SAMPLE 18.1 Acceleration in polar coordinates. A bug walks along the spiral section of a natural shell. The path of the bug is described by the equation $R = R_0 e^{a\theta}$ where $a = 0.182$ and $R_0 = 5\,\mathrm{mm}$. The bug's radial distance from the center of the spiral is seen to be increasing at a constant rate of $2\,\mathrm{mm/s}$. Find the x and y components of the acceleration of the bug at $\theta = \pi$.

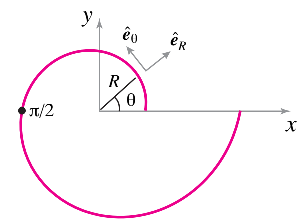

Figure 18.13

Solution In polar coordinates, the acceleration of a particle in planar motion is

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

Since we know the position of the bug,

$$\begin{aligned} R &= R_0\, e^{a\theta}, \\ \dot{R} &= R_0\, a e^{a\theta}\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{\dot{R}}{R_0 a\, e^{a\theta}}, \\ \ddot{R} &= R_0\, a e^{a\theta}\ddot{\theta} + R_0\, a^2 e^{a\theta}\dot{\theta}^2. \end{aligned}$$

Since the radial distance R of the bug is increasing at a constant rate $\dot{R} = 2\ \mathrm{mm}/s, \ddot{R} = 0$, that is,

$$\begin{aligned} R_0\, a e^{a\theta}(\ddot{\theta} + a\dot{\theta}^2) &= 0 \\ \Rightarrow \qquad \ddot{\theta} &= -a\dot{\theta}^2 \\ &= -\frac{\dot{R}^2}{R_0^2 a\, e^{2a\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \\ &= \left(0 - R_0 e^{a\theta}\cdot\frac{\dot{R}^2}{R_0^2 a^2 e^{2a\theta}}\right)\hat{e}_R + \left(\frac{2\dot{R}^2}{R_0 a\, e^{a\theta}} + R_0 e^{a\theta}\cdot\frac{-\dot{R}^2}{R_0^2 a\, e^{2a\theta}}\right)\hat{e}_\theta \\ &= \frac{\dot{R}^2}{R_0 a\, e^{a\theta}}\left[-\frac{1}{a}\hat{e}_R + (2-1)\hat{e}_\theta\right]. \end{aligned}$$

Now substituting $R_0 = 5\,\mathrm{mm}$, $a = 0.182$, $\dot{R} = 2\,\mathrm{mm}/s$, and $\theta = \pi$ in the above expression, we get

$$\vec{a} = (-13.63\,\mathrm{mm/s^2})\,\hat{e}_R + (2.48\,\mathrm{mm/s^2})\,\hat{e}_\theta.$$

But, at $\theta = \pi$

$$\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} = -\hat{\imath} \quad \text{and} \quad \hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} = -\hat{\jmath},$$

therefore,

$$\begin{aligned} \vec{a} &= (13.63\,\mathrm{mm/s^2})\,\hat{\imath} - (2.48\,\mathrm{mm/s^2})\,\hat{\jmath}, \\ \Rightarrow \quad a_x &= 13.63\,\mathrm{mm/s^2} \quad \text{and} \quad a_y = -2.48\,\mathrm{mm/s^2}. \end{aligned}$$

$$a_x = 13.63\,\mathrm{mm/s^2},\ a_y = -2.48\,\mathrm{mm/s^2}$$

SAMPLE 18.2 Going back and forth between (x, y) and (R, θ). Given the position of a particle in polar coordinates (R, θ) and its radial and angular velocity $(\dot{R}, \dot{\theta})$ and radial and angular acceleration $(\ddot{R}, \ddot{\theta})$, find $(\dot{x}, \dot{y})$ and $(\ddot{x}, \ddot{y})$. Also, find the inverse relationship.

