# 17.4 Mechanics of contacting bodies: rolling and sliding

A typical machine part has forces that come from contact with other parts. In fact, with the major exceptions of

- Gravity,
- Electromagnetic forces inside motors, and
- Magnetic attraction/repulsion

most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free-body diagrams have been contact forces: The force of the ground on a wheel, of an axle on a bearing, of any part on any other part it touches.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid objects in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book):

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}, \tag{17.40}$$

$$\dot{\vec{\boldsymbol{H}}}_{/\text{C}} = \vec{\boldsymbol{r}}_{\text{cm/C}} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}) + I\dot{\omega}\hat{\boldsymbol{k}} \qquad (\text{where} I = I_{zz}^{\text{cm}}) \tag{17.41}$$

$$E_{\text{K}} = m_{\text{tot}}v_{cm}^2/2 + I\omega^2/2 \tag{17.42}$$

The key to success, as usual, is the drawing of appropriate free-body diagrams (see Chapter 3). For rolling and sliding the two cases one needs to consider as possible are, self-evidently, rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than $\mu N$, and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of $\mu N$ but oppose the relative motion.

Work-energy relations and impulse-momentum relations are useful to solve some problems both with and without slip. In pure rolling contact the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated. The rate of loss of kinetic and potential energy is

$$\text{Rate of frictional dissipation} = P_{diss} = F_{\text{friction}} \cdot v_{\text{slip}} \tag{17.43}$$

where $v_{\text{slip}}$ is the relative velocity of the contacting slipping points. If either the friction force (ideal lubrication) or sliding velocity (no slip) is zero there is no dissipation.

As for various problems throughout the text, it is often a savings of calculation to use angular momentum balance (or moment balance in statics) relative to a point where there are unknown reaction forces. For rolling and slipping problems this often means making use of contact points.

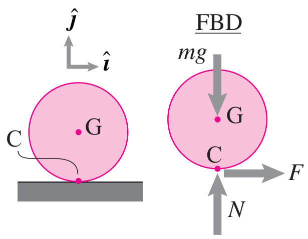



Figure 17.49: A ball rolls or slides on level ground.

**Example: Pure rolling on level ground**
A ball or wheel rolling on level ground, with no air friction etc, rolls at constant speed (see fig. 17.49). This is most directly deduced from angular momentum balance about the contact point C:

$$\begin{aligned}
\vec{\boldsymbol{M}}_C = \dot{\vec{\boldsymbol{H}}}_C \quad &\Rightarrow \quad \vec{\boldsymbol{r}}_{G/C} \times -mg\hat{\boldsymbol{\jmath}} &&= \quad \vec{\boldsymbol{r}}_{G/C} \times m\vec{\boldsymbol{a}}_G + \dot{\omega} I_{zz}^{cm}\hat{\boldsymbol{k}} \\
&\Rightarrow \quad \vec{\boldsymbol{0}} &&= \quad R\hat{\boldsymbol{\jmath}} \times (-m\dot{\omega}R\hat{\boldsymbol{\imath}}) + \dot{\omega} I_{zz}^{cm}\hat{\boldsymbol{k}} \\
\text{dotting with } \hat{\boldsymbol{k}} \quad &\Rightarrow \quad \dot{\omega} = 0 &&\Rightarrow \quad \omega = \text{constant.}
\end{aligned}$$

Because for rolling $v_G = -\omega R$ we thus have that $v_G$ is a constant. [The result can also be obtained by combining angular momentum balance about the center-of-mass with linear momentum balance.]

Finally, linear momentum balance gives the reaction force at C to be $\vec{\boldsymbol{F}}_{tot} = F\hat{\boldsymbol{\imath}} + n\hat{\boldsymbol{\jmath}} = mg\hat{\boldsymbol{\jmath}}$. So,

assuming point contact, there is no rolling resistance.

**Example: Bowling ball with initial sliding**
A bowling ball is released with an initial speed of $v_0$ and no rotation rate. What is its subsequent motion? To start with, the motion is incompatible with rolling, the bottom of the ball is sliding to the right. So there is a frictional force which opposes motion and $F = -\mu N$ (see fig. 17.49). Linear and angular momentum balance give:

$$\begin{aligned}
\text{LMB:} \qquad &\Rightarrow \quad \{-F\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}} - mg\hat{\boldsymbol{\jmath}} = ma\hat{\boldsymbol{\imath}}\} \\
\{\}\cdot\hat{\boldsymbol{\jmath}} \qquad &\Rightarrow \quad N = mg \\
\{\}\cdot\hat{\boldsymbol{\imath}} \qquad &\Rightarrow \quad a = -\mu g \\
\text{AMB}_{/G}\text{:} \qquad &\Rightarrow \quad -R\mu mg = I_{zz}^{cm}\dot{\omega}
\end{aligned}$$

$$\Rightarrow \qquad v = v_0 - \mu g t \qquad \text{and} \qquad \omega = -\mu R m g t / I_{zz}^{cm}$$

Thus the forward speed of the ball decreases linearly with time while the counter-clockwise angular velocity decreases linearly with time.

