SAMPLE 17.27 For two masses conservation of momentum is expressed by the vector equation $m_1\vec{\boldsymbol{v}}_1 + \mathrm{m}_2\vec{\boldsymbol{v}}_2 = m_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+$. Suppose $\vec{\boldsymbol{v}}_1 = \vec{\mathbf{0}}$, $\vec{\boldsymbol{v}}_2 = -v_0\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}}_1^+ = v_1^+\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}_2^+ = v_{2_t}^+\hat{\boldsymbol{e}}_t + v_{2_n}^+\hat{\boldsymbol{e}}_n$, where $\hat{\boldsymbol{e}}_t = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_n = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

1. Obtain two independent scalar equations from the momentum equation corresponding to projections in the $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_t$ directions.
2. Assume that you are given another equation $v^+{}_{2_t} = -v_0\sin\theta$. Set up a matrix equation to solve for v_1^+, $v_{2_t}^+$, and $v_{2_n}^+$ from the three equations.

Solution

1. The given equation of conservation of linear momentum is

$$\begin{aligned} m_1 \underbrace{\vec{\boldsymbol{v}}_1}_{\mathbf{0}} + \mathrm{m}_2\vec{\boldsymbol{v}}_2 &= m_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+ \\ \text{or} \qquad -m_2v_0\hat{\boldsymbol{\jmath}} &= m_1v_1^+\hat{\boldsymbol{\imath}} + m_2(v_{2_t}^+\hat{\boldsymbol{e}}_t + v_{2_n}^+\hat{\boldsymbol{e}}_n). \end{aligned} \tag{17.58}$$

Dotting both sides of eqn. (17.58) with $\hat{\boldsymbol{e}}_n$ gives

$$-m_2v_0\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\jmath}})}^{\cos\theta} = m_1v_1^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\imath}})}^{-\sin\theta} + m_2v_{2_t}^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_t)}^{0} + m_2v_{2_n}^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_n)}^{1}$$

$$\text{or} \qquad -m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+. \tag{17.59}$$

Dotting both sides of eqn. (17.58) with $\hat{\boldsymbol{e}}_t$ gives

$$-m_2v_0\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\jmath}})}^{\sin\theta} = m_1v_1^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\imath}})}^{\cos\theta} + m_2v_{2_t}^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_t)}^{1} + m_2v_{2_n}^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_n)}^{0}$$

$$\text{or} \qquad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+. \tag{17.60}$$

$$-m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+, \quad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+$$

2. Now, we rearrange eqn. (17.59) and 17.60 along with the third given equation, $v^+{}_{2_t} = -v_0\sin\theta$, so that all unknowns are on the left hand side and the known quantities are on the right hand side of the equal sign. These equations, in matrix form, are as follows.

$$\begin{bmatrix} -m_1\sin\theta & 0 & m_2 \\ -m_1\cos\theta & m_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v_1^+ \\ v_{2_t}^+ \\ v_{2_n}^+ \end{Bmatrix} = \begin{Bmatrix} -m_2v_0\cos\theta \\ -m_2v_0\sin\theta \\ -v_0\sin\theta \end{Bmatrix}.$$

This equation can be easily solved on a computer for the unknowns.

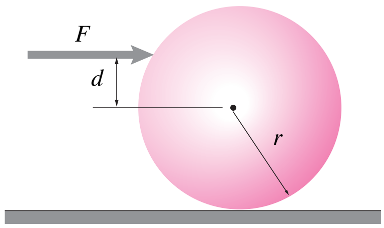

Figure 17.78

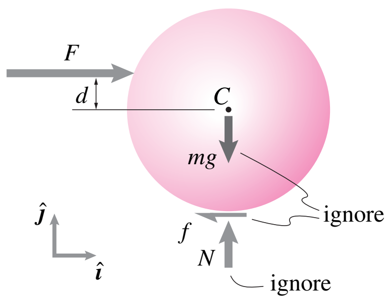

Figure 17.79: FBD of the ball during the strike. The nonimpulsive forces mg, N, and f can be ignored in comparison to the strike force F.

SAMPLE 17.28 Cueing a billiard ball. A billiard ball is cued by striking it horizontally at a distance $d = 10\,\text{mm}$ above the center of the ball. The ball has mass $m = 0.2\,\text{kg}$ and radius $r = 30\,\text{mm}$. Immediately after the strike, the center-of-mass of the ball moves with linear speed $v = 1\,\text{m/s}$. Find the angular speed of the ball immediately after the strike. Ignore friction between the ball and the table during the strike.

Solution Let the force imparted during the strike be F. Since the ball is cued by giving a blow with the cue, F is an impulsive force. Impulsive forces, such as F, are in general so large that all non-impulsive forces are negligible in comparison during the time such forces act. Therefore, we can ignore all other forces (mg, N, f) acting on the ball from its free-body diagram during the strike.

