17.4 Mechanics of contacting bodies: rolling and sliding

A typical machine part has forces that come from contact with other parts. In fact, with the major exceptions of

- Gravity,
- Electromagnetic forces inside motors, and
- Magnetic attraction/repulsion

most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free-body diagrams have been contact forces: The force of the ground on a wheel, of an axle on a bearing, of any part on any other part it touches.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid objects in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book):

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}, \tag{17.40}$$

$$\dot{\vec{\boldsymbol{H}}}_{/\text{C}} = \vec{\boldsymbol{r}}_{\text{cm}/\text{C}} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}) + I\dot{\omega}\hat{\boldsymbol{k}} \qquad (\text{where} I = I_{zz}^{\text{cm}}) \tag{17.41}$$

$$E_{\text{K}} = m_{\text{tot}}v_{cm}^2/2 + I\omega^2/2 \tag{17.42}$$

The key to success, as usual, is the drawing of appropriate free-body diagrams (see Chapter 3). For rolling and sliding the two cases one needs to consider as possible are, self-evidently, rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than μN, and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of μN but oppose the relative motion.

Work-energy relations and impulse-momentum relations are useful to solve some problems both with and without slip. In pure rolling contact the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated. The rate of loss of kinetic and potential energy is

$$\text{Rate of frictional dissipation} = P_{diss} = F_{\text{friction}} \cdot v_{\text{slip}} \tag{17.43}$$

where v_{slip} is the relative velocity of the contacting slipping points. If either the friction force (ideal lubrication) or sliding velocity (no slip) is zero there is no dissipation.

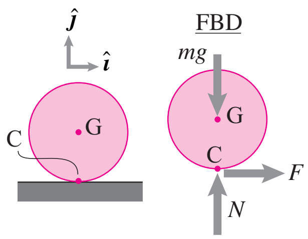

Figure 17.49: A ball rolls or slides on level ground.

As for various problems throughout the text, it is often a savings of calculation to use angular momentum balance (or moment balance in statics) relative to a point where there are unknown reaction forces. For rolling and slipping problems this often means making use of contact points.

Example: Pure rolling on level ground

A ball or wheel rolling on level ground, with no air friction etc, rolls at constant speed (see fig. 17.49). This is most directly deduced from angular momentum balance about the contact point C:

$$\begin{array}{lcccc} \vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} & \Rightarrow & \vec{\boldsymbol{r}}_{\mathrm{G/C}} \times -mg\hat{\boldsymbol{\jmath}} & = & \vec{\boldsymbol{r}}_{\mathrm{G/C}} \times m\vec{\boldsymbol{a}}_{\mathrm{G}} + \dot{\omega} I_{zz}^{\mathrm{cm}} \hat{\boldsymbol{k}} \\ & \Rightarrow & \vec{\boldsymbol{0}} & = & R\hat{\boldsymbol{\jmath}} \times (-m\dot{\omega}R\hat{\boldsymbol{\imath}}) + \dot{\omega} I_{zz}^{\mathrm{cm}} \hat{\boldsymbol{k}} \\ \text{dotting with } \hat{\boldsymbol{k}} & \Rightarrow & \dot{\omega} = 0 & \Rightarrow & \omega = \text{constant.} \end{array}$$

Because for rolling $v_G = -\omega R$ we thus have that v_G is a constant. [The result can also be obtained by combining angular momentum balance about the center-of-mass with linear momentum balance.]

Finally, linear momentum balance gives the reaction force at C to be $\vec{\boldsymbol{F}}_{tot} = F\hat{\boldsymbol{\imath}} + n\hat{\boldsymbol{\jmath}} = mg\hat{\boldsymbol{\jmath}}$. So,

assuming point contact, there is no rolling resistance.

Example: Bowling ball with initial sliding

A bowling ball is released with an initial speed of v_0 and no rotation rate. What is its subsequent motion? To start with, the motion is incompatible with rolling, the bottom of the ball is sliding to the right. So there is a frictional force which opposes motion and $F = -\mu N$ (see fig. 17.49). Linear and angular momentum balance give:

$$\begin{array}{lccl} \text{LMB:} & & \Rightarrow & \{-F\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}} - mg\hat{\boldsymbol{\jmath}} = ma\hat{\boldsymbol{\imath}}\} \\ & \{\}\cdot\hat{\boldsymbol{\jmath}} & \Rightarrow & N = mg \\ & \{\}\cdot\hat{\boldsymbol{\imath}} & \Rightarrow & a = -\mu g \\ \text{AMB}_{/G}: & & \Rightarrow & -R\mu mg = I_{zz}^{\mathrm{cm}}\dot{\omega} \end{array}$$

$$\Rightarrow \quad v = v_0 - \mu g t \quad \text{and} \quad \omega = -\mu R m g t / I_{zz}^{\mathrm{cm}}$$

Thus the forward speed of the ball decreases linearly with time while the counter-clockwise angular velocity decreases linearly with time.