This solution is only appropriate so long as there is rightward slip, $v_G > -\omega R$. Just like for a sliding block, there is no impetus for reversal, and the ball switches to pure rolling when

$$v = -\omega R \Rightarrow v_0 - \mu g t = -\left(-\mu R m g t / I_{zz}^{cm}\right) R \Rightarrow t = \frac{v_0}{\mu g \left(1 + \frac{mR^2}{I_{zz}^{cm}}\right)}.$$

Note that the energy lost during sliding is less than $\mu mg$ times the distance the center of the ball moves during slip.

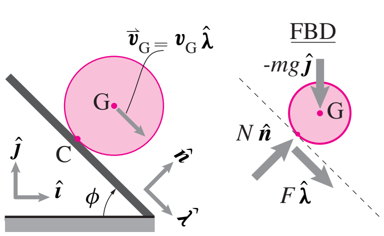



Figure 17.50: A ball rolls or slides down a slope.

**Example: Ball rolling down hill.**
Assuming rolling we can find the acceleration of a ball as it rolls downhill (see fig. 17.50). We start out with the kinematic observations that $\vec{\boldsymbol{a}}_G = a_G\hat{\boldsymbol{\lambda}}$, that

$R\omega = -v_G$ and that $R\dot{\omega} = -a_G$. Angular momentum balance about the stationary point on the ground instantaneously coinciding with the contact point gives

$$
\begin{aligned}
\text{AMB}_{/C} \quad &\Rightarrow \quad \vec{r}_{G/C} \times (-mg\hat{\boldsymbol{\jmath}}) = \vec{r}_{G/C} \times m\vec{a}_G + I^{cm}_{zz}\dot{\omega}\hat{\boldsymbol{k}} \\
& \qquad \{-R\sin\phi mg\hat{\boldsymbol{k}} = (R\hat{\boldsymbol{n}}) \times (ma_G\hat{\boldsymbol{\lambda}}) + I^{cm}_{zz}\dot{\omega}\hat{\boldsymbol{k}}\} \\
\{\}\cdot\hat{\boldsymbol{k}} \quad &\Rightarrow \quad -Rmg\sin\phi = -Rma_G - I^{cm}_{zz}a_G/R \\
&\Rightarrow \quad a_G = \frac{g\sin\phi}{1+I^{cm}_{zz}/(mR^2)}.
\end{aligned}
$$

Which is less than the acceleration of a block sliding on a ramp without friction: $a = g\sin\phi$ (unless the mass of the rolling ball is concentrated at the center with $I^{cm}_{zz} = 0$). Note that a very small ball rolls just as slowly. In the limit as the ball radius goes to zero the behavior does not approach that of a point mass that slides; the rolling remains significant.

**Example: Ball rolling down hill: energy approach**

We can find the acceleration of the rolling ball using power balance or conservation of energy. For example

$$
\begin{aligned}
0 = \frac{d}{dt}E_T \Rightarrow \qquad 0 &= \dot{E}_K + \dot{E}_P \\
&= \frac{d}{dt}\left(mv^2/2 + I^{cm}_{zz}\omega^2/2\right) + \frac{d}{dt}(mgy) \\
&= mv\dot{v} + I^{cm}_{zz}\omega\dot{\omega} + mg\dot{y} \\
&= mv\dot{v} + I^{cm}_{zz}(v/R)\dot{v}/R - mg(\sin\phi)v \\
\text{assuming } v \neq 0 \Rightarrow \qquad 0 &= (m + I^{cm}_{zz}/R^2)\dot{v} - mg\sin\phi \\
\Rightarrow \qquad \dot{v} &= \frac{g\sin\phi}{1+I^{cm}_{zz}/(mR^2)}
\end{aligned}
$$

as before.

**Example: Does the ball slide?**

How big is the coefficient of friction $\mu$ needed to prevent slip for a ball rolling down a hill? Use angular momentum balance to find the normal and frictional components of the contact force, using the rolling example above.

$$
\begin{aligned}
\text{LMB } (\vec{F}_{tot} = m\vec{a}_G) \qquad &\Rightarrow \quad \{N\hat{\boldsymbol{n}} + F\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{\jmath}} = ma_G\hat{\boldsymbol{\lambda}}\} \\
\{\}\cdot\hat{\boldsymbol{n}} \quad &\Rightarrow \quad N = mg\cos\phi \\
\{\}\cdot\hat{\boldsymbol{\lambda}} \quad &\Rightarrow \quad F + mg\sin\phi = m\frac{g\sin\phi}{1+I^{cm}_{zz}/(mR^2)} \\
& \qquad F = \frac{-mg\sin\phi}{1+mR^2/I^{cm}_{zz}} \\
\text{Critical condition:} \qquad &\Rightarrow \quad \mu = \frac{|F|}{N} = \frac{\tan\phi}{1+mR^2/I^{cm}_{zz}}
\end{aligned}
$$

If $I^{cm}_{zz}$ is very small (the mass concentrated near the center of the ball) then small friction is needed to prevent rolling. For a uniform rubber ball on pavement (with $\mu \approx 1$ and $I^{cm}_{zz} \approx 2mR^2/5$) the steepest slope for rolling without slip is a steep $\phi = \tan^{-1}(7/2) \approx 74^\circ$. A metal hoop on the other hand (with $\mu \approx .3$ and $I^{cm}_{zz} \approx mR^2$) will only roll without slip for slopes less than about $\phi = \tan^{-1}(.6) \approx 31^\circ$.

