17.5 Rigid-object collision mechanics

This section extends the particle collisions discussion in Section 13.2 that starts on page 665.

2D collisions

For collisions between rigid bodies with more general motions before and after the collisions we depend on the three key ideas

I. Collision forces are big,
II. Collisions are quick, and
III. The laws of mechanics apply during the collision.

There are two extra assumptions that are needed in simple analysis:

IV. Collision forces are few. For a given rigid body there is one, or at most two non-negligible collision forces. This is the real import of idea (I) above. Because collision forces are big most other forces can be neglected.
V. The collision force(s) act at a well defined point which does not move during the collision.

Based on these assumptions one then uses linear and angular momentum balance in their time-integrated form.

Example: Two bodies in space
Two bodies collide at point C. The impulse acting on body 2 is $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}_{\text{coll}}\, dt$. If the mass and inertia properties of both bodies is known, as are the velocities and rotation rates before the collision we have the following linear and angular momentum balance equations for the two bodies:

$$
\begin{aligned}
-\vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}^+_{G1} - \vec{\boldsymbol{v}}^-_{G1}\right)\\
\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}^+_{G2} - \vec{\boldsymbol{v}}^-_{G2}\right)\\
\vec{\boldsymbol{r}}_{C/G1} \times (-\vec{\boldsymbol{P}}) &= I^{\text{cm}\,1}_{zz}\left(\omega^+_1 - \omega^-_1\right)\hat{\boldsymbol{k}}\\
\vec{\boldsymbol{r}}_{C/G2} \times \vec{\boldsymbol{P}} &= I^{\text{cm}\,2}_{zz}\left(\omega^+_2 - \omega^-_2\right)\hat{\boldsymbol{k}}.
\end{aligned}
\tag{17.57}
$$

These make up 6 scalar equations (2 for each momentum equation, 1 for each angular momentum equation). There are 8 scalar unknowns: $\vec{\boldsymbol{v}}^+_{G1}$ (2), $\vec{\boldsymbol{v}}^+_{G2}$ (2), ω^+_1 (1), ω^+_2 (1), and $\vec{\boldsymbol{P}}$ (2). Thus the motion after the collision cannot be determined.

[Note that linear and angular momentum balance for the system would give equations which could be obtained by adding and subtracting combinations of the equations above. So adding system momentum balance equations does not add information (*i.e.,* adds linearly dependent equations).]

So, as for 1-D collisions, momentum balance is not enough to determine the outcome of the collision. Eqns. 17.57 aren't enough. A thousand different models and assumptions could be added to make the system solvable. But there are only two cases that are non-controversial and also relatively simple: 1) sticking collisions, and 2) frictionless collisions.

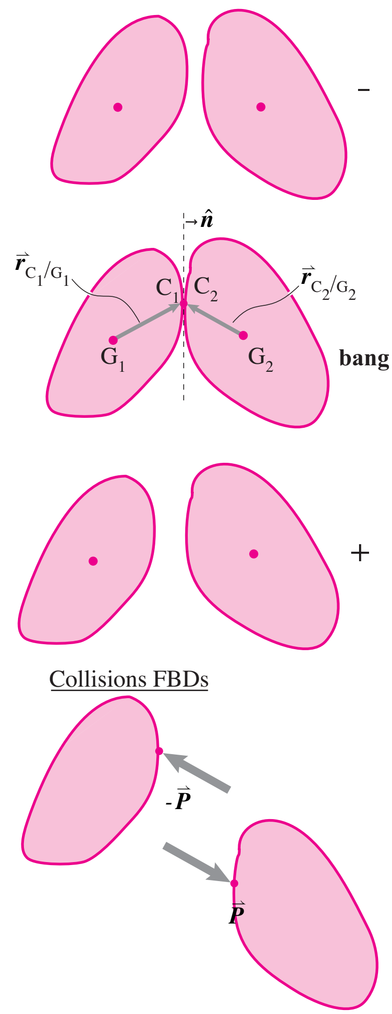

Figure 17.75: Two bodies collide at point C. The only non-negligible collision impulse is $\vec{\boldsymbol{P}}$ acting on body 2 (and $-\vec{\boldsymbol{P}}$ on body 1) at point C. The material points on the contacting bodies are C_1 and C_2. The outward normal to body 1 at C_1 is $\hat{\boldsymbol{n}}$.