Solution

- **Polar to Cartesian:** In polar coordinates, we are given $R, \theta, \dot{R}, \dot{\theta}, \ddot{R}$, and $\ddot{\theta}$. We need to find $\dot{x}, \dot{y}, \ddot{x}$, and $\ddot{y}$. Let us consider the velocity first. The velocity of a point is $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ in cartesian coordinates and $\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$ in polar coordinates. Thus,

$$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta.$$

where $\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$. Dotting this equation with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} \dot{x} &= \dot{R}\,\overbrace{(\hat{e}_R \cdot \hat{\imath})}^{\cos\theta} + R\dot{\theta}\,\overbrace{(\hat{e}_\theta \cdot \hat{\imath})}^{-\sin\theta} \\ \dot{y} &= \dot{R}\,\underbrace{(\hat{e}_R \cdot \hat{\jmath})}_{\sin\theta} + R\dot{\theta}\,\underbrace{(\hat{e}_\theta \cdot \hat{\jmath})}_{\cos\theta} \end{aligned}$$

or,

$$\begin{aligned} \dot{x} &= \dot{R}\cos\theta + R\dot{\theta}(-\sin\theta) \\ \dot{y} &= \dot{R}\sin\theta + R\dot{\theta}\cos\theta \end{aligned}$$

or,

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{Bmatrix} \dot{R} \\ \dot{\theta} \end{Bmatrix}. \tag{18.5}$$

Thus given $\dot{R}$ and $\dot{\theta}$ at (R, θ), we can find $\dot{x}$ and $\dot{y}$. Similarly, from the acceleration formula, $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, we derive

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{Bmatrix} \ddot{R} \\ \ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} -R\dot{\theta}^2 \\ 2\dot{R}\dot{\theta} \end{Bmatrix} \right). \tag{18.6}$$

It is not necessary to split the terms on the right hand side. We could have kept them together as $(\ddot{R} - R\dot{\theta}^2)$ and $(2\dot{R}\dot{\theta} + R\ddot{\theta})$ but we split them to keep the radial acceleration term $\ddot{R}$ and angular acceleration $\ddot{\theta}$ in evidence.

- **Cartesian to Polar:** Given $\dot{x}, \dot{y}, \ddot{x}$, and $\ddot{y}$ at (x, y), we can now find $\dot{R}, \dot{\theta}, \ddot{R}$, and $\ddot{\theta}$ easily by inverting eqn. (18.5) and eqn. (18.6):

$$\begin{Bmatrix} \dot{R} \\ \dot{\theta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}. \tag{18.7}$$

$$\begin{Bmatrix} \ddot{R} \\ \ddot{\theta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} - \begin{Bmatrix} -R\dot{\theta}^2 \\ 2\dot{R}\dot{\theta} \end{Bmatrix} \right). \tag{18.8}$$

Note that in eqn. (18.8) we need $\dot{R}$ and $\dot{\theta}$ in order to compute $\ddot{R}$ and $\ddot{\theta}$. This, however, is no problem since we have $\dot{R}$ and $\dot{\theta}$ from eqn. (18.7). Of course, R and θ are required too, which are easily computed as $R = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

SAMPLE 18.3 Velocity in path coordinates. The path of a particle, stuck at the edge of a disk rolling on a level ground with constant speed, is called a cycloid. The parametric equations of a cycloid described by a particle is $x = t - \sin t$, $y = 1 - \cos t$ where t is a dimensionless time. Find the velocity of the particle at

1. $t = \frac{\pi}{2}$,
2. $t = \pi$, and
3. $t = 2\pi$

and express the velocity in terms of path basis vectors $(\hat{\boldsymbol{e}}_t, \hat{\boldsymbol{e}}_n)$.

Solution The position of the particle is given:

$$
\begin{aligned}
\vec{\boldsymbol{r}} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \\
&= (t - \sin t)\hat{\boldsymbol{\imath}} + (1 - \cos t)\hat{\boldsymbol{\jmath}} \\
\Rightarrow \quad \vec{\boldsymbol{v}} &\equiv \frac{d\vec{\boldsymbol{r}}}{dt} \qquad (18.9)\\
&= (1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}, \\
\text{and} \quad v &= |\vec{\boldsymbol{v}}| \qquad (18.10)\\
&= \sqrt{(1 - \cos t)^2 + \sin^2 t} \\
&= \sqrt{2 - 2\cos t}. \qquad (18.11)
\end{aligned}
$$

In terms of path basis vectors, the velocity is given by

$$\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t \quad \text{where} \quad \hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/v.$$

Here,

$$\hat{\boldsymbol{e}}_t = \frac{(1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}}{\sqrt{2 - 2\cos t}}. \qquad (18.12)$$