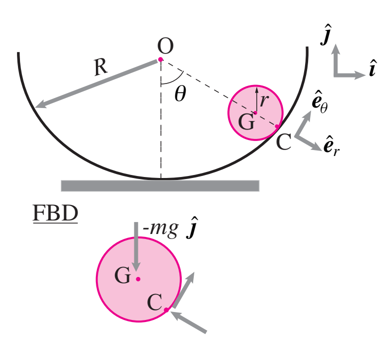



Figure 17.51: A ball rolls in a round cross-section bowl.

**Example: Oscillations of a ball in a bowl.**
A round ball can oscillate back and forth in the bottom of a circular cross section bowl or pipe (see fig. 17.51). Similarly, a cylindrical object can roll inside a pipe. What is the period of oscillation? Start with angular momentum balance about the contact point

$$\begin{aligned}\vec{\boldsymbol{r}}_{G/C} \times (-mg\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{r}}_{G/C} \times m\vec{\boldsymbol{a}}_G + I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}} \\ rmg\sin\theta\hat{\boldsymbol{k}} &= -r\hat{\boldsymbol{e}}_r \times \left(m\left((R-r)\ddot{\theta}\hat{\boldsymbol{e}}_\theta - (R-r)\dot{\theta}^2\hat{\boldsymbol{e}}_r\right)\right) \\ &\quad + I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

Evaluating the cross products (using that $\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$) and using the kinematics from the previous section (that $(R-r)\dot{\theta} = -r\omega$) and dotting the left and right sides with $\hat{\boldsymbol{k}}$ gives

$$(R-r)\ddot{\theta} = \frac{-g\sin\theta}{1 + I_{zz}^{cm}/mr^2},$$

the tangential acceleration is the same as would have been predicted by putting the ball on a constant slope of $-\theta$. Using the small angle approximation that $\sin\theta = \theta$ the equation can be rearranged as a standard harmonic oscillator equation

$$\ddot{\theta} + \left(\frac{g}{(R-r)(1 + I_{zz}^{cm}/mr^2)}\right)\theta = 0.$$

If all the ball's mass were concentrated in its middle, keeping $r$ a fixed non-zero size, (so $I_{zz}^{cm} = 0$, like a lead pellet inside a styrofoam ball) this is the same as for a simple pendulum with length $R-r$. For any parameter values the period of small oscillation is

$$T = 2\pi\sqrt{\frac{(R-r)(1 + I_{zz}^{cm}/mr^2)}{g}}.$$

For a marble or ball bearing in a sideways glass (with $R - r \approx 4\,\text{cm} = .04\,\text{m}$, $I_{zz}^{cm}/mr^2 \approx 2/5$ and $g \approx 10\,\text{m/s}^2$) this gives about one oscillation every half second. See page 1029 for the energy approach to this problem.

**SAMPLE 17.16** **Equation of motion of a driven wheel** Consider a wheel of mass $m$ and radius $R$ driven by an axle force $\vec{F}$ as shown in the figure. The wheel rolls to the left without slipping. Write the equation of motion of the wheel.

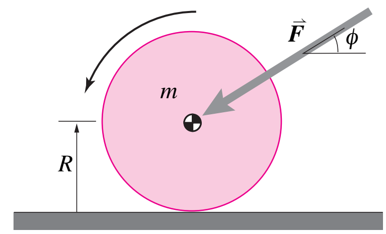



Figure 17.52

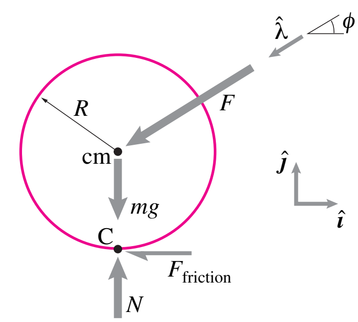



Figure 17.53: FBD of a wheel with mass $m$. Force $F$ is applied by the axle.

**Solution** The free-body diagram of the wheel is shown in figure 17.53. We can write the equation of motion of the wheel in terms of either the center-of-mass position $x$ or the angular displacement of the wheel $\theta$. Since in pure rolling, these two variables share a simple relationship ($x = R\theta$), we can easily get the equation of motion in terms of $x$ if we have the equation in terms of $\theta$ and vice versa. Let $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{k}}$.

