 18.7: The base vectors for path coordinates are tangent and perpendicular to the path. The osculating circle is a circle that is tangent to the path and has the same curvature as the path. The circle is in the plane containing both the tangent to the path and the rate of change of that tangent. The osculating circle is not defined for a straight-line path.

Another way to picture the tangent plane is to find the circle in space that is tangent to the path and which turns at the same rate and in the same direction as the path turns. This circle, which touches the path so intimately, is called the *osculating* or 'kissing' circle. The tangent plane is the plane of this circle. (See fig. 18.7).

The unit normal $\hat{\boldsymbol{e}}_n$ is the unit vector which is perpendicular to the unit tangent and is in the tangent plane. It is pointed in the direction from the edge of the osculating circle towards the center of the circle as shown in fig. 18.7. For 2-D motion in the xy plane the osculating plane is the xy plane and the osculating circle is in the xy plane. The path base vectors are unit vectors that vary along the path, always tangent and normal to the path (see fig. 18.8).

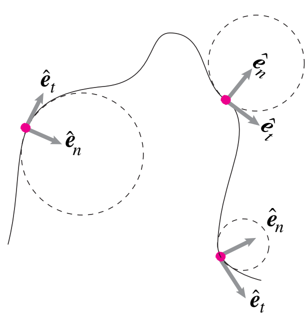


Figure 18.8: A path in the plane with the path base vectors and osculating circle marked at three points.

Formal definition of path basis vectors

The path of a particle $\vec{\boldsymbol{r}}(t)$ can also be parameterized by arc length s along the path, as explained in any introductory calculus text. So the path in space is also $\vec{\boldsymbol{r}}(s)$[1], where the arc length s is the path "coordinate". The unit tangent is:

$$\hat{\boldsymbol{e}}_t \equiv \frac{d\vec{\boldsymbol{r}}(s)}{ds}.$$

Using the chain rule with $\vec{\boldsymbol{r}}(s(t))$ this is also

$$\hat{\boldsymbol{e}}_t = \frac{d\vec{\boldsymbol{r}}(t)}{dt}\frac{dt}{ds} = \frac{\vec{\boldsymbol{v}}}{v}.$$

[1] Mathematicians often do not like this engineering sloppiness of notation. That is, the sequence of points $\vec{\boldsymbol{r}}$ that a particle traverses as both time t and accumulated arc length s increase is not the same mathematical function of both the arguments s and t. For example, when $t = 6$ s the position is not the same as when $s = 6$ cm. Here we don't think it is worth the trouble of writing $\vec{\boldsymbol{r}} = \mathbf{f}(t)$ and $\vec{\boldsymbol{r}} = \mathbf{g}(s)$.

To define the unit normal let's first define the *curvature* $\vec{\boldsymbol{\kappa}}$ of the path as the rate of change of the tangent (rate in terms of arc length).

$$\vec{\boldsymbol{\kappa}} \equiv \frac{d\hat{\boldsymbol{e}}_t}{ds}$$

The unit normal $\hat{\boldsymbol{e}}_n$ is the unit vector in the direction of the curvature

$$\hat{\boldsymbol{e}}_n = \frac{\vec{\boldsymbol{\kappa}}}{|\vec{\boldsymbol{\kappa}}|}.$$

Finally, the binormal $\hat{\boldsymbol{e}}_b$ is the unit vector perpendicular to $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$:

$$\hat{\boldsymbol{e}}_b \equiv \hat{\boldsymbol{e}}_t \times \hat{\boldsymbol{e}}_n.$$

For 2-D motion the binormal $\hat{\boldsymbol{e}}_b$ is always in the $\hat{\boldsymbol{k}}$ direction. The radius of the osculating circle ρ is

$$\rho = \frac{1}{|\vec{\boldsymbol{\kappa}}|}.$$

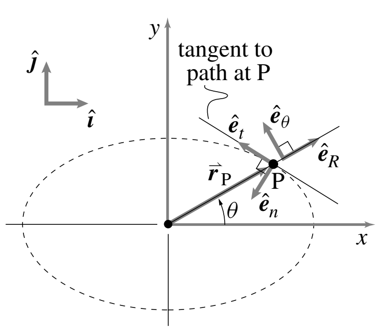


Figure 18.9: Particle P on an elliptical path.

Note that, in general, the polar and path coordinate basis vectors are not parallel; i.e., $\hat{\boldsymbol{e}}_n$ is *not* parallel to $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_t$ is *not* parallel to $\hat{\boldsymbol{e}}_\theta$. For example, consider a particle moving on an elliptical path in the plane shown in fig. 18.9. In this case, the polar coordinate and path coordinate basis vectors are only parallel where the major and minor axes intersect the path.

Velocity and acceleration in path coordinates.