Sticking collisions

A 'perfectly-plastic' sticking collision is one where the relative velocities of the two contacting points are assumed to go suddenly to zero. That is

$$\vec{\boldsymbol{v}}^+_{C1} = \vec{\boldsymbol{v}}^+_{C2}$$

Writing $\vec{\boldsymbol{v}}^+_{\mathrm{C1}} = \vec{\boldsymbol{v}}^+_{\mathrm{G1}} + (\omega^+_1 \hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{C/G1}}$ and similarly for $\vec{\boldsymbol{v}}_{\mathrm{C2}}$ thus adds a vector equation (2 scalar equations) to the equation set 17.57. This gives 8 equations in 8 unknowns.

A little cleverness can reduce the problem to one of solving only 4 equations in 4 unknowns. Linear momentum balance for the system, angular momentum balance for the system and angular momentum balance for object 2 make up 4 scalar equations. None of these equations includes the impulse $\vec{\boldsymbol{P}}$. Because the system moves as if hinged at C_1 after the collision, the state of motion after the system is fully characterized by $\vec{\boldsymbol{v}}^+_{\mathrm{G1}}$, ω^+_1, and ω^+_2. Thus we have 4 equations in 4 unknowns.

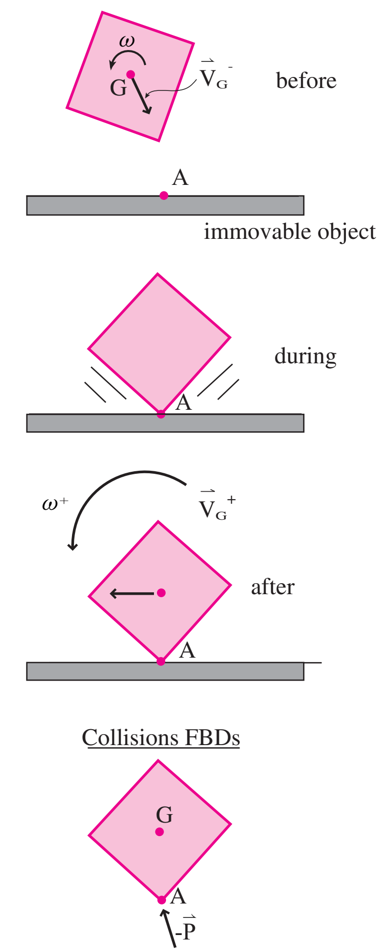

Figure 17.76: Sticking collision with an immovable object. The box sticks at A and then rotates about A. Angular momentum about point A is conserved in the collision.

Example: One body is hugely massive: collision with an immovable object

If body 2, say, is huge compared to body 1 then it can be taken to be immovable and collision problems can be solved by only considering body 1 (see fig. 17.76). In the case of a sticking collision the full state of the system after the collision is determined by ω^+_1. This can be found from the single scalar equation obtained from angular momentum balance about the collision point.

$$\begin{aligned} \vec{\boldsymbol{H}}^-_{\mathrm{A}} &= \vec{\boldsymbol{H}}^+_{\mathrm{A}} \\ \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}^-_{\mathrm{G}} + I^{\mathrm{cm}}_{zz}\omega^-\hat{\boldsymbol{k}} &= \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}^+_{\mathrm{G}} + I^{\mathrm{cm}}_{zz}\omega^+\hat{\boldsymbol{k}} \end{aligned}$$