Since all the forces are shown in the free-body diagram, we can readily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force, $-mg\hat{\boldsymbol{j}}$, the friction force $-F_{\text{friction}}\hat{\boldsymbol{\imath}}$, and the normal reaction of the ground $N\hat{\boldsymbol{j}}$, all pass through the contact point $C$ and therefore, produce no moment about this point). We have

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{/\text{C}}$$

where

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\text{C}} &= \overbrace{\vec{\boldsymbol{r}}_{cm/C}}^{R\hat{\jmath}} \times (F\hat{\boldsymbol{\lambda}}) \\
&= R\hat{\boldsymbol{j}} \times F(-\cos\phi\,\hat{\boldsymbol{\imath}} - \sin\phi\,\hat{\boldsymbol{j}}) \\
&= FR\cos\phi\,\hat{\boldsymbol{k}}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\text{C}} &= \vec{\boldsymbol{r}}_{\text{cm/C}} \times m\vec{\boldsymbol{a}}_{\text{cm}} + I_{zz}^{\text{cm}}\dot{\vec{\boldsymbol{\omega}}} \\
&= R\hat{\boldsymbol{j}} \times m\underbrace{\ddot{x}}_{-\dot{\omega}R}\hat{\boldsymbol{\imath}} + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
&= m\dot{\omega}R^2\hat{\boldsymbol{k}} + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
&= (I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
FR\cos\phi\,\hat{\boldsymbol{k}} &= (I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}
\end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute $\ddot{\theta} = -\ddot{x}/R$ in the equation of motion above to get the equation of motion in terms of the center-of-mass displacement $x$ as

$$\ddot{x} = -\frac{FR^2\cos\phi}{I_{zz}^{\text{cm}} + mR^2}.$$

$$\boxed{\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}}$$

**Comments:** We could have, of course, used linear momentum balance with angular momentum balance about the center-of-mass to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the $x$ and $y$ directions involving all forces shown in the free-body diagram . The angular momentum balance , on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

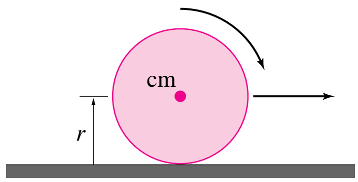



Figure 17.54

**SAMPLE 17.17** **Energy and power of a rolling wheel.** A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft· lbf. Assume the wheel to be a thin, uniform disk.

1. Find the rate of rotation of the wheel.
2. Find the average power required to bring the wheel to a complete stop in 5 s.

**Solution**

1. Let $\omega$ be the rate of rotation of the wheel. Since the wheel rotates without slip, its center-of-mass moves with speed $v_{cm} = \omega r$. The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$
\begin{aligned}
E_K &= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I^{cm} \omega^2 \\
&= \frac{1}{2} m \omega^2 r^2 + \frac{1}{2} I^{cm} \omega^2 \\
&= \frac{1}{2} (m r^2 + \underbrace{I^{cm}}_{\frac{1}{2} m r^2}) \omega^2 \\
&= \frac{3}{4} m r^2 \omega^2 \\
\Rightarrow \quad \omega^2 &= \frac{4 E_K}{3 m r^2} \\
&= \frac{4 \times 1700\,\mathrm{ft \cdot lbf}}{3 \times 20\,\mathrm{lbm} \cdot 1\,\mathrm{ft}^2} \\
&= \frac{4 \times 1700 \times 32.2\,\cancel{\mathrm{lb}}\mathrm{m} \cdot \cancel{\mathrm{f}}\mathrm{t/s^2}}{3 \times 20\,\cancel{\mathrm{lb}}\mathrm{m} \cdot \cancel{\mathrm{f}}\mathrm{t}} \\
&= 3649.33 \frac{1}{\mathrm{s}^2} \\
\Rightarrow \quad \omega &= 60.4\,\mathrm{rad/s}.
\end{aligned}
$$

$\boxed{\omega = 60.4\,\mathrm{rad/s}}$

**Note:** This rotational speed, by the way, is extremely high. At this speed the center-of-mass moves at 60.4 ft/s!

2. Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the average power is,

$$
\begin{aligned}
P &= \frac{E_{K_1} - E_{K_2}}{\Delta t} \\
&= \frac{1700\,\mathrm{ft \cdot lbf} - 0}{5\,\mathrm{s}} = 340\,\mathrm{ft \cdot lbf/s} \\
&= 340\,\mathrm{ft \cdot lbf/s} \cdot \frac{1\,\mathrm{hp}}{550\,\mathrm{ft \cdot lbf/s}} \\
&= 0.62\,\mathrm{hp}
\end{aligned}
$$

$\boxed{P = 0.62\,\mathrm{hp}}$

**SAMPLE 17.18 Equation of motion of a rolling wheel from energy balance.** Consider the wheel with mass $m$ from figure 17.55. The free-body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance. Assume CCW (counter clockwise) rotation is positive.

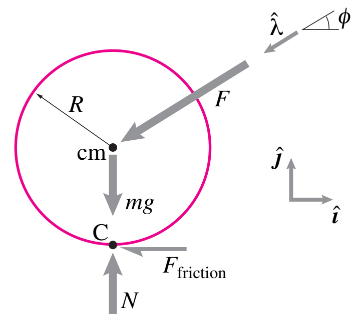



Figure 17.55: FBD of a rolling wheel.