This solution is only appropriate so long as there is rightward slip, $v_G > -\omega R$. Just like for a sliding block, there is no impetus for reversal, and the ball switches to pure rolling when

$$v = -\omega R \Rightarrow v_0 - \mu g t = -\left(-\mu R m g t / I_{zz}^{\mathrm{cm}}\right) R \Rightarrow t = \frac{v_0}{\mu g \left(1 + \frac{mR^2}{I_{zz}^{\mathrm{cm}}}\right)}.$$

Note that the energy lost during sliding is less than μmg times the distance the center of the ball moves during slip.

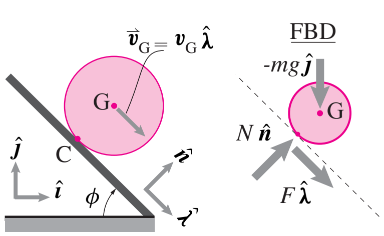

Figure 17.50: A ball rolls or slides down a slope.

Example: Ball rolling down hill.

Assuming rolling we can find the acceleration of a ball as it rolls downhill (see fig. 17.50). We start out with the kinematic observations that $\vec{\boldsymbol{a}}_{\mathrm{G}} = a_{\mathrm{G}}\hat{\boldsymbol{\lambda}}$, that

$R\omega = -v_{\mathrm{G}}$ and that $R\dot{\omega} = -a_{\mathrm{G}}$. Angular momentum balance about the stationary point on the ground instantaneously coinciding with the contact point gives

$$
\begin{aligned}
\mathrm{AMB}_{/\mathrm{C}} \quad &\Rightarrow \quad \vec{\boldsymbol{r}}_{\mathrm{G}/\mathrm{C}} \times (-mg\hat{\boldsymbol{\jmath}}) = \vec{\boldsymbol{r}}_{\mathrm{G}/\mathrm{C}} \times m\vec{\boldsymbol{a}}_{\mathrm{G}} + I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
& \qquad \{-R\sin\phi mg\hat{\boldsymbol{k}} = (R\hat{\boldsymbol{n}}) \times (ma_{\mathrm{G}}\hat{\boldsymbol{\lambda}}) + I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{\boldsymbol{k}}\} \\
\{\}\cdot\hat{\boldsymbol{k}} \quad &\Rightarrow \quad -Rmg\sin\phi = -Rma_{\mathrm{G}} - I_{zz}^{\mathrm{cm}}a_{\mathrm{G}}/R \\
&\Rightarrow \quad a_{\mathrm{G}} = \tfrac{g\sin\phi}{1+I_{zz}^{\mathrm{cm}}/(mR^2)}.
\end{aligned}
$$

Which is less than the acceleration of a block sliding on a ramp without friction: $a = g\sin\phi$ (unless the mass of the rolling ball is concentrated at the center with $I_{zz}^{\mathrm{cm}} = 0$). Note that a very small ball rolls just as slowly. In the limit as the ball radius goes to zero the behavior does not approach that of a point mass that slides; the rolling remains significant.

Example: Ball rolling down hill: energy approach

We can find the acceleration of the rolling ball using power balance or conservation of energy. For example

$$
\begin{aligned}
0 = \tfrac{d}{dt}E_{\mathrm{T}} \Rightarrow \qquad 0 &= \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} \\
&= \tfrac{d}{dt}\left(mv^2/2 + I_{zz}^{\mathrm{cm}}\omega^2/2\right) + \tfrac{d}{dt}(mgy) \\
&= mv\dot{v} + I_{zz}^{\mathrm{cm}}\omega\dot{\omega} + mg\dot{y} \\
&= mv\dot{v} + I_{zz}^{\mathrm{cm}}(v/R)\dot{v}/R - mg(\sin\phi)v \\
\text{assuming } v \neq 0 \Rightarrow \qquad 0 &= (m + I_{zz}^{\mathrm{cm}}/R^2)\dot{v} - mg\sin\phi \\
\Rightarrow \qquad \dot{v} &= \tfrac{g\sin\phi}{1+I_{zz}^{\mathrm{cm}}/(mR^2)}
\end{aligned}
$$

as before.

Example: Does the ball slide?