Although it is not necessarily easy to compute the path basis vectors $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$, they lead to simple expressions for the velocity and acceleration:

$$\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t = \frac{ds}{dt}\hat{\boldsymbol{e}}_t \tag{18.3}$$

$$\begin{aligned} \vec{\boldsymbol{a}} &= \frac{d}{dt}\vec{\boldsymbol{v}} = \frac{d}{dt}\left(v\hat{\boldsymbol{e}}_t\right) = \dot{v}\hat{\boldsymbol{e}}_t + v\dot{\hat{\boldsymbol{e}}}_t \\ &= \dot{v}\hat{\boldsymbol{e}}_t + v\frac{d\hat{\boldsymbol{e}}_t}{ds}\frac{ds}{dt} \; = \dot{v}\hat{\boldsymbol{e}}_t + v\vec{\boldsymbol{\kappa}}v \\ &= \underbrace{\dot{v}\hat{\boldsymbol{e}}_t}_{\vec{\boldsymbol{a}}_t} + \underbrace{\frac{v^2}{\rho}\hat{\boldsymbol{e}}_n}_{\vec{\boldsymbol{a}}_n}. \end{aligned} \tag{18.4}$$

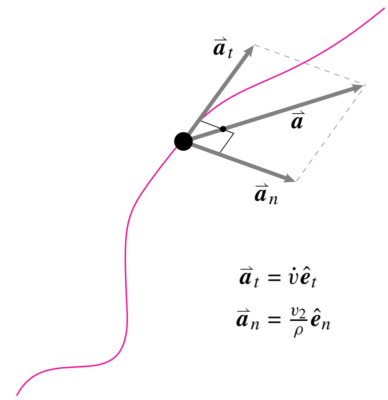


Figure 18.10: The acceleration of any particle can be broken into a part that is parallel to its path and a part perpendicular to its path.

This formula for velocity is intuitive: velocity is speed times a unit vector in the direction of motion. The formula for acceleration is more interesting. It says that the acceleration of any particle at any time is given by the same formula as the formula for acceleration of a particle going around in circles at non-constant rate.

- The first term $\dot{v}\hat{\boldsymbol{e}}_t$ is tangent to the path (also tangent to the osculating circle), as shown in fig. 18.10. If we draw the osculating circle at the point of interest and fix it in space, then this first term is $\rho\ddot{\theta}\hat{\boldsymbol{e}}_\theta$ in a polar coordinate system centered at the center of that fixed circle.
- The term $v^2/\rho\hat{\boldsymbol{n}}$ is directed towards the center of the osculating circle. This term is associated with change of direction and does not vanish even if the speed is constant. This normal acceleration is perpendicular to the path.

That is, the two terms in the path-coordinate acceleration formula correspond exactly to the two terms for acceleration for a particle going in circles.

Example: Estimating the direction of acceleration
By looking at the path of a particle (*e.g.,* see fig. 18.11) and just knowing whether it is speeding up or slowing down one can estimate the direction of the acceleration.

If the particle is known to be speeding up at A then $\dot{v} > 0$ so the tangential acceleration is in the direction of the velocity. Without thinking about the normal acceleration you know that the acceleration vector is pointed in the half plane of directions shown.

If at point B nothing is known about the rate of change of speed, you still know that the acceleration must be in the half plane shown because that is the direction of $\vec{\boldsymbol{\kappa}}$ and $\hat{\boldsymbol{e}}_n$.

If at C you know that the particle is slowing down then you know that $\dot{v} < 0$. But you also can see the curve is to the right so $\hat{\boldsymbol{e}}_n$ is to the right. So the acceleration must be in the quadrant shown.

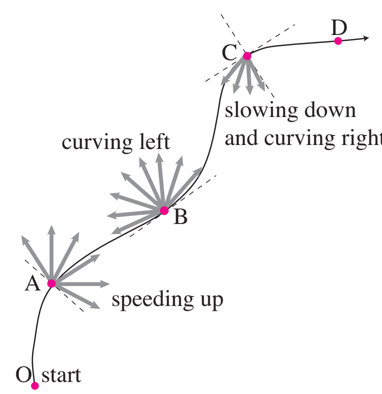


Figure 18.11: A particle moves from 0 to D on the path shown. At A it is speeding up so $\dot{v} > 0$. At B it is turning to the left so $\hat{\boldsymbol{e}}_n$ points to the left. At C it is slowing down and turning to the right. So the acceleration must be a positive vector pointed in the quadrant shown.

One can use the information about the curvature to further restrict the possible accelerations at point A also. At point B there is nothing more to know unless you know how the speed is changing with time.

Earlier we found the curvature by assuming the particle's path was parameterized by arc length s. A second way of calculating the curvature $\vec{\boldsymbol{\kappa}}$ (and then the unit normal $\hat{\boldsymbol{e}}_n$) is to calculate the normal part of the acceleration. First calculate the acceleration. Then subtract from the acceleration that part which is parallel to the velocity.

$$\vec{\boldsymbol{a}}_n = \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t = \vec{\boldsymbol{a}} - \frac{(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{v}}}{v^2}$$

The normal acceleration is $\vec{\boldsymbol{a}}_n = v^2\vec{\boldsymbol{\kappa}}$ so

$$\vec{\boldsymbol{\kappa}} = \frac{\vec{\boldsymbol{a}}_n}{v^2} = \frac{\vec{\boldsymbol{a}}}{v^2} - \frac{(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{v}}}{v^4}.$$

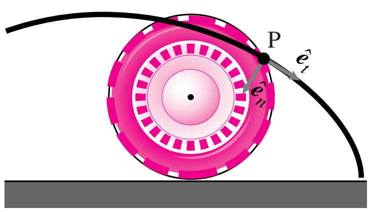
Recipes for path coordinates.