**Solution** From energy balance, we have

$$P = \dot{E}_{\text{K}}$$

where

$$\begin{aligned}
P &= \sum \vec{F}_i \cdot \vec{v}_i \\
&= -F_{\text{friction}}\hat{\imath} \cdot \overbrace{\vec{v}_C}^{\vec{0}} + N\hat{\jmath} \cdot \overbrace{\vec{v}_C}^{\vec{0}} - mg\hat{\jmath} \cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} + F\hat{\lambda} \cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} \\
&= -mgv(\underbrace{\hat{\imath} \cdot \hat{\jmath}}_{0}) + Fv(\underbrace{\hat{\lambda} \cdot \hat{\imath}}_{-\cos\phi}) \\
&= -Fv\cos\phi
\end{aligned}$$

and

$$\begin{aligned}
\dot{E}_{\text{K}} &= \frac{d}{dt}\Big(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{zz}^{\text{cm}}\overbrace{\omega^2}^{(\dot{x}/R)^2}\Big) \\
&= \frac{1}{2}\frac{d}{dt}\left[\left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}^2\right] \\
&= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}\ddot{x}.
\end{aligned}$$

Thus,

$$\begin{aligned}
-Fv\cos\phi &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}\ddot{x} \\
\text{or} \quad -F\cancel{\dot{x}}\cos\phi &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\cancel{\dot{x}}\ddot{x} \\
\Rightarrow \quad \ddot{x} &= -\frac{F\cos\phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.
\end{aligned}$$

We can also write the equation of motion in terms of $\theta$ by replacing $\ddot{x}$ with $-\ddot{\theta}R$ giving,

$$\ddot{\theta} = \frac{(F/R)\cos\phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.$$

$$\boxed{\ddot{x} = -\frac{F\cos\phi}{m + I_{zz}^{\text{cm}}/R^2}}$$

**Comments:** In the equations above (for calculating $P$), we have set $\vec{v}_C = \vec{0}$ because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center-of-mass. So, the only power supplied to the wheel is due to the force $F\hat{\lambda}$ acting at the center-of-mass.

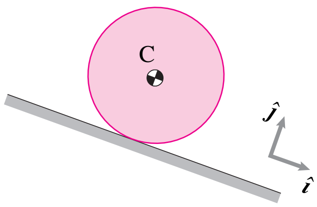



Figure 17.56

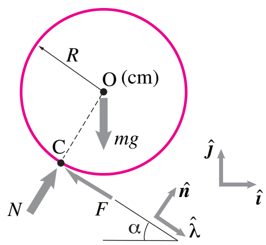



Figure 17.57

**SAMPLE 17.19 Equation of motion of a rolling disk on an incline.** A uniform circular disk of mass $m = 1\,\text{kg}$ and radius $R = 0.4\,\text{m}$ rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center-of-mass in 2 s.

**Solution** The free-body diagram of the disk is shown in fig. 17.57. In addition to the base unit vectors $\hat{\imath}$ and $\hat{\jmath}$, let us use unit vectors $\hat{\lambda}$ and $\hat{n}$ along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the other hand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

$$\sum \vec{M}_C = \dot{\vec{H}}_{/C}$$

where

$$\begin{aligned}\sum \vec{M}_C &= \vec{r}_{O/C} \times m\vec{g} = R\hat{n} \times (-mg\hat{\jmath}) \\ &= -Rmg\sin\alpha\hat{k}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{H}}_{/C} &= -I_{zz}^{cm}\dot{\omega}\hat{k} + \overbrace{\vec{r}_{O/C}}^{R\hat{n}} \times m\overbrace{\vec{a}_{cm}}^{R\dot{\omega}\hat{\lambda}} \\ &= -I_{zz}^{cm}\dot{\omega}\hat{k} + mR^2\dot{\omega}(\hat{n}\times\hat{\lambda}) \\ &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k}.\end{aligned}$$

Thus,

$$\begin{aligned}-Rmg\sin\alpha\hat{k} &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k} \\ \Rightarrow \quad \dot{\omega} &= \frac{g\sin\alpha}{R[1 + I_{zz}^{cm}/(mR^2)]}.\end{aligned}$$

$$\boxed{\dot{\omega} = \frac{g\sin\alpha}{R[1+I_{zz}^{cm}/(mR^2)]}}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for $\omega$ and $\theta$ by simply integrating this equation and substituting the initial conditions $\omega(t = 0) = 0$ and $\theta(t = 0) = 0$. Let us write the equation of motion as $\dot{\omega} = \beta$ where $\beta = g\sin\alpha/R(1 + I_{zz}^{cm}/mR^2)$. Then,

$$\begin{aligned}\omega &\equiv \dot{\theta} = \beta t + C_1 \\ \theta &= \frac{1}{2}\beta t^2 + C_1 t + C_2.\end{aligned}$$