Now, from the linear momentum balance of the ball we get

$$F\hat{\imath} = \dot{\vec{L}} \quad \text{or} \quad (F\hat{\imath})\mathrm{d}t = \mathrm{d}\vec{L} \quad \Rightarrow \quad \int (F\hat{\imath})\mathrm{d}t = \vec{L}_2 - \vec{L}_1$$

where $L_2 - L_1 = \Delta\vec{L}$ is the net change in the linear momentum of the ball during the strike. Since the ball is at rest before the strike, $\vec{L}_1 = m \underbrace{\vec{v}_1}_{0} = \vec{0}$. Immediately after the strike, $\vec{v} = v\hat{\imath} = 1\,\text{m/s}$.

$$\text{Thus} \quad \vec{L}_2 = m\vec{v} = 0.2\,\text{kg}\cdot 1\,\text{m/s}\hat{\imath} = 0.2\,\text{N}\cdot\text{s}\hat{\imath}.$$

$$\text{Hence} \quad \int (F\hat{\imath})\mathrm{d}t = 0.2\,\text{N}\cdot\text{s}\hat{\imath} \quad \text{or} \quad \int F\mathrm{d}t = 0.2\,\text{N}\cdot\text{s}. \tag{17.61}$$

To find the angular speed we apply the angular momentum balance. Let ω be the angular speed immediately after the strike and $\vec{\omega} = \omega\hat{k}$. Now,

$$\sum \vec{M}_{\text{cm}} = \dot{\vec{H}}_{\text{cm}} \quad \Rightarrow \quad \int \sum \vec{M}_{\text{cm}}\,\mathrm{d}t = \int \mathrm{d}\vec{H}_{\text{cm}} = (\vec{H}_{\text{cm}})_2 - (\vec{H}_{\text{cm}})_1.$$

$$\begin{aligned}
\text{Because} \quad \vec{H}_{\text{cm}} &= I_{cm}^{zz}\vec{\omega} \quad \text{and just before the strike, } \vec{\omega} = \vec{0},\\
(\vec{H}_{\text{cm}})_1 &\equiv \text{angular momentum just before the strike } = \vec{0}\\
(\vec{H}_{\text{cm}})_2 &\equiv \text{angular momentum just after the strike } = I_{cm}^{zz}\omega\hat{k},\\
\int \sum \vec{M}_{\text{cm}}\,\mathrm{d}t &= I_{cm}^{zz}\omega\hat{k} = \frac{2}{5}mr^2\omega\hat{k} \text{ (since for a sphere, } I_{cm}^{zz} = \tfrac{2}{5}mr^2).
\end{aligned}$$

$$\begin{aligned}
\text{But} \quad & \sum \vec{M}_{\text{cm}} = -Fd\hat{k},\\
\text{therefore} \quad & -\int (Fd)\mathrm{d}t\hat{k} = \frac{2}{5}mr^2\omega\hat{k}\\
\text{or} \quad & -\underbrace{d}_{\text{constant}}\int F\mathrm{d}t = \frac{2}{5}mr^2\omega \quad \Rightarrow \quad \omega = -\frac{5d}{2mr^2}\int F\mathrm{d}t.
\end{aligned}$$

Substituting the given values and $\int F\mathrm{d}t = 0.2\,\text{N}\cdot\text{s}$ from equation 17.61 we get

$$\omega = -\frac{5(0.01\,\text{m})}{2\cdot 0.2\,\text{kg}\cdot(0.03\,\text{m})^2}\cdot 0.2\,\text{N}\cdot\text{s} = -27.78\,\text{rad/s}.$$

The negative value makes sense because the ball will spin clockwise after the strike, but we assumed that ω was anticlockwise.

$$\boxed{\omega = -27.78\,\text{rad/s}.}$$

SAMPLE 17.29 **Falling stick.** A uniform bar of length ℓ and mass m falls on the ground at an angle θ as shown in the figure. Just before impact at point C, the entire bar has the same velocity v directed vertically downwards. Assume that the collision at C is plastic, *i.e.*, end C of the bar gets stuck to the ground upon impact.

1. Find the angular velocity of the bar just after impact.
2. Assuming θ to be small, find the velocity of end B of the bar just after impact.

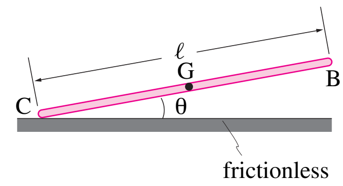

Figure 17.80

Solution We are given that the impact at point C is plastic. That is, end C of the bar has zero velocity after impact. Thus end C gets stuck to the ground. Then we expect the rod to rotate about point C as the rest of the bar moves (perhaps faster) to touch the ground. The free-body diagram of the bar is shown in fig. 17.81 during the impact at point C. Note that we can ignore the force of gravity in comparison to the large impulsive force F_c due to impact at C.

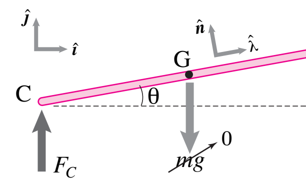

Figure 17.81: The free-body diagram of the bar during collision. The impulsive force at the point of impact C is so large that the force of gravity can be completely ignored in comparison.