Assume that you know the position as a function of time in either cartesian or polar coordinates. Then, say, at a particular time of interest when the particle is at $\vec{\boldsymbol{r}}$, you can calculate the velocity of the particle using:

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \quad \text{or} \quad \vec{\boldsymbol{v}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}$$

and the acceleration using

$$\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}} \quad \text{or} \quad \vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}.$$



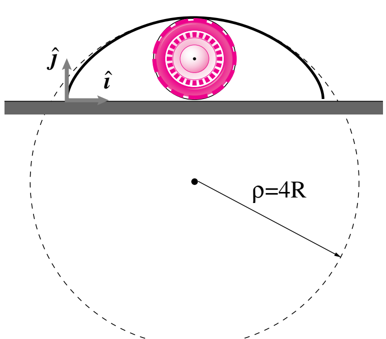


Figure 18.12: The path of a rock in a tire of radius R is shown as are the unit tangent, the unit normal and the osculating circle at the top position.

From these expressions we can calculate all the quantities used in the path coordinate description. So we repeat what we have already said but in an algorithmic form. Here is one set of steps one can follow. This recipe is of little practical use, but does show that the motion explicitly determines the path base vectors as well as the osculating circle.

1. Calculate $\hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$.
2. Calculate $a_t = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}}/v$.
3. Calculate $\vec{\boldsymbol{a}}_n = \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{v}}/v^2$.
4. Calculate $\hat{\boldsymbol{e}}_n = \frac{\vec{\boldsymbol{a}}_n}{|\vec{\boldsymbol{a}}_n|}$
5. Calculate the radius of curvature as $\rho = \frac{|\vec{\boldsymbol{v}}|^2}{|\vec{\boldsymbol{a}}_n|}$.
6. Write a parametric equation for the osculating circle as

$$\vec{\boldsymbol{r}}_{\text{osculating}} = \underbrace{(\vec{\boldsymbol{r}} + \rho\hat{\boldsymbol{e}}_n)}_{\text{center}} + \underbrace{\rho(-\cos\phi\hat{\boldsymbol{e}}_n + \sin\phi\hat{\boldsymbol{e}}_t)}_{\text{circle}}$$

where ϕ is the parameter used to parameterize the points on the circle $\vec{\boldsymbol{r}}_{\text{osculating}}$ of the point on the curve $\vec{\boldsymbol{r}}$. As ϕ ranges from 0 to 2π the point $\vec{\boldsymbol{r}}_{\text{osculating}}$ goes from $\vec{\boldsymbol{r}}$ around the circle and back. The plane of the osculating circle is determined by $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$. For planar curves, the osculating circle is in the plane of the curve.

Example: Particle on the rim of a tire
A particle P on the rim of a tire whose center is moving at constant speed v has position r given by

$$\vec{\boldsymbol{r}} = \left(vt - R\sin(vt/R)\right)\hat{\boldsymbol{\imath}} + R\left(1 - \cos(vt/R)\right)\hat{\boldsymbol{\jmath}}$$

where the origin is at the ground contact time $t = 0$. When the particle is at its highest point $vt = \pi R$ and

$$\vec{\boldsymbol{v}} = 2v\hat{\boldsymbol{\imath}} \quad \text{and} \quad \vec{\boldsymbol{a}} = -(v^2/R)\hat{\boldsymbol{\jmath}}.$$

At that midpoint

$$\begin{aligned} &\hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}, \quad \hat{\boldsymbol{e}}_n = -\hat{\boldsymbol{\jmath}}, \vec{\boldsymbol{\kappa}} = -(1/(4R))\hat{\boldsymbol{\jmath}} \\ \text{and} \quad &(2v)^2/\rho = v^2/R \Rightarrow \rho = 4R \end{aligned}$$

as shown in fig. 18.12. The osculating circle has 4 times the radius of the tire. Note the intimacy of the osculating circle's kiss with the cycloidal path.

Summary of polar(cylindrical) coordinates

See also the inside back cover, table II, row 3 for future reference.

$$\begin{aligned}
\vec{\boldsymbol{r}} &= R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{v}} &= v_R\hat{\boldsymbol{e}}_R + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{k}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{a}} &= a_R\hat{\boldsymbol{e}}_R + a_\theta\hat{\boldsymbol{e}}_\theta + a_z\hat{\boldsymbol{k}} = (\ddot{R} - \dot{\theta}^2 R)\hat{\boldsymbol{e}}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{e}}_R &= (\vec{\boldsymbol{r}} - z\hat{\boldsymbol{k}})/|\vec{\boldsymbol{r}} - z\hat{\boldsymbol{k}}| \\
\hat{\boldsymbol{e}}_\theta &= \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_R
\end{aligned}$$

For 2-D problems just set $z = 0$, $\dot{z} = 0$, and $\ddot{z} = 0$ in these equations.