Substituting the values of t in equations 18.11 and 18.12 we get

1. at $t = \frac{\pi}{2}$:

$$v = \sqrt{2}, \quad \hat{\boldsymbol{e}}_t = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \quad \vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t.$$

$$\vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \tfrac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$$

2. at $t = \pi$:

$$v = 2, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t.$$

$$\vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}$$

3. at $t = 2\pi$:

$$v = 0, \quad \hat{\boldsymbol{e}}_t = \text{undefined}, \quad \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}.$$

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

SAMPLE 18.4 Path coordinates in 2-D. A particle traverses a limacon $R = (1 + 2\cos\theta)\,\text{ft}$, with constant angular speed $\dot\theta = 3\,\text{rad/s}$.

1. Find the normal and tangential accelerations (a_t and a_n) of the particle at $\theta = \frac{\pi}{2}$.
2. Find the radius of the osculating circle and draw the circle at $\theta = \frac{\pi}{2}$.

Solution

1. The equation of the path is

$$R = (1 + 2\cos\theta)\,\text{ft}.$$

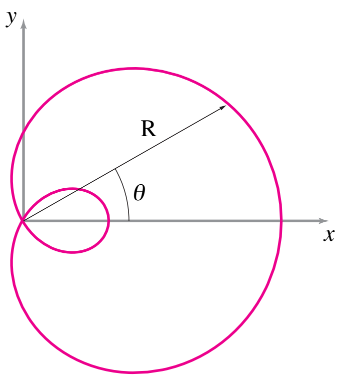

Figure 18.14: The limacon $R = (1 + 2\cos\theta)\,\text{ft}$.

The path is shown in Fig. 18.14. Since the equation of the path is given in polar coordinates, we can calculate the velocity and acceleration using the polar coordinate formulae:

$$\begin{aligned} \vec{v} &= \dot{R}\hat{e}_R + R\dot\theta\hat{e}_\theta && (18.13)\\ \vec{a} &= (\ddot{R} - R\dot\theta^2)\hat{e}_R + (2\dot{R}\dot\theta + R\ddot\theta)\hat{e}_\theta. && (18.14)\end{aligned}$$

So, we need to find $\dot{R}, \ddot{R}, \ddot\theta$ for computing $\vec{v}$ and $\vec{a}$. From the given equation for R

$$\begin{aligned} R &= (1 + 2\cos\theta)\,\text{ft}\\ \Rightarrow\quad \dot{R} &= -(2\,\text{ft})\sin\theta\,\dot\theta\\ \Rightarrow\quad \ddot{R} &= -(2\,\text{ft})\sin\theta\,\ddot\theta\ - (2\,\text{ft})\cos\theta\,\dot\theta^2\\ &= -(2\,\text{ft})\,\dot\theta^2\cos\theta\end{aligned}$$

where we set $\ddot\theta = 0$ because $\dot\theta$ = constant. Substituting these expressions in Eqn. (18.13) and (18.14), we get

$$\begin{aligned} \vec{v} &= -(2\,\text{ft})\dot\theta\sin\theta\hat{e}_R + \dot\theta(1 + \cos\theta)\,\text{ft}\hat{e}_\theta\\ \vec{a} &= (\ddot{R} - R\dot\theta^2)\hat{e}_R + (2\dot{R}\dot\theta + R\ddot\theta)\hat{e}_\theta\\ &= [-2\dot\theta^2\cos\theta - (1 + 2\cos\theta)\dot\theta^2]\,\text{ft}\,\hat{e}_R + (-2\dot\theta^2\sin\theta)\,\text{ft}\,\hat{e}_\theta\\ &= -\dot\theta^2[(1 + 4\cos\theta)\,\hat{e}_R + (2\sin\theta)\,\hat{e}_\theta]\,\text{ft}\end{aligned}$$

which give velocity and acceleration at any θ. Now substituting $\theta = \pi/2$ we get the velocity and acceleration at the desired point:

$$\begin{aligned} \vec{v}|_{\frac{\pi}{2}} &= \dot\theta[-2\sin\frac{\pi}{2}\hat{e}_R + (1 + 2\cos\frac{\pi}{2})\hat{e}_\theta]\,\text{ft}\\ &= 3\,\text{ft/s}(-2\hat{e}_R + \hat{e}_\theta)\\ \vec{a}|_{\frac{\pi}{2}} &= -9\,\text{ft/s}^2(\hat{e}_R + 2\hat{e}_\theta).\end{aligned}$$