1. Now, if we carry out angular momentum balance about point C, there will be no net moment acting on the bar, and therefore, angular momentum about the impact point C is conserved. Distinguishing the kinematic quantities before and after impact with superscripts '-' and '+', respectively, we get from the conservation of angular momentum about point C,

$$\begin{aligned}\vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{H}}_C^+\\ I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}^- + \vec{\boldsymbol{r}}_{\mathrm{G/C}}\times m\vec{\boldsymbol{v}}_G^- &= I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{\boldsymbol{r}}_{\mathrm{G/C}}\times m\vec{\boldsymbol{v}}_G^+.\end{aligned}$$

Now, we know that $\vec{\boldsymbol{\omega}}^- = \vec{\boldsymbol{0}}$ since every point on the bar has the same vertical velocity $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\jmath}}$, and that just after impact, $\vec{\boldsymbol{v}}_G^+ = \vec{\boldsymbol{\omega}}^+\times\vec{\boldsymbol{r}}_{\mathrm{G/C}}$ where we can take $\vec{\boldsymbol{\omega}}^+ = \omega(-\hat{\boldsymbol{k}})$. Thus, 1

$$\begin{aligned}\vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{r}}_{\mathrm{G/C}}\times m\vec{\boldsymbol{v}}_G^- = (\ell/2)\hat{\boldsymbol{\lambda}}\times mv(-\hat{\boldsymbol{\jmath}})\\ &= -\frac{mv\ell}{2}\cos\theta\hat{\boldsymbol{k}} \qquad (\text{since } \hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})\\ \vec{\boldsymbol{H}}_C^+ &= I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{\boldsymbol{r}}_{\mathrm{G/C}}\times m(\vec{\boldsymbol{\omega}}^+\times\vec{\boldsymbol{r}}_{\mathrm{G/C}})\\ &= -I_{zz}^{\mathrm{cm}}\omega\hat{\boldsymbol{k}} + (\ell/2)\hat{\boldsymbol{\lambda}}\times m\underbrace{(-\omega\hat{\boldsymbol{k}}\times\ell/2\hat{\boldsymbol{\lambda}})}_{\omega\ell/2(-\hat{\boldsymbol{n}})}\\ &= -\frac{1}{12}m\ell^2\omega\hat{\boldsymbol{k}} - \frac{1}{4}m\ell^2\omega\hat{\boldsymbol{k}} = -\frac{1}{3}m\ell^2\omega\hat{\boldsymbol{k}}.\end{aligned}$$

Now, equating $\vec{\boldsymbol{H}}_C^-$ and $\vec{\boldsymbol{H}}_C^+$ we get

$$\omega = \frac{3v}{2\ell}\cos\theta, \quad\Rightarrow\quad \vec{\boldsymbol{\omega}} = -\frac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}.$$

$$\boxed{\vec{\boldsymbol{\omega}} = -\frac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}}$$

1 Since C is a fixed point for the motion of the bar after impact, we could calculate $\vec{\boldsymbol{H}}_C^+$ as follows.

$$\vec{\boldsymbol{H}}_C^+ = I_{zz}^C\vec{\boldsymbol{\omega}} = \underbrace{\frac{1}{3}m\ell^2}_{I_{zz}^C}\omega(-\hat{\boldsymbol{k}}).$$

2. The velocity of the end B is now easily found using $\vec{\boldsymbol{v}}_{\mathrm{B}} = \vec{\boldsymbol{v}}_{\mathrm{C}} + \vec{\boldsymbol{v}}_{\mathrm{B/C}} = \vec{\boldsymbol{v}}_{\mathrm{B/C}}$ and $\vec{\boldsymbol{v}}_{\mathrm{B/C}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\mathrm{B/C}}$. Thus,

$$\begin{aligned}\vec{\boldsymbol{v}}_{\mathrm{B/C}} &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\mathrm{B/C}} = -\omega\hat{\boldsymbol{k}}\times\ell\hat{\boldsymbol{\lambda}}\\ &= -\omega\ell\hat{\boldsymbol{n}} = -\frac{3v}{2}\cos\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\end{aligned}$$

but, for small θ, $\cos\theta\approx 1$, and $\sin\theta\approx 0$. Therefore, $\vec{\boldsymbol{v}}_{\mathrm{B/C}} = -\frac{3v}{2}\hat{\boldsymbol{\jmath}}$. Thus, end B of the bar speeds up by one and a half times its original speed due to the plastic impact at C.

$$\boxed{\vec{\boldsymbol{v}}_{\mathrm{B/C}} = -(3/2)v\hat{\boldsymbol{\jmath}}}$$

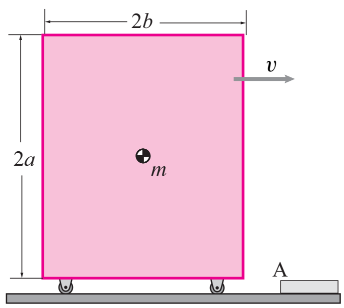

Figure 17.82

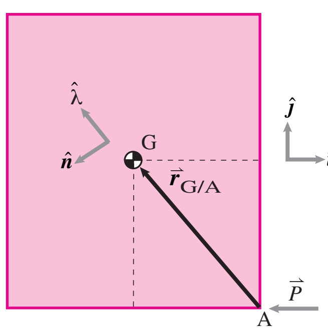

Figure 17.83: The free-body diagram of the box during collision.