Summary of path coordinates

See the inside back cover table II, row 4 and the text under the table for future reference:

$$\vec{\boldsymbol{r}} = \text{no simple expression in terms of path base vectors}$$

$$\vec{\boldsymbol{v}} = \dot{s}\hat{\boldsymbol{e}}_t = v\hat{\boldsymbol{e}}_t$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_t + \vec{\boldsymbol{a}}_n$$

$$\vec{\boldsymbol{a}}_t = a_t\hat{\boldsymbol{e}}_t = \dot{v}\hat{\boldsymbol{e}}_t = \ddot{s}\hat{\boldsymbol{e}}_t = (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t$$

$$\vec{\boldsymbol{a}}_n = a_n\hat{\boldsymbol{e}}_n = (v^2/\rho)\hat{\boldsymbol{e}}_n = \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t$$

$$\hat{\boldsymbol{e}}_t = d\vec{\boldsymbol{r}}/ds = \vec{\boldsymbol{v}}/v$$

$$\hat{\boldsymbol{e}}_n = \vec{\boldsymbol{\kappa}}/|\vec{\boldsymbol{\kappa}}| = \rho\vec{\boldsymbol{\kappa}} = (\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t)/|\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t|$$

$$\hat{\boldsymbol{e}}_b = \hat{\boldsymbol{e}}_t \times \hat{\boldsymbol{e}}_n$$

$$\vec{\boldsymbol{\kappa}} = d\hat{\boldsymbol{e}}_t/ds = (\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t)/v^2 \qquad \rho = 1/|\vec{\boldsymbol{\kappa}}|$$

Both polar coordinates and path coordinates define base vectors in terms of the motion of a particle of interest relative to a fixed coordinate system.

SAMPLE 18.1 Acceleration in polar coordinates. A bug walks along the spiral section of a natural shell. The path of the bug is described by the equation $R = R_0 e^{a\theta}$ where $a = 0.182$ and $R_0 = 5\,\text{mm}$. The bug's radial distance from the center of the spiral is seen to be increasing at a constant rate of 2 mm/ s. Find the x and y components of the acceleration of the bug at $\theta = \pi$.

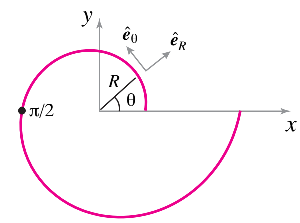


Figure 18.13

Solution In polar coordinates, the acceleration of a particle in planar motion is

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

Since we know the position of the bug,

$$\begin{aligned} R &= R_0\, e^{a\theta}, \\ \dot{R} &= R_0\, a e^{a\theta}\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{\dot{R}}{R_0 a\, e^{a\theta}}, \\ \ddot{R} &= R_0\, a e^{a\theta}\ddot{\theta} + R_0\, a^2 e^{a\theta}\dot{\theta}^2. \end{aligned}$$

Since the radial distance R of the bug is increasing at a constant rate $\dot{R} = 2\ \text{mm}/s, \ddot{R} = 0$, that is,

$$\begin{aligned} R_0\, a e^{a\theta}(\ddot{\theta} + a\dot{\theta}^2) &= 0 \\ \Rightarrow \qquad \ddot{\theta} &= -a\dot{\theta}^2 \\ &= -\frac{\dot{R}^2}{R_0^2 a\, e^{2a\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \\ &= \left(0 - R_0 e^{a\theta}\cdot\frac{\dot{R}^2}{R_0^2 a^2 e^{2a\theta}}\right)\hat{e}_R + \left(\frac{2\dot{R}^2}{R_0 a\, e^{a\theta}} + R_0 e^{a\theta}\cdot\frac{-\dot{R}^2}{R_0^2 a\, e^{2a\theta}}\right)\hat{e}_\theta \\ &= \frac{\dot{R}^2}{R_0 a\, e^{a\theta}}\left[-\frac{1}{a}\hat{e}_R + (2-1)\hat{e}_\theta\right]. \end{aligned}$$

Now substituting $R_0 = 5\,\text{mm}$, $a = 0.182$, $\dot{R} = 2\,\text{mm}/s$, and $\theta = \pi$ in the above expression, we get

$$\vec{a} = (-13.63\,\text{mm/s}^2)\,\hat{e}_R + (2.48\,\text{mm/s}^2)\,\hat{e}_\theta.$$

But, at $\theta = \pi$

$$\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} = -\hat{\imath} \quad \text{and} \quad \hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} = -\hat{\jmath},$$

therefore,

$$\begin{aligned} \vec{a} &= (13.63\,\text{mm/s}^2)\,\hat{\imath} - (2.48\,\text{mm/s}^2)\,\hat{\jmath}, \\ \Rightarrow \quad a_x &= 13.63\,\text{mm/s}^2 \quad \text{and} \quad a_y = -2.48\,\text{mm/s}^2. \end{aligned}$$

$$a_x = 13.63\,\text{mm/s}^2,\ a_y = -2.48\,\text{mm/s}^2$$

SAMPLE 18.2 Going back and forth between (x, y) and (R, θ). Given the position of a particle in polar coordinates (R, θ) and its radial and angular velocity $(\dot{R}, \dot{\theta})$ and radial and angular acceleration $(\ddot{R}, \ddot{\theta})$, find $(\dot{x}, \dot{y})$ and $(\ddot{x}, \ddot{y})$. Also, find the inverse relationship.