Substituting the given initial conditions $\dot{\theta}(0) = 0$ and $\theta(0) = 0$, we get $C_1 = 0$ and $C_2 = 0$, which implies that $\theta = \frac{1}{2}\beta t^2$. Now, in pure rolling, $x = R\theta$. Therefore,

$$\begin{aligned}x(t) &= R\theta(t) = \frac{1}{2}\beta t^2 = \cancel{R}\cdot\frac{1}{2}\frac{g\sin\alpha}{\cancel{R}(1 + I_{zz}^{cm}/mR^2)}t^2 \\ &= \frac{1}{2}\frac{g\sin\alpha}{1 + \frac{\frac{1}{2}mR^2}{mR^2}}t^2 = \frac{1}{3}(g\sin\alpha)t^2 \\ x(2\,\text{s}) &= \frac{1}{3}\cdot 9.8\,\text{m/s}^2\cdot\sin(30^\circ)\cdot(2\,\text{s})^2 = 6.53\,\text{m}.\end{aligned}$$

$$\boxed{x(2\,\text{s}) = 6.53\,\text{m}}$$

**SAMPLE 17.20 Using Work and energy in pure rolling.** Consider the disk of Sample 17.19 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center-of-mass when the disk is 2 m down the inclined plane.

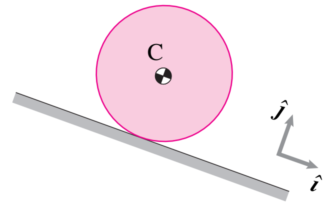



Figure 17.58

**Solution** We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center-of-mass after it has travelled 2 malong the incline. We can, of course, solve this problem using equation of motion, by first solving for the time $t$ the disk takes to travel the given distance and then evaluating the expression for speed $\omega(t)$ or $x(t)$ at that $t$. However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let $\omega_1$ and $\omega_2$ be the initial and final (after rolling down by $d = 2\,\mathrm{m}$) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

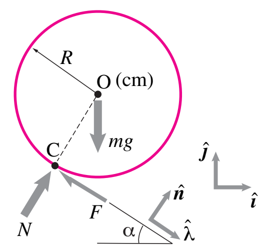



Figure 17.59

$$E_K = \frac{1}{2} m \overbrace{v_{cm}^2}^{(\omega R)^2} + \frac{1}{2} I_{zz}^{cm} \omega^2 = \frac{1}{2}(mR^2 + I_{zz}^{cm})\omega^2.$$

Therefore,

$$\Delta E_K = E_{K_2} - E_{K_1} = \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2). \tag{17.44}$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of the mass-center by $d$ along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{\jmath}) \cdot (d\hat{\lambda}) = -mgd(\overbrace{\hat{\jmath} \cdot \hat{\lambda}}^{-\sin\alpha}) = mgd \sin\alpha. \tag{17.45}$$

From work-energy principle (integral form of power balance, $P = \dot{E}_K$ ), we know that $W = \Delta E_K$. Therefore, from eqn. (17.44) and eqn. (17.45), we get

$$\begin{aligned}
mgd\sin\alpha &= \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2) \\
\Rightarrow \quad \omega_2^2 &= \omega_1^2 + \frac{2mgd\sin\alpha}{mR^2 + I_{zz}^{cm}} = \omega_1^2 + \frac{2gd\sin\alpha}{R^2\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)} \\
&= \omega_1^2 + \frac{4gd\sin\alpha}{3R^2}.
\end{aligned}$$

Substituting the values of $g, d, \alpha, R$, etc., and setting $\omega_1 = 0$, we get

$$\begin{aligned}
\omega_2^2 &= \frac{4 \cdot (9.8\,\mathrm{m/s^2}) \cdot (2\,\mathrm{m}) \cdot (\sin(30^\circ))}{3 \cdot (0.4\,\mathrm{m})^2} = 81.67/s^2 \\
\Rightarrow \quad \omega_2 &= 9.04\,\mathrm{rad/s}.
\end{aligned}$$

The corresponding speed of the center-of-mass is

$$v_{cm} = \omega_2 R = 9.04\,\mathrm{rad/s} \cdot 0.4\,\mathrm{m} = 3.61\,\mathrm{m/s}.$$

$$\boxed{v_{cm} = 3.61\,\mathrm{m/s}}$$

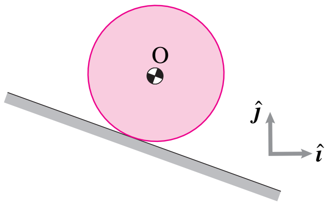



Figure 17.60

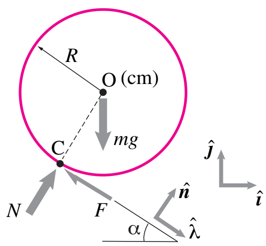



Figure 17.61

**SAMPLE 17.21 Impulse and momentum calculations in pure rolling.** Consider the disk of Sample 17.19 rolling down the incline again. Find an expression for the rolling speed ($\omega$) of the disk after a finite time $\Delta t$, given the rolling speed $\omega_1$ at some instant $t_1$.