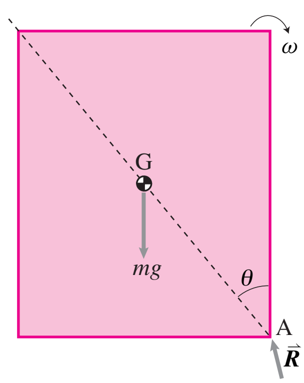

Figure 17.84: The free-body diagram of the box just after the collision is over.

SAMPLE 17.30 **Tipping box.** A box of mass m and dimensions $2a$ and $2b$ moves along a horizontal surface with uniform speed v. Suddenly, it bumps into an obstacle at A. Assume that the impact is plastic and point A is at the lowest level of the box. What is the maximum v the box can have so that it does not tip over after the impact.

Solution Whether the box can tip or not depends on whether it gets sufficient initial angular speed just after collision to overcome the restoring moment due to gravity about the point of rotation A. So, first we need to find the angular velocity of the box immediately following the collision. The free-body diagram of the box during collision is shown in fig. 17.83. There is an impulse $\vec{P}$ acting at the point of impact. If we carry out the angular momentum balance about point A, we see that the impulse at A produces no moment impulse about A, and therefore, the angular momentum about point A has to be conserved. That is, $\vec{H}_{A}{}^{+} = \vec{H}_{A}{}^{-}$. Now,

$$\vec{H}_{A}{}^{-} = \vec{r}_{G/A} \times m\vec{v}_{G}{}^{-} = (-b\hat{\imath} + a\hat{\jmath}) \times mv\hat{\imath} = -mav\hat{k}$$

Let the box have angular velocity $\vec{\omega}^{+} = \omega\hat{k}$ just after impact. Then,

$$\begin{aligned}
\vec{H}_{A}{}^{+} &= I_{zz}^{cm}\vec{\omega}^{+} + \vec{r}_{G/A} \times m\vec{v}_{G}{}^{+} = I_{zz}^{cm}\omega\hat{k} + r\hat{\lambda} \times m(\omega\hat{k} \times r\hat{\lambda}) \\
&= I_{zz}^{cm}\omega\hat{k} + mr^2\omega\hat{k} = \frac{1}{12}(4a^2 + 4b^2)m\omega\hat{k} + m(a^2 + b^2)\omega\hat{k} \\
&= \frac{4}{3}(a^2 + b^2)m\omega\hat{k}.
\end{aligned}$$

Now equating the two momenta, we get

$$\omega^{+} = -\frac{3a}{4(a^2 + b^2)}\,v \quad \Rightarrow \quad \vec{\omega}^{+} = -\frac{3a}{4(a^2 + b^2)}\,v\hat{k}. \tag{17.62}$$

Thus we know the angular velocity immediately after impact. Now let us find out if it is enough to get over the hill, so to speak. Once the impact is over (in a few milliseconds), the usual forces show up on the free-body diagram (see fig. 17.84).

In this post-collision part of the motion energy is conserved. Calling just after collision '+' and the top position 't' (where G is directly above A), using that $I_{zz}^{A} = \frac{4}{3}m(a^2+b^2)$ and solving for the critical condition, that the block has just enough energy to get to the top position, we have,

$$\begin{aligned}
E_{P}{}^{+} + E_{K}{}^{+} &= E_{K}{}^{t} + E_{P}{}^{t} \\
E_{P}{}^{t} - E_{P}{}^{+} &= -(E_{K}{}^{t} - E_{K}{}^{+}) \\
mg(\sqrt{(b^2 + a^2)} - b) &= -\left(0 - \frac{1}{2}(\omega^{+})^2 I_{zz}^{A}\right) \\
(\omega^{+})^2 &= \frac{3}{2}g\frac{\left(1 - \frac{b}{\sqrt{(b^2+a^2)}}\right)}{\sqrt{(b^2 + a^2)}}.
\end{aligned}$$

This tells us how big ω^{+} has to be. Substituting this ω^{+} into eqn. (17.62) we find how big v has to be. We find that the box does not tip over so long as v is small enough, as given by

$$\boxed{v \leq \frac{2}{a}\sqrt{2g(b^2 + a^2)^{3/2}\left(1 - \frac{b}{\sqrt{(b^2+a^2)}}\right)/3}.}$$

SAMPLE 17.31 **Ball hits the bat.** A uniform bar of mass $m_2 = 1\,\text{kg}$ and length $2\ell = 1\,\text{m}$ hangs vertically from a hinge at A. A ball of mass $m_1 = 0.25\,\text{kg}$ comes and hits the bar horizontally at point D with speed $v = 5\,\text{m/s}$. The point of impact D is located at $d = 0.75\,\text{m}$ from the hinge point A. Assume that the collision between the ball and the bar is plastic.

1. Find the velocity of point D on the bar immediately after impact.
2. Find the impulse on the bar at D due to the impact.
3. Find and plot the impulsive reaction at the hinge point A as a function of d, the distance of the point of impact from the hinge point. What is the value of d which makes the impulse at A zero?