Solution

- **Polar to Cartesian:** In polar coordinates, we are given $R, \theta, \dot{R}, \dot{\theta}, \ddot{R}$, and $\ddot{\theta}$. We need to find $\dot{x}, \dot{y}, \ddot{x}$, and $\ddot{y}$. Let us consider the velocity first. The velocity of a point is $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ in cartesian coordinates and $\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$ in polar coordinates. Thus,
$$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta.$$
where $\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$. Dotting this equation with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get
$$\begin{aligned}\dot{x} &= \dot{R}\,\overbrace{(\hat{e}_R \cdot \hat{\imath})}^{\cos\theta} + R\dot{\theta}\,\overbrace{(\hat{e}_\theta \cdot \hat{\imath})}^{-\sin\theta}\\ \dot{y} &= \dot{R}\,\underbrace{(\hat{e}_R \cdot \hat{\jmath})}_{\sin\theta} + R\dot{\theta}\,\underbrace{(\hat{e}_\theta \cdot \hat{\jmath})}_{\cos\theta}\end{aligned}$$
or,
$$\begin{aligned}\dot{x} &= \dot{R}\cos\theta + R\dot{\theta}(-\sin\theta)\\ \dot{y} &= \dot{R}\sin\theta + R\dot{\theta}\cos\theta\end{aligned}$$
or,
$$\begin{Bmatrix}\dot{x}\\ \dot{y}\end{Bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\begin{Bmatrix}\dot{R}\\ \dot{\theta}\end{Bmatrix}. \tag{18.5}$$
Thus given $\dot{R}$ and $\dot{\theta}$ at (R, θ), we can find $\dot{x}$ and $\dot{y}$. Similarly, from the acceleration formula, $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, we derive
$$\begin{Bmatrix}\ddot{x}\\ \ddot{y}\end{Bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\begin{Bmatrix}\ddot{R}\\ \ddot{\theta}\end{Bmatrix} + \begin{Bmatrix}-R\dot{\theta}^2\\ 2\dot{R}\dot{\theta}\end{Bmatrix}\right). \tag{18.6}$$
It is not necessary to split the terms on the right hand side. We could have kept them together as $(\ddot{R} - R\dot{\theta}^2)$ and $(2\dot{R}\dot{\theta} + R\ddot{\theta})$ but we split them to keep the radial acceleration term $\ddot{R}$ and angular acceleration $\ddot{\theta}$ in evidence.

- **Cartesian to Polar:** Given $\dot{x}, \dot{y}, \ddot{x}$, and $\ddot{y}$ at (x, y), we can now find $\dot{R}, \dot{\theta}, \ddot{R}$, and $\ddot{\theta}$ easily by inverting eqn. (18.5) and eqn. (18.6):
$$\begin{Bmatrix}\dot{R}\\ \dot{\theta}\end{Bmatrix} = \begin{bmatrix}1 & 0\\ 0 & \frac{1}{R}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{y}\end{Bmatrix}. \tag{18.7}$$
$$\begin{Bmatrix}\ddot{R}\\ \ddot{\theta}\end{Bmatrix} = \begin{bmatrix}1 & 0\\ 0 & \frac{1}{R}\end{bmatrix}\left(\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{y}\end{Bmatrix} - \begin{Bmatrix}-R\dot{\theta}^2\\ 2\dot{R}\dot{\theta}\end{Bmatrix}\right). \tag{18.8}$$
Note that in eqn. (18.8) we need $\dot{R}$ and $\dot{\theta}$ in order to compute $\ddot{R}$ and $\ddot{\theta}$. This, however, is no problem since we have $\dot{R}$ and $\dot{\theta}$ from eqn. (18.7). Of course, R and θ are required too, which are easily computed as $R = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

SAMPLE 18.3 Velocity in path coordinates. The path of a particle, stuck at the edge of a disk rolling on a level ground with constant speed, is called a cycloid. The parametric equations of a cycloid described by a particle is $x = t - \sin t$, $y = 1 - \cos t$ where t is a dimensionless time. Find the velocity of the particle at

1. $t = \frac{\pi}{2}$,
2. $t = \pi$, and
3. $t = 2\pi$

and express the velocity in terms of path basis vectors $(\hat{\boldsymbol{e}}_t, \hat{\boldsymbol{e}}_n)$.