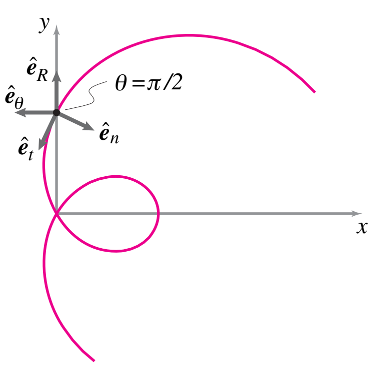

Figure 18.15: The unit vectors $\hat{e}_R, \hat{e}_\theta$, and $\hat{e}_t, \hat{e}_n$ at $\theta = \pi/2$.

Thus we know the velocity and the acceleration of the particle in polar coordinates. Now we proceed to find the tangential and the normal components of acceleration (acceleration in path coordinates). In path coordinates

$$\begin{aligned} \vec{a} &= \vec{a}_t + \vec{a}_n\\ &\equiv a_t\hat{e}_t + a_n\hat{e}_n\end{aligned}$$

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors in the directions of the tangent and the principle normal of the path. We compute these unit vectors as follows.

$$
\begin{aligned}
\hat{\boldsymbol{e}}_t &= \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} \\
&= (\frac{-6\,\text{ft/s})\hat{\boldsymbol{e}}_R + (3\,\text{ft/s})\hat{\boldsymbol{e}}_\theta}{\sqrt{45}\ \text{ft/s}} \\
&= -\frac{2}{\sqrt{5}}\hat{\boldsymbol{e}}_R + \frac{1}{\sqrt{5}}\hat{\boldsymbol{e}}_\theta.
\end{aligned}
$$

So,

$$
\begin{aligned}
\vec{\boldsymbol{a}}_t &= (\vec{\boldsymbol{a}}\cdot\hat{\boldsymbol{e}}_t)\,\hat{\boldsymbol{e}}_t \\
&= 9\,\text{ft/s}^2\left(-\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)\hat{\boldsymbol{e}}_t \\
&= \vec{\boldsymbol{0}},
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{\boldsymbol{a}}_n &= \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_t \\
&= -9\,\text{ft/s}^2(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\boldsymbol{e}}_n &= \frac{\vec{\boldsymbol{a}}_n}{|\vec{\boldsymbol{a}}_n|} \\
&= -\frac{1}{\sqrt{5}}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= 9\sqrt{5}\ \text{ft/s}^2\hat{\boldsymbol{e}}_n \\
\Rightarrow \quad a_t &= 0 \quad \text{and} \quad a_n = 20.12\,\text{ft/s}^2.
\end{aligned}
$$

$a_t = 0,\ a_n = 20.12\,\text{ft/s}^2$

2. In path coordinates the acceleration is also expressed as

$$\vec{\boldsymbol{a}} = \dot{v}\hat{\boldsymbol{e}}_t + \frac{v^2}{\rho}\hat{\boldsymbol{e}}_n$$

where ρ is the radius of the osculating circle. Since we already know the speed v and the normal component of acceleration a_n we can easily compute the radius of the osculating circle.

$$
\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow \quad \rho &= \frac{v^2}{a_n} = \frac{45(\,\text{ft/s})^2}{9\sqrt{5}\,\text{ft/s}^2} = \sqrt{5}\,\text{ft}.
\end{aligned}
$$

$\rho = 2.24\,\text{ft}$

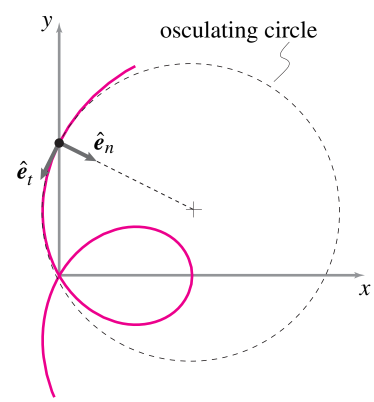

Figure 18.16: The osculating circle of radius $\rho = 5^{1/2}$ ft at $\theta = \pi/2$. Note that $\hat{\boldsymbol{e}}_n$ points to the center of the osculating circle.