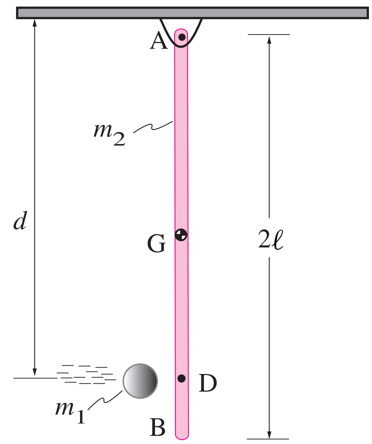

Figure 17.85

Solution The free-body diagram of the ball and the bar as a single system is shown in fig. 17.86 during impact. There is only one external impulsive force $\vec{F}_A$ acting at the hinge point A. We take the ball and the bar together here so that the impulsive force acting between the ball and the bar becomes internal to the system and we are left with only one external force at A. Then, the angular momentum balance about point A yields $\dot{\vec{H}}_A = \vec{0}$ since there is no net moment about A. Thus the angular momentum about A is conserved during the impact.

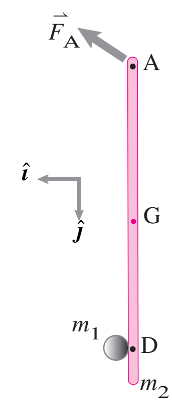

Figure 17.86: The free-body diagram of the ball and the bar together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram.

1. Let us distinguish the kinematic quantities just before impact and immediately after impact with superscripts '-' and '+', respectively. Then, from the conservation of angular momentum about point A, we get $\vec{H}_A^- = \vec{H}_A^+$. Now,

$$\begin{aligned}
\vec{H}_A^- &= (\vec{H}_A^-)_{\text{ball}} + (\vec{H}_A^-)_{\text{bar}} \\
&= \vec{r}_{D/A} \times m_1 \vec{v}^- + I_{zz}^A \vec{\omega}^- \\
&= d\hat{\jmath} \times m_1 v(-\hat{\imath}) + \vec{0} = m_1 dv\hat{k}.
\end{aligned}$$

Similarly,

$$\vec{H}_A^+ = \vec{r}_{D/A} \times m_1 \vec{v}^+ + I_{zz}^A \vec{\omega}^+$$

but, $\vec{v}^+ = \vec{\omega}^+ \times \vec{r}_{D/A} = -\omega d\hat{\imath}$, where $\vec{\omega}^+ = \omega\hat{k}$ (let). Hence,

$$\begin{aligned}
\vec{H}_A^+ &= d\hat{\jmath} \times m_1(-\omega d\hat{\imath}) + \frac{1}{3}m_2(2\ell)^2\omega\hat{k} \\
&= (m_1 d^2 + \frac{4}{3}m_2\ell^2)\omega\hat{k}.
\end{aligned}$$

Equating the two momenta, we get

$$\begin{aligned}
\omega &= \frac{m_1 dv}{m_1 d^2 + (4/3)m_2\ell^2} \\
&= \frac{v}{d\left(1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2\right)} \\
\Rightarrow \quad \vec{v}_D &= \vec{\omega}^+ \times \vec{r}_{D/A} = \omega d(-\hat{\imath}) \\
&= -\frac{v}{1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\imath}.
\end{aligned}$$

Now, substituting the given numerical values, $v = 5\,\text{m/s}$, $m_1 = 0.25\,\text{kg}$, $m_2 = 1\,\text{kg}$, $\ell = 0.5\,\text{m}$, and $d = 0.75\,\text{m}$, we get $\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}$

$$\boxed{\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}}$$

2. To find the impulse at D due to the impact, we can consider either the ball or the bar separately, and find the impulse by evaluating the change in the linear momentum of the body. Let us consider the ball since it has only one impulse acting on it. The

free-body diagram of the ball during impact is shown in fig. 17.87. From the linear impulse-momentum relationship we get,

$$\begin{aligned}\vec{\boldsymbol{P}}_D = \int \vec{\boldsymbol{F}}_{\mathrm{D}}\, \mathrm{d}t &= \vec{\boldsymbol{L}}^+ - \vec{\boldsymbol{L}}^- = m_1(\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}^-) \\ &= m_1\left(-\frac{v}{1+\frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\imath}}\right) \\ &= m_1 v\left(1 - \frac{1}{1+\frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\right)\hat{\boldsymbol{\imath}}.\end{aligned}$$

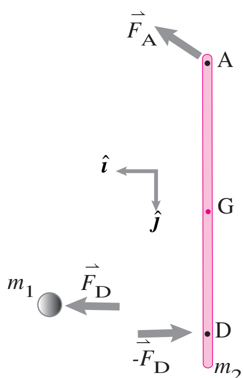

Figure 17.87: Separate free-body diagrams of the ball and the bar during collision.

Substituting the given numerical values, we get $\vec{\boldsymbol{P}}_D = 0.73\,\mathrm{kg{\cdot}m/s}\hat{\boldsymbol{\imath}}$. The impulse on the bar is equal and opposite. Therefore, the impulse on the bar is $-\vec{\boldsymbol{P}}_D = -0.73\,\mathrm{kg{\cdot}m/s}\hat{\boldsymbol{\imath}}$.