Solution The position of the particle is given:

$$\begin{aligned} \vec{\boldsymbol{r}} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \\ &= (t - \sin t)\hat{\boldsymbol{\imath}} + (1 - \cos t)\hat{\boldsymbol{\jmath}} \\ \Rightarrow \quad \vec{\boldsymbol{v}} &\equiv \frac{d\vec{\boldsymbol{r}}}{dt} \qquad (18.9) \\ &= (1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}, \\ \text{and} \quad v &= |\vec{\boldsymbol{v}}| \qquad (18.10) \\ &= \sqrt{(1 - \cos t)^2 + \sin^2 t} \\ &= \sqrt{2 - 2\cos t}. \qquad (18.11) \end{aligned}$$

In terms of path basis vectors, the velocity is given by

$$\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t \quad \text{where} \quad \hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/v.$$

Here,

$$\hat{\boldsymbol{e}}_t = \frac{(1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}}{\sqrt{2 - 2\cos t}}. \qquad (18.12)$$

Substituting the values of t in equations 18.11 and 18.12 we get

1. at $t = \frac{\pi}{2}$:

$$v = \sqrt{2}, \quad \hat{\boldsymbol{e}}_t = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \quad \vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t.$$

$$\boxed{\vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \tfrac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

2. at $t = \pi$:

$$v = 2, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t.$$

$$\boxed{\vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}}$$

3. at $t = 2\pi$:

$$v = 0, \quad \hat{\boldsymbol{e}}_t = \text{undefined}, \quad \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}.$$

$$\boxed{\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}}$$

SAMPLE 18.4 Path coordinates in 2-D. A particle traverses a limacon $R = (1 + 2\cos\theta)\,\mathrm{ft}$, with constant angular speed $\dot\theta = 3\,\mathrm{rad/s}$.

1. Find the normal and tangential accelerations (a_t and a_n) of the particle at $\theta = \frac{\pi}{2}$.
2. Find the radius of the osculating circle and draw the circle at $\theta = \frac{\pi}{2}$.

Solution

1. The equation of the path is

$$R = (1 + 2\cos\theta)\,\mathrm{ft}.$$

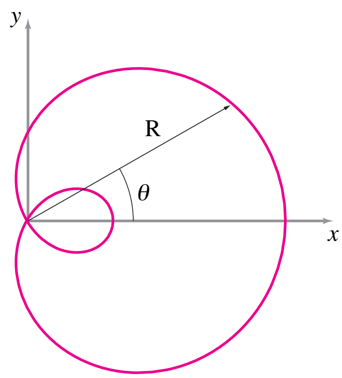


Figure 18.14: The limacon $R = (1 + 2\cos\theta)\,\mathrm{ft}$.

The path is shown in Fig. 18.14. Since the equation of the path is given in polar coordinates, we can calculate the velocity and acceleration using the polar coordinate formulae:

$$\begin{aligned} \vec{v} &= \dot R\hat{e}_R + R\dot\theta\hat{e}_\theta && (18.13)\\ \vec{a} &= (\ddot R - R\dot\theta^2)\hat{e}_R + (2\dot R\dot\theta + R\ddot\theta)\hat{e}_\theta. && (18.14) \end{aligned}$$

So, we need to find $\dot R, \ddot R, \ddot\theta$ for computing $\vec{v}$ and $\vec{a}$. From the given equation for R

$$\begin{aligned} R &= (1 + 2\cos\theta)\,\mathrm{ft}\\ \Rightarrow \quad \dot R &= -(2\,\mathrm{ft})\sin\theta\,\dot\theta\\ \Rightarrow \quad \ddot R &= -(2\,\mathrm{ft})\sin\theta\,\ddot\theta \;-(2\,\mathrm{ft})\cos\theta\,\dot\theta^2\\ &= -(2\,\mathrm{ft})\,\dot\theta^2\cos\theta \end{aligned}$$

where we set $\ddot\theta = 0$ because $\dot\theta =$ constant. Substituting these expressions in Eqn. (18.13) and (18.14), we get

$$\begin{aligned} \vec{v} &= -(2\,\mathrm{ft})\dot\theta\sin\theta\hat{e}_R + \dot\theta(1 + \cos\theta)\,\mathrm{ft}\hat{e}_\theta\\ \vec{a} &= (\ddot R - R\dot\theta^2)\hat{e}_R + (2\dot R\dot\theta + R\ddot\theta)\hat{e}_\theta\\ &= [-2\dot\theta^2\cos\theta - (1 + 2\cos\theta)\,\dot\theta^2]\,\mathrm{ft}\,\hat{e}_R + (-2\dot\theta^2\sin\theta)\,\mathrm{ft}\,\hat{e}_\theta\\ &= -\dot\theta^2[(1 + 4\cos\theta)\,\hat{e}_R + (2\sin\theta)\,\hat{e}_\theta]\,\mathrm{ft} \end{aligned}$$

which give velocity and acceleration at any θ. Now substituting $\theta = \pi/2$ we get the velocity and acceleration at the desired point:

$$\begin{aligned} \vec{v}|_{\frac{\pi}{2}} &= \dot\theta[-2\sin\frac{\pi}{2}\hat{e}_R + (1 + 2\cos\frac{\pi}{2})\hat{e}_\theta]\,\mathrm{ft}\\ &= 3\,\mathrm{ft/s}(-2\hat{e}_R + \hat{e}_\theta)\\ \vec{a}|_{\frac{\pi}{2}} &= -9\,\mathrm{ft/s^2}(\hat{e}_R + 2\hat{e}_\theta). \end{aligned}$$