**Solution** Once again, this problem can be solved by integrating the equation of motion (as done in Sample 17.19). However, we will solve this problem here using impulse-momentum relationship. Let the desired angular speed at $t_2 = t_1 + \Delta t$ be $\omega_2$. We need to find $\omega_2$, given the $\omega_1$ at $t = t_1$. Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relationship, $\sum \vec{F} \cdot \Delta t = \vec{L}_2 - \vec{L}_1$, we have

$$(-F\hat{\lambda} + N\hat{\imath} - mg\hat{\jmath})\Delta t = m(v_2 - v_1)\hat{\lambda}. \tag{17.46}$$

Dotting eqn. (17.46) with $\hat{\lambda}$ gives

$$\begin{aligned}
(-F - mg\underbrace{(\hat{\jmath}\cdot\hat{\lambda})}_{-\sin\alpha})\Delta t &= m(v_2 - v_1) \\
(-F + mg\sin\alpha)\Delta t &= mR(\omega_2 - \omega_1). \qquad (17.47)
\end{aligned}$$

Similarly, the angular impulse-momentum relationship about the mass-center, $\vec{M}_{\text{O}}\Delta t = (\vec{H}_{/\text{O}})_2 - (\vec{H}_{/\text{O}})_1$, gives

$$\begin{aligned}
(-FR\hat{k})\Delta t &= -I_{zz}^{\text{cm}}(\omega_2 - \omega_1)\hat{k} \\
\Rightarrow \quad FR\Delta t &= I_{zz}^{\text{cm}}(\omega_2 - \omega_1). \qquad (17.48)
\end{aligned}$$

Note that the other forces ($N$ and $mg$) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force $F$ from eqn. (17.47) and eqn. (17.48) by multiplying eqn. (17.47) with $R$ and adding to eqn. (17.48):

$$\begin{aligned}
(-F + mg\sin\alpha)\Delta t \cdot R + FR\Delta t &= mR(\omega_2 - \omega_1)\cdot R + I_{zz}^{\text{cm}}(\omega_2 - \omega_1) \\
\text{or} \qquad mgR\sin\alpha\Delta t &= (I_{zz}^{\text{cm}} + mR^2)(\omega_2 - \omega_1) \\
\text{or} \qquad g\sin\alpha\Delta t &= R\left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)(\omega_2 - \omega_1) \\
\Rightarrow \qquad \omega_2 &= \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)}\Delta t.
\end{aligned}$$

$$\boxed{\omega_2 = \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)}\Delta t}$$

**Note:** From the answer obtained, we clearly see that $\omega_2 = \omega_1 + \dot{\omega}\Delta t$ where $\dot{\omega}$ is the angular acceleration given by the expression $\dot{\omega} = \frac{g\sin\alpha}{R(1+I_{zz}^{\text{cm}}/mR^2)}$. This is the same expression we obtained for the angular acceleration in Sample 17.19.

**SAMPLE 17.22 Falling ladder.** A ladder AB, modeled as a uniform rigid rod of mass $m$ and length $\ell$, rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at $\theta = \theta_o$ ($\theta_o < \pi/2$). Assume the motion to be planar (in the vertical plane).

1. As the ladder falls, what is the path of the center-of-mass of the ladder?
2. Find the equation of motion (e.g., a differential equation in terms of $\theta$ and its time derivatives) for the ladder.
3. How does the angular speed $\omega$ ($= \dot{\theta}$) depend on $\theta$?

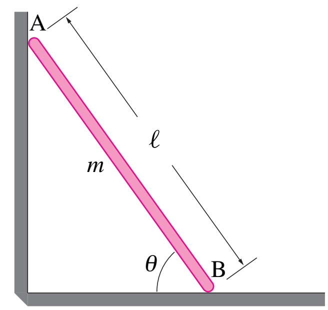



Figure 17.62: A ladder, modeled as a uniform rod of mass $m$ and length $\ell$, falls from a rest position at $\theta = \theta_o$ ($< \pi/2$) such that its ends slide along frictionless vertical and horizontal surfaces.

**Solution** Since the ladder is modeled by a uniform rod AB, its center-of-mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle $\theta$ (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

1. **Path of the center-of-mass:** Let the origin of our $x$-$y$ coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 17.63). The position vector of the center-of-mass G may be written as

$$\begin{aligned} \vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell\cos\theta\,\hat{\imath} + \frac{\ell}{2}(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= \frac{\ell}{2}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \end{aligned} \tag{17.49}$$

Thus the coordinates of the center-of-mass are

$$x_G = \frac{\ell}{2}\cos\theta \quad \text{and} \quad y_G = \frac{\ell}{2}\sin\theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius $\frac{\ell}{2}$. Therefore, the center-of-mass of the ladder follows a circular path of radius $\frac{\ell}{2}$ centered at the origin. Of course, the center-of-mass traverses only that part of the circle which lies between its initial position at $\theta = \theta_o$ and the final position at $\theta = 0$.