$\boxed{\text{Impulse at D} = -0.73\,\mathrm{kg{\cdot}m/s}\hat{\boldsymbol{\imath}}}$

3. Now that we know the impulse at D, we can easily find the impulse at A by applying impulse-momentum relationship to the bar. Since, the bar is stationary just before impact, its initial momentum is zero. Thus, for the bar,

$$\int (\vec{\boldsymbol{F}}_{\mathrm{A}} - \vec{\boldsymbol{F}}_{\mathrm{D}})\mathrm{d}t = \vec{\boldsymbol{L}}^+ - \vec{\boldsymbol{L}}^- = \vec{\boldsymbol{L}}^+ = m_2\vec{\boldsymbol{v}}^+_{\mathrm{cm}}.$$

Denoting the impulse at A with $\vec{\boldsymbol{P}}_A$, the mass ratio m_2/m_1 by m_r, and the length ratio ℓ/d by q, and noting that $\vec{\boldsymbol{v}}^+_{\mathrm{cm}} = \omega\hat{\boldsymbol{k}} \times \ell\hat{\boldsymbol{\jmath}} = -\omega\ell\hat{\boldsymbol{\imath}}$, we get

$$\begin{aligned}\vec{\boldsymbol{P}}_A &\equiv \int \vec{\boldsymbol{F}}_{\mathrm{A}}\, \mathrm{d}t = \int \vec{\boldsymbol{F}}_{\mathrm{D}}\, \mathrm{d}t + m_2(-\omega\ell\hat{\boldsymbol{\imath}}) \\ &= m_1 v\left(1 - \frac{1}{1+\frac{4}{3}m_r q^2}\right)\hat{\boldsymbol{\imath}} - m_2\ell\frac{v}{d\left(1+\frac{4}{3}m_r q^2\right)}\hat{\boldsymbol{\imath}} \\ &= m_1 v\left(\frac{\frac{4}{3}m_r q^2}{1+\frac{4}{3}m_r q^2}\right)\hat{\boldsymbol{\imath}} - m_2 v\left(\frac{q}{1+\frac{4}{3}m_r q^2}\right)\hat{\boldsymbol{\imath}} \\ &= \frac{(4/3)m_2 q^2 - m_2 q}{1+\frac{4}{3}m_r q^2}v\hat{\boldsymbol{\imath}} = \frac{q(4q-3)}{3\left(1+\frac{4}{3}m_r q^2\right)}m_2 v\hat{\boldsymbol{\imath}}.\end{aligned}$$

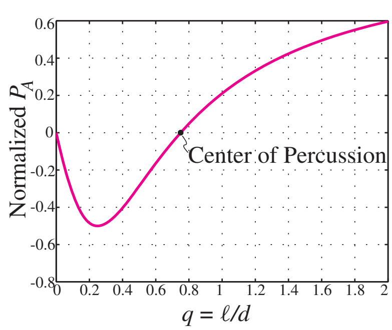

Figure 17.88: Plot of normalized impulse at A as a function of $q = \ell/d$.

Now, we are ready to graph the impulse at A as a function of $q \equiv \ell/d$. However, note that a better quantity to graph will be $P_A/(m_1 v)$, that is, the nondimensional impulse at A, normalized with respect to the initial linear momentum $m_1 v$ of the ball. The plot is shown in fig. 17.88. It is clear from the plot, as well as from the expression for $\vec{\boldsymbol{P}}_A$, that the impulse at A is zero when $q = 3/4$ or $d = 4\ell/3 = 2/3(2\ell)$, that is, when the ball strikes at two thirds the length of the bar. Note that this location of the impact point is independent of the mass ratio m_r.

$\boxed{d = 2/3(2\ell) \text{ for } \vec{\boldsymbol{P}}_A = \vec{\boldsymbol{0}}}$

Comment: This particular point of impact D (when $d = 2/3(2\ell)$) which induces no impulse at the support point A is called the *center of percussion.* If you imagine the bar to be a bat or a racquet and point A to be the location of your grip, then hitting a ball at D gives you an impulse-free shot. In sports, point D is called a *sweet spot.*

SAMPLE 17.32 **Flying dish collides with solar panel.** A uniform rectangular plate of dimensions $2a = 2\,\mathrm{m}$ and $2b = 1\,\mathrm{m}$ and mass $m_P = 2\,\mathrm{kg}$ drifts in space at a uniform speed $v_p = 10\,\mathrm{m/s}$ (relative to some local Newtonian reference frame) in the direction shown. A circular disk of radius $R = 0.25\,\mathrm{m}$ and mass $m_D = 1\,\mathrm{kg}$ is heading towards the plate at a linear speed $v_0 = v_D = 1\,\mathrm{m/s}$ directed normal to the facing edge of the plate. In addition, the disk is spinning at $\omega_D = 5\,\mathrm{rad/s}$ in the clockwise direction. The plate and the disk collide at point A of the plate, located at $d = 0.8\,\mathrm{m}$ from the center of the long edge. Assume that the collision is frictionless and purely elastic. Find the linear and angular velocities of the plate and the disk immediately after the collision.