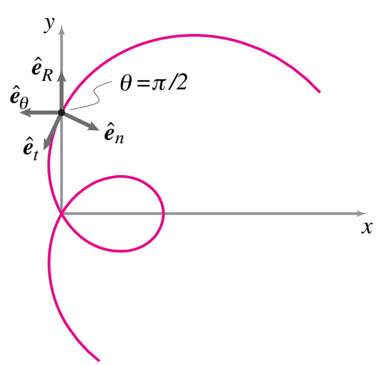


Figure 18.15: The unit vectors $\hat{e}_R, \hat{e}_\theta$, and $\hat{e}_t, \hat{e}_n$ at $\theta = \pi/2$.

Thus we know the velocity and the acceleration of the particle in polar coordinates. Now we proceed to find the tangential and the normal components of acceleration (acceleration in path coordinates). In path coordinates

$$\begin{aligned} \vec{a} &= \vec{a}_t + \vec{a}_n\\ &\equiv a_t\hat{e}_t + a_n\hat{e}_n \end{aligned}$$

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors in the directions of the tangent and the principle normal of the path. We compute these unit vectors as follows.

$$
\begin{aligned}
\hat{\boldsymbol{e}}_t &= \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} \\
&= (\frac{-6\,\text{ft/s})\hat{\boldsymbol{e}}_R + (3\,\text{ft/s})\hat{\boldsymbol{e}}_\theta}{\sqrt{45}\ \text{ft/s}} \\
&= -\frac{2}{\sqrt{5}}\hat{\boldsymbol{e}}_R + \frac{1}{\sqrt{5}}\hat{\boldsymbol{e}}_\theta.
\end{aligned}
$$

So,

$$
\begin{aligned}
\vec{\boldsymbol{a}}_t &= (\vec{\boldsymbol{a}}\cdot\hat{\boldsymbol{e}}_t)\,\hat{\boldsymbol{e}}_t \\
&= 9\,\text{ft/s}^2\left(-\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)\hat{\boldsymbol{e}}_t \\
&= \vec{\boldsymbol{0}},
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{\boldsymbol{a}}_n &= \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_t \\
&= -9\,\text{ft/s}^2(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\boldsymbol{e}}_n &= \frac{\vec{\boldsymbol{a}}_n}{|\vec{\boldsymbol{a}}_n|} \\
&= -\frac{1}{\sqrt{5}}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= 9\sqrt{5}\ \text{ft/s}^2\hat{\boldsymbol{e}}_n \\
\Rightarrow \quad a_t &= 0 \quad \text{and} \quad a_n = 20.12\,\text{ft/s}^2.
\end{aligned}
$$

$a_t = 0,\ a_n = 20.12\,\text{ft/s}^2$

2. In path coordinates the acceleration is also expressed as

$$\vec{\boldsymbol{a}} = \dot{v}\hat{\boldsymbol{e}}_t + \frac{v^2}{\rho}\hat{\boldsymbol{e}}_n$$

where ρ is the radius of the osculating circle. Since we already know the speed v and the normal component of acceleration a_n we can easily compute the radius of the osculating circle.

$$
\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow \quad \rho &= \frac{v^2}{a_n} = \frac{45(\,\text{ft/s})^2}{9\sqrt{5}\,\text{ft/s}^2} = \sqrt{5}\,\text{ft}.
\end{aligned}
$$

$\rho = 2.24\,\text{ft}$

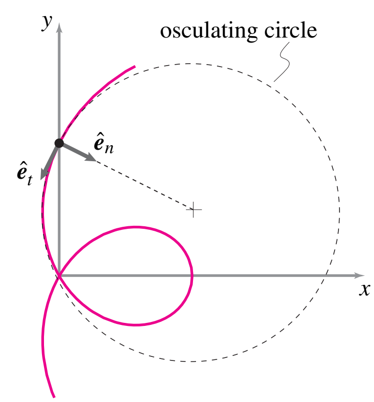


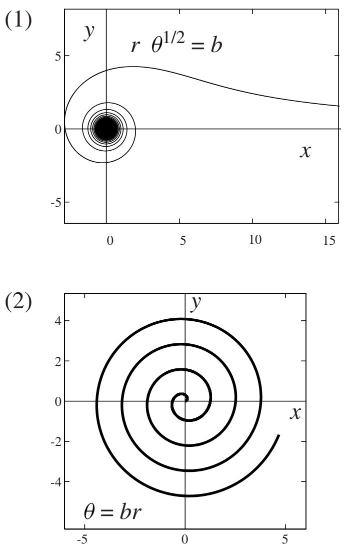
Figure 18.16: The osculating circle of radius $\rho = 5^{1/2}\,\text{ft}$ at $\theta = \pi/2$. Note that $\hat{\boldsymbol{e}}_n$ points to the center of the osculating circle.