How big is the coefficient of friction μ needed to prevent slip for a ball rolling down a hill? Use angular momentum balance to find the normal and frictional components of the contact force, using the rolling example above.

$$
\begin{aligned}
\mathrm{LMB}\ (\vec{\boldsymbol{F}}_{\mathrm{tot}} = m\vec{\boldsymbol{a}}_{\mathrm{G}}) \qquad &\Rightarrow \quad \{N\hat{\boldsymbol{n}} + F\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{\jmath}} = ma_{\mathrm{G}}\hat{\boldsymbol{\lambda}}\} \\
\{\}\cdot\hat{\boldsymbol{n}} \quad &\Rightarrow \quad N = mg\cos\phi \\
\{\}\cdot\hat{\boldsymbol{\lambda}} \quad &\Rightarrow \quad F + mg\sin\phi = m\tfrac{g\sin\phi}{1+I_{zz}^{\mathrm{cm}}/(mR^2)} \\
& \qquad F = \tfrac{-mg\sin\phi}{1+mR^2/I_{zz}^{\mathrm{cm}}} \\
\text{Critical condition:} \qquad &\Rightarrow \quad \mu = \tfrac{|F|}{N} = \tfrac{\tan\phi}{1+mR^2/I_{zz}^{\mathrm{cm}}}
\end{aligned}
$$

If I_{zz}^{cm} is very small (the mass concentrated near the center of the ball) then small friction is needed to prevent rolling. For a uniform rubber ball on pavement (with $\mu \approx 1$ and $I_{zz}^{\mathrm{cm}} \approx 2mR^2/5$) the steepest slope for rolling without slip is a steep $\phi = \tan^{-1}(7/2) \approx 74^{\circ}$. A metal hoop on the other hand (with $\mu \approx .3$ and $I_{zz}^{\mathrm{cm}} \approx mR^2$) will only roll without slip for slopes less than about $\phi = \tan^{-1}(.6) \approx 31^{\circ}$.

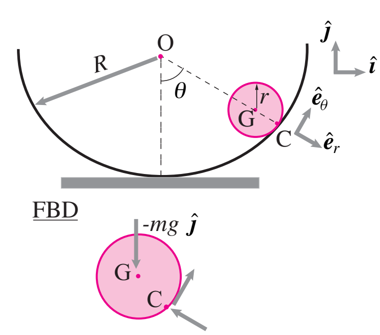

Figure 17.51: A ball rolls in a round cross-section bowl.

Example: Oscillations of a ball in a bowl.
A round ball can oscillate back and forth in the bottom of a circular cross section bowl or pipe (see fig. 17.51). Similarly, a cylindrical object can roll inside a pipe. What is the period of oscillation? Start with angular momentum balance about the contact point

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{G/C}} \times (-mg\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{r}}_{\mathrm{G/C}} \times m\vec{\boldsymbol{a}}_{\mathrm{G}} + I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
rmg\sin\theta\hat{\boldsymbol{k}} &= -r\hat{\boldsymbol{e}}_r \times \left(m\left((R-r)\ddot{\theta}\hat{\boldsymbol{e}}_\theta - (R-r)\dot{\theta}^2\hat{\boldsymbol{e}}_r\right)\right) \\
&\quad + I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}
$$

Evaluating the cross products (using that $\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$) and using the kinematics from the previous section (that $(R-r)\dot{\theta} = -r\omega$) and dotting the left and right sides with $\hat{\boldsymbol{k}}$ gives

$$(R-r)\ddot{\theta} = \frac{-g\sin\theta}{1 + I_{zz}^{\mathrm{cm}}/mr^2},$$

the tangential acceleration is the same as would have been predicted by putting the ball on a constant slope of $-\theta$. Using the small angle approximation that $\sin\theta = \theta$ the equation can be rearranged as a standard harmonic oscillator equation

$$\ddot{\theta} + \left(\frac{g}{(R-r)(1 + I_{zz}^{\mathrm{cm}}/mr^2)}\right)\theta = 0.$$

If all the ball's mass were concentrated in its middle, keeping r a fixed non-zero size, (so $I_{zz}^{\mathrm{cm}} = 0$, like a lead pellet inside a styrofoam ball) this is the same as for a simple pendulum with length $R - r$. For any parameter values the period of small oscillation is

$$T = 2\pi\sqrt{\frac{(R-r)(1 + I_{zz}^{\mathrm{cm}}/mr^2)}{g}}.$$

For a marble or ball bearing in a sideways glass (with $R - r \approx 4\,\mathrm{cm} = .04\,\mathrm{m}$, $I_{zz}^{\mathrm{cm}}/mr^2 \approx 2/5$ and $g \approx 10\,\mathrm{m/s^2}$) this gives about one oscillation every half second. See page 1029 for the energy approach to this problem.