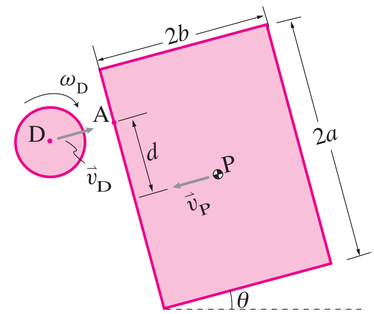

Figure 17.89

Solution We are given the linear and angular velocities of the two bodies before their collision. We have to find their linear and angular velocities after the collision. Let the linear velocities of the plate and the disk after the collision be $\vec{\boldsymbol{v}}_P^+$ and $\vec{\boldsymbol{v}}_D^+$, respectively. We assume that the collision takes place in the plane of the two bodies (that is, all motions before and after collision are planar). Then the unknown angular velocities of the plate and the disk after the collision can be assumed as $\vec{\boldsymbol{\omega}}_P^+ = \omega_P^+\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\omega}}_D^+ = \omega_D^+\hat{\boldsymbol{k}}$. Thus, in total, we have 6 scalar unknowns here — 4 for linear velocities of the disk and the plate (each velocity has two components) and 2 for the two angular velocities.

Since we are trying to find velocities after a collision, given the velocities before the collision, we can use linear and angular momentum-impulse relations. Let $\vec{\boldsymbol{P}}$ be the impulse acting during the collision. Note that the collision is frictionless and hence $\vec{\boldsymbol{P}}$ must act normal to the two colliding surfaces. Therefore, the direction of the impulse is known, only its magnitude is not known. Thus, in total, we have 7 unknowns now. To solve for them, we need 7 independent equations.

We have 6 independent equations from the linear and angular impulse-momentum balance for the two bodies (3 each). We need one more equation. That equation is the relationship between the normal components of the relative velocities of approach and departure with the coefficient of restitution e (=1 for elastic collision). Thus we have 7 equations for 7 unknowns. First we set them up, then we solve them with a calculator or computer.

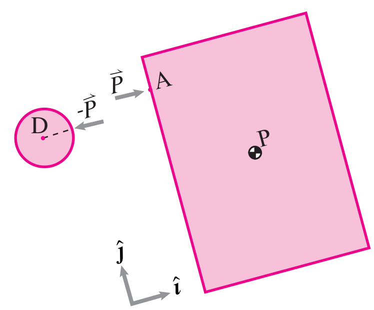

Figure 17.90: Separate free-body diagrams of the disk and the plate during collision. The impulse $\vec{\boldsymbol{P}}$ acts normal to the colliding surfaces and hence we can assume that on the plate, $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{\imath}}$. Therefore, the impulse on the disk is $-P\hat{\boldsymbol{\imath}}$.

The free-body diagrams of the disk and the plate together and the two separately are shown in fig. 17.91 and 17.90, respectively. Using an xy coordinate system oriented as shown in fig. 17.91, we can write

$$
\begin{array}{lrcl}
\text{LMB for disk:} & m_D(\vec{\boldsymbol{v}}_D^+ - \vec{\boldsymbol{v}}_D^-) & = & -P\hat{\boldsymbol{\imath}} \\
\text{LMB for plate:} & m_P(\vec{\boldsymbol{v}}_P^+ - \vec{\boldsymbol{v}}_P^-) & = & P\hat{\boldsymbol{\imath}} \\
\text{AMB for disk:} & I_D^{\mathrm{cm}}(\vec{\boldsymbol{\omega}}_D^+ - \vec{\boldsymbol{\omega}}_D^-) & = & \vec{\boldsymbol{0}} \\
\text{AMB for plate:} & I_P^{\mathrm{cm}}(\vec{\boldsymbol{\omega}}_P^+ - \vec{\boldsymbol{\omega}}_P^-) & = & \vec{\boldsymbol{r}}_{\mathrm{A/P}} \times P\hat{\boldsymbol{\imath}} \\
\text{kinematics:} & \vec{\boldsymbol{v}}_{A_D}^+ - \vec{\boldsymbol{v}}_{A_P}^+ & = & e(\vec{\boldsymbol{v}}_{A_P}^- - \vec{\boldsymbol{v}}_{A_D}^-)
\end{array}
$$

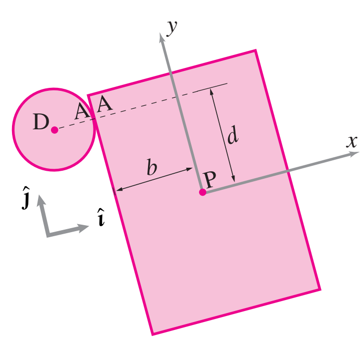

Figure 17.91: The free-body diagram of the disk and the plate together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram .

where, in the last equation $\vec{\boldsymbol{v}}_{A_D}$ and $\vec{\boldsymbol{v}}_{A_P}$ refer to the velocities of the material points located at A on the disk and on the plate, respectively. Other linear velocities in the equations above refer to the velocities at the center-of-mass of the corresponding bodies. We are given that $\vec{\boldsymbol{v}}_D^- = v_D\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}}_P^- = -v_P\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{\omega}}_D^- = -\omega_D\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{\omega}}_P^- = \vec{\boldsymbol{0}}$. Let us assume that $\vec{\boldsymbol{\omega}}_D^+ = \omega_D^+\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{\omega}}_P^+ = \omega_P^+\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{v}}_D^+ = v_{D_x}^+\hat{\boldsymbol{\imath}} + v_{D_y}^+\hat{\boldsymbol{\jmath}}$, and similarly, $\vec{\boldsymbol{v}}_P^+ = v_{P_x}^+\hat{\boldsymbol{\imath}} + v_{P_y}^+\hat{\boldsymbol{\jmath}}$. Then,