Because the state of the system before the collision is assumed known (the left "-" side of the equation, and because the post-collision (+) state is a rotation about A, this equation is one scalar equation in the one unknown ω^+. Note that $\vec{\boldsymbol{H}}^+_{\mathrm{A}}$ could also be evaluated as $\vec{\boldsymbol{H}}^+_{\mathrm{A}} = \omega^+ I^A_{zz}\hat{\boldsymbol{k}}$. So one way of expressing the post-collision state is as

$$\omega^+ = \frac{\left(\vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}^-_{\mathrm{G}} + I^{\mathrm{cm}}_{zz}\omega^-\hat{\boldsymbol{k}}\right)\cdot\hat{\boldsymbol{k}}}{I^A_{zz}} \quad \text{and} \quad \vec{\boldsymbol{v}}^+_{\mathrm{G}} = \omega^+\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/A}}.$$

Note also that the same $\vec{\boldsymbol{r}}_{\mathrm{G/A}}$ is used on the right and left sides of the equation because only the velocity and not the position is assumed to jump during the collision.

The collision impulse $\vec{\boldsymbol{P}}$ can then be found from linear momentum balance as

$$\vec{\boldsymbol{P}} = m\left(\vec{\boldsymbol{v}}^+_{\mathrm{G}} - \vec{\boldsymbol{v}}^-_{\mathrm{G}}\right).$$

Sticking collisions are used as models of projectiles hitting targets, of robot and animal limbs making contact with the ground, of monkeys and acrobats grabbing hand holds, and of some particularly dead and frictional collisions between solids (such as when a car trips on a curb).

Frictionless collisions

The second special case is that of a frictionless collision. Here we add two assumptions:

1. There is no friction so $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. The number of unknowns is thus reduced from 8 to 7.
2. There is a coefficient of (normal) restitution e.

The normal restitution coefficient is taken as a property of the colliding bodies. It is a given number with $0 < e < 1$ with this defining equation:

$$(\vec{\boldsymbol{v}}^{+}_{\mathrm{C2}} - \vec{\boldsymbol{v}}^{+}_{\mathrm{C1}}) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}^{-}_{\mathrm{C2}} - \vec{\boldsymbol{v}}^{-}_{\mathrm{C1}}) \cdot \hat{\boldsymbol{n}}.$$

This says that the normal part of the relative velocity of the contacting points reverses sign and its magnitude is attenuated by e. This adds a scalar equation to the set Eqns. 17.57 thus giving 7 scalar equations (4 momentum, 2 angular momentum, 1 restitution) for 7 unknowns (4 velocity components, 2 angular velocities and the normal impulse).

The most popular application of the frictionless collision model is for billiard or pool balls, or carom pucks. These things have relatively small coefficients of friction.

We state without proof that a frictionless collision with $e = 1$ conserves energy.

Example: Pool balls

Assume one ball approaches the other with initial velocity $\vec{\boldsymbol{v}}^{+}_{\mathrm{G1}} = v\hat{\boldsymbol{\imath}}$ and has an elastic frictionless collision with the other ball at a collision angle of θ as shown in fig. 17.77. Defining $\hat{\boldsymbol{n}} \equiv \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$ we have that $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. To determine the outcome of the equation we have the angular momentum balance equations (about the center-of-mass) which trivially tell us that

$$\omega_1^+ = \omega_2^+ = 0$$

because the balls start with no spin and the frictionless collision impulses $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ and $-\vec{\boldsymbol{P}} = -P\hat{\boldsymbol{n}}$ have no moment about the center-of-mass. Linear momentum balance for each of the balls

$$\begin{aligned} -P\hat{\boldsymbol{n}} &= m\vec{\boldsymbol{v}}^{+}_{\mathrm{G1}} - mv\hat{\boldsymbol{\imath}} \\ P\hat{\boldsymbol{n}} &= m\vec{\boldsymbol{v}}^{+}_{\mathrm{G2}} - \vec{\boldsymbol{0}} \end{aligned}$$

gives 4 scalar equations which are supplemented by the restitution equation (using $e = 1$)