18.1 Polar coordinates and path coordinates

18.1.1 A particle moves along the two paths (1) and (2) as shown.

a) In each case, determine the velocity of the particle in terms of b, θ, and $\dot{\theta}$. *

b) Find the x and y coordinates of the path as functions of b and r or b and θ. *



Problem 18.1.1

18.1.2 For the particle path (1) in problem 18.1.1, find the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

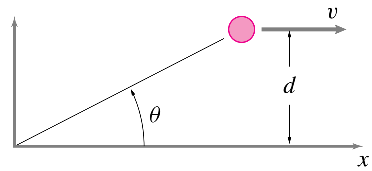
18.1.3 For the particle path (2) in problem 18.1.1, find the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

18.1.4 A body moves with constant velocity V in a straight line parallel to and at a distance d from the x-axis.

a) Calculate $\dot{\theta}$ in terms of V, d, and θ.

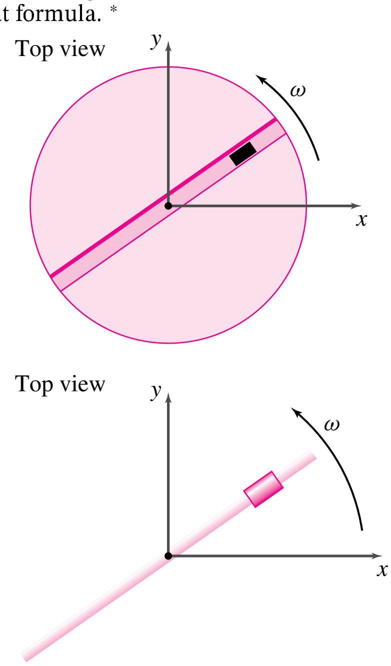
b) Calculate the $\hat{\boldsymbol{e}}_\theta$ component of acceleration.

Carefully define, using a sketch and/or words, any variables, coordinate systems, and reference frames you use. Express your answers using any convenient coordinate system (just make sure its orientation has been clearly defined).



Problem 18.1.4

18.1.5 Picking apart the polar coordinate formula for velocity. This problem concerns a small mass m that sits in a slot in a turntable. Alternatively you can think of a small bead that slides on a rod. The mass always stays in the slot (or on the rod). Assume the mass is a little bug that can walk as it pleases on the rod (or in the slot) and you control how the turntable/rod rotates. Name two situations in which one of the terms is zero but the other is not in the two term polar coordinate formula for velocity, $\dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$. You should thus gain some insight into the meaning of each of the two terms in that formula. *



Problem 18.1.5

18.1.6 Picking apart the polar coordinate formula for acceleration. Reconsider the configurations in problem 18.1.5. This time, name four situations in which all of the terms, but one, in the four term polar coordinate formula for acceleration, $\vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta$, are zero. Each situation should pick out a different term. You should thus gain some insight into the meaning of each of the four terms in that formula. *

18.1.7 The two differential equations:

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \end{aligned}$$

have the general solution

$$\begin{aligned} R &= \sqrt{d^2 + (v(t - t_0))^2} \\ \theta &= \theta_0 + \tan^{-1}(v(t - t_0)/d) \end{aligned}$$

where θ_0, d, t_0, and v are arbitrary constants. This solution could be checked by plugging back into the differential equations — you need not do this (tedious) substitution. The solution describes a curve in the plane. That is, if for a range of values of t, the values of R and θ were calculated and then plotted using polar coordinates, a curve would be drawn. *What can you say about the shape of this curve?*

[Hints:

a) Actually make a plot using some random values of the constants and see what the plot looks like.

b) Write the equation $\mathbf{F} = m\mathbf{a}$ for a particle in polar coordinates and think of a force that would be relevant to this problem.

c) The answer is something simple.]

18.1.8 A car driver on a very boring highway is carefully monitoring her speed. Over a one hour period, the car travels on a curve with constant radius of curvature, $\rho = 25$ mi, and its speed increases uniformly from 50 mph to 60 mph. What is the acceleration of the car half-way through this one hour period, in path coordinates?

18.1.9 Find expressions for $\hat{\boldsymbol{e}}_t$, a_t, a_n, $\hat{\boldsymbol{e}}_n$, and the radius of curvature ρ, at any position (or time) on the given particle paths for

a) problem 12.1.11, *

b) problem 12.1.12,

c) problem 18.2.4,

d) problem 12.1.14,

e) problem 12.1.13, and

f) problem 12.1.10.

18.1.10 A particle travels at non-constant speed on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$. Carefully sketch the ellipse for particular values of a and b. For various positions of the particle on the path, sketch the position vector $\vec{r}(t)$; the polar coordinate basis vectors $\hat{e}_r$ and $\hat{e}_\theta$; and the path coordinate basis vectors $\hat{e}_n$ and $\hat{e}_t$. At what points on the path are $\hat{e}_r$ and $\hat{e}_n$ parallel(or $\hat{e}_\theta$ and $\hat{e}_t$ parallel)?