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{A_D}^- &= \vec{\boldsymbol{v}}_D^- + \vec{\boldsymbol{\omega}}_D^- \times \vec{\boldsymbol{r}}_{\mathrm{A/O}} = v_D\hat{\boldsymbol{\imath}} - \omega_D R\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{v}}_{A_D}^+ &= \vec{\boldsymbol{v}}_D^+ + \vec{\boldsymbol{\omega}}_D^+ \times \vec{\boldsymbol{r}}_{\mathrm{A/O}} = v_{D_x}^+\hat{\boldsymbol{\imath}} + (v_{D_y}^+ + \omega_D^+ R)\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{v}}_{A_P}^- &= \vec{\boldsymbol{v}}_P^- = -v_P\hat{\boldsymbol{\imath}} \\
\vec{\boldsymbol{v}}_{A_P}^+ &= \vec{\boldsymbol{v}}_P^+ + \vec{\boldsymbol{\omega}}_P^+ \times \vec{\boldsymbol{r}}_{\mathrm{A/P}} = (v_{P_x}^+ - \omega_P^+ d)\hat{\boldsymbol{\imath}} + (v_{P_y}^+ - \omega_P^+ b)\hat{\boldsymbol{\jmath}}.
\end{aligned}
$$

Substituting these quantities in the kinematics equation above and dotting with $\hat{\imath}$, the unit normal at A, we get

$$v^+_{D_x} - v^+_{P_x} + \omega^+_P d = \underbrace{e}_{1}(-v_P - v_D) = -v_P - v_D. \tag{17.63}$$

Now, let us extract the scalar equations from the impulse-momentum equations for the disk and the plate by dotting with appropriate unit vectors.

Dotting LMB for the disk with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_D(v^+_{D_x} - v_D) = -P \tag{17.64}$$

$$m_D v^+_{D_y} = 0. \tag{17.65}$$

Dotting LMB for the plate with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_P(v^+_{P_x} + v_P) = P \tag{17.66}$$

$$m_P v^+_{P_y} = 0. \tag{17.67}$$

Dotting AMB for the disk and the plate with $\hat{k}$, we get

$$I^{\mathrm{cm}}_D(\omega^+_D + \omega_D) = 0 \tag{17.68}$$

$$I^{\mathrm{cm}}_P \omega^+_P = -Pd. \tag{17.69}$$

We have all the equations we need. Let us rearrange these equations in a matrix form, taking the known quantities to the right and putting all unknowns to the left side. We then, write eqns. (17.64)–(17.69), and then eqn. (17.63) as

$$\begin{bmatrix} m_D & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_P & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & m_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{\mathrm{cm}}_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{\mathrm{cm}}_P & d \\ 1 & 0 & -1 & 0 & 0 & d & 0 \end{bmatrix} \begin{Bmatrix} v^+_{D_x} \\ v^+_{D_y} \\ v^+_{P_x} \\ v^+_{P_y} \\ \omega^+_D \\ \omega^+_P \\ P \end{Bmatrix} = \begin{Bmatrix} m_D v_D \\ 0 \\ -m_P v_P \\ 0 \\ -I^{\mathrm{cm}}_D \omega_D \\ 0 \\ -v_P - v_D \end{Bmatrix}.$$

Substituting the given numerical values for the masses and the pre-collision velocities, and the moments of inertia, $I^{\mathrm{cm}}_D = (1/2)m_D R^2$ and $I^{\mathrm{cm}}_P = (1/12)m_P(4a^2 + 4b^2)$, and then solving the matrix equation on a computer, we get,

$$\begin{aligned} \vec{\boldsymbol{v}}^+_D &= -8.70\,\mathrm{m/s}\hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{v}}^+_P = -5.15\,\mathrm{m/s}\hat{\boldsymbol{\imath}} \\ \vec{\boldsymbol{\omega}}^+_D &= -5\,\mathrm{rad/s}\hat{\boldsymbol{k}}, \quad \vec{\boldsymbol{\omega}}^+_P = -9.31\,\mathrm{rad/s}\hat{\boldsymbol{k}} \\ P &= 9.70\,\mathrm{kg{\cdot}m/s}. \end{aligned}$$

You can easily check that the results obtained satisfy the conservation of linear momentum for the plate and the disk taken together as one system.

Comments: In this particular problem, the equations are simple enough to be solved by hand. For example, eqns. (17.65), (17.67), and (17.68) are trivial to solve and immediately give, $v^+_{D_y} = 0, v^+_{P_y} = 0$, and $\omega^+_D = \omega^D = -5\,\mathrm{rad/s}$. The rest of the equations can be solved by usual eliminations and substitutions, etc. The key is that you have 7 linearly independent equations for the 7 unknowns.