$$\begin{aligned} (\Delta\vec{\boldsymbol{v}}^{+}) \cdot \hat{\boldsymbol{n}} &= -e\,(\Delta\vec{\boldsymbol{v}}^{-}) \cdot \hat{\boldsymbol{n}} \\ \Rightarrow -v\cos\theta &= \vec{\boldsymbol{v}}^{+}_{\mathrm{G2}} \cdot \hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}^{+}_{\mathrm{G1}} \cdot \hat{\boldsymbol{n}} \end{aligned}$$

which together make 5 scalar equations in the 5 scalar unknowns $\vec{\boldsymbol{v}}^{+}_{\mathrm{G1}}$, $\vec{\boldsymbol{v}}^{+}_{\mathrm{G2}}$, and P (each vector has 2 unknown components). These have the solution

$$\begin{aligned} \vec{\boldsymbol{v}}^{+}_{\mathrm{G1}} &= v\sin\theta(\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}), \\ \vec{\boldsymbol{v}}^{+}_{\mathrm{G2}} &= v\cos\theta(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}), \quad \text{and} \\ P &= mv\cos\theta. \end{aligned}$$

The solution can be checked by plugging back into the momentum and restitution equations. Also, as promised, this $e = 1$ solution conserves kinetic energy. The solution has the interesting property that the outgoing trajectories of the

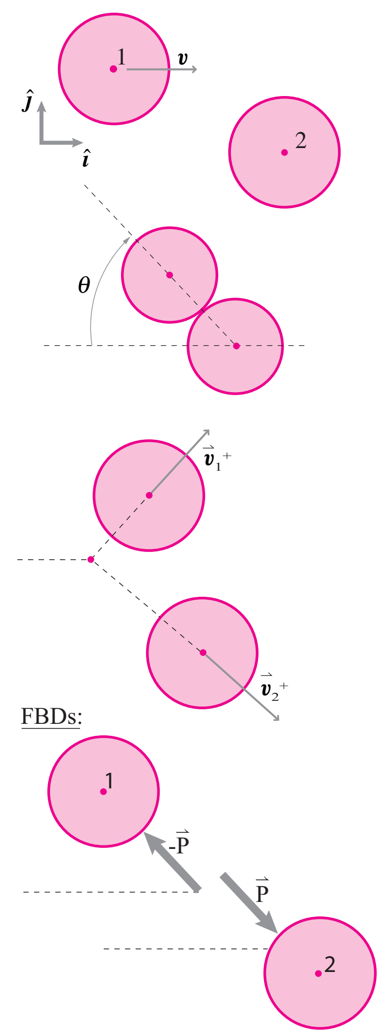

Figure 17.77: Frictionless collision between two identical round objects. Ball one is initially moving to the right, ball 2 is initially stationary. The impulse of ball 1 *on* ball 2 is $\vec{\boldsymbol{P}}$.

two balls are orthogonal for all θ but $\theta = 0$ in which case ball 1 comes to rest in the collision. [The solution can be found graphically by looking for two outgoing vectors which add to the original velocity of mass 1, where the sum of the squares of the outgoing speeds must add to the square of the incoming speed.]

Frictional collisions

For a collision with friction, but not so much that total sticking is accurate, the modeling is complex and subtle. As of this writing there are no standard acceptable ways of dealing with such situations. Commercial simulation packages should be used for such with skeptical caution. They are generally defective in that they can predict only a limited range of phenomena and/or they can create energy even with innocent input parameters.

Why is it hard to find a good collision law

Ideally one would like a rule to determine how bodies move after a collision from how they move before the collision. Such a rule would be called a collision law or a constitutive relation for collisions. That accurate collision laws are rare at best might be surmised from the basic problem that the phrase *rigid body collisions* is in some sense a contradiction in terms, an oxymoron. The force generated in the contact comes from material deformation, and deformation is just what we generally try to neglect when doing rigid body mechanics.

There is a temptation to say that one wants to continue to neglect deformation during the collision, but for an infinitesimal contact region. And some collision laws are formulated with this approach. Even then, there are no reliable models for the deformation in that small region, and such laws are doomed to inaccuracy in situations where the deformation is not so limited.

For complex shaped bodies touching at various points that are generally not known *a priori*, no collision law is reliably accurate.