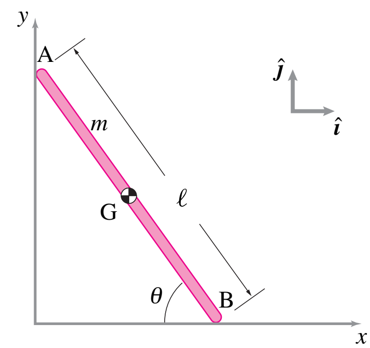



Figure 17.63:

2. **Equation of motion:** The free-body diagram of the ladder is shown in Fig. 17.64. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the ladder we get

$$N_1\hat{\imath} + (N_2 - mg)\hat{\jmath} = m\vec{a}_G = m\ddot{\vec{r}}_G.$$

Differentiating eqn. (17.49) twice we get $\ddot{\vec{r}}_G$ as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\hat{\imath} + (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\jmath}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by $\hat{\imath}$ and then by $\hat{\jmath}$ we get

$$\begin{aligned} N_1 &= -\frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg. \end{aligned}$$

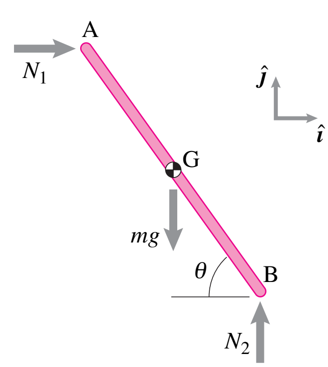



Figure 17.64: The free-body diagram of the ladder.

Next, we write the angular momentum balance for the ladder about its center-of-mass, $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$, where

$$\begin{aligned}
\sum \vec{M}_{/G} &= \left(-N_1 \frac{\ell}{2}\sin\theta + N_2 \frac{\ell}{2}\cos\theta\right)\hat{k} \\
&= \frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\frac{\ell}{2}\sin\theta\hat{k} \\
&\quad + \left[\frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg\right]\frac{\ell}{2}\cos\theta\hat{k} \\
&= \left(\frac{1}{4}m\ell^2\ddot{\theta} + \frac{1}{2}mg\ell\cos\theta\right)\hat{k}
\end{aligned}$$

and

$$\dot{\vec{H}}_{/G} = I_{zz/G}\dot{\vec{\omega}} = \frac{1}{12}m\ell^2\ddot{\theta}(-\hat{k}),$$

where $\dot{\vec{\omega}} = \ddot{\theta}(-\hat{k})$ because $\theta$ is measured positive in the clockwise direction $(-\hat{k})$. Now, equating the two quantities $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ and dotting both sides with $\hat{k}$ we get

$$\begin{aligned}
\frac{1}{4}\cancel{m}\ell^2\ddot{\theta} + \frac{1}{2}\cancel{m}g\ell\cos\theta &= -\frac{1}{12}\cancel{m}\ell^2\ddot{\theta} \\
\text{or} \qquad \left(\frac{1}{12} + \frac{1}{4}\right)\ell^2\ddot{\theta} &= -\frac{1}{2}g\ell\cos\theta \\
\text{or} \qquad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta
\end{aligned} \tag{17.50}$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for $\theta(t)$.

3. **Angular Speed of the ladder:** To solve for the angular speed $\omega\ (=\dot{\theta})$ as a function of $\theta$ we need to express eqn. (17.50) in terms of $\omega$, $\theta$, and derivatives of $\omega$ with respect to $\theta$. Now,

$$\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}.$$

Substituting in eqn. (17.50) and integrating both sides from the initial rest position to an arbitrary position $\theta$ we get

$$\begin{aligned}
\int_0^{\omega}\omega\, d\omega &= -\int_{\theta_0}^{\theta}\frac{3g}{2\ell}\cos\theta\, d\theta \\
\Rightarrow \qquad \frac{1}{2}\omega^2 &= -\frac{3g}{2\ell}(\sin\theta - \sin\theta_0) \\
\Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}.
\end{aligned}$$

Since end B is sliding to the right, $\theta$ is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, *i.e.,*

$$\vec{\omega} = \dot{\theta}(-\hat{k}) = -\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}\,\hat{k}.$$

**Note:** To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. If a real ladder were sliding against a slippery wall and floor, would it not lose contact? The answer is yes, it would. One way of finding when the contact would be lost is to calculate the normal reaction $N_1$ and finding out at what value of $\theta$ it passes through zero. The answer depends on $\theta_0$. For example, if $\theta_0 = 80^{\circ}$, then $N_1$ is zero at about $\theta = 41°$. You can verify this result by substituting the expressions for $\ddot{\theta}$ and $\dot{\theta}$ in the expression for $N_1$ and solving for $\theta$ when $N_1 = 0$.

**SAMPLE 17.23** **The falling ladder again.** Consider the falling ladder of Sample 17.10 again. The mass of the ladder is $m$ and the length is $\ell$. The ladder is released from rest at $\theta = 80^{\circ}$.

1. At the instant when $\theta = 45^{\circ}$, find the speed of the center-of-mass of the ladder using energy.
2. Derive the equation of motion of the ladder using work-energy balance.

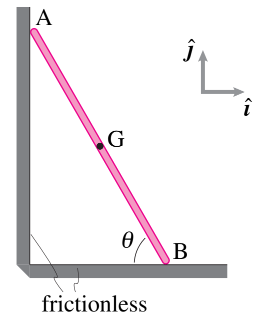



Figure 17.65

### Solution

1. Since there is no friction, there is no loss of energy between the two states: $\theta_0 = 80^{\circ}$ and $\theta_f = 45^{\circ}$. The only external forces on the ladder are $N_1$, $N_2$, and $mg$ as shown in the free-body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls, $N_1$ and $N_2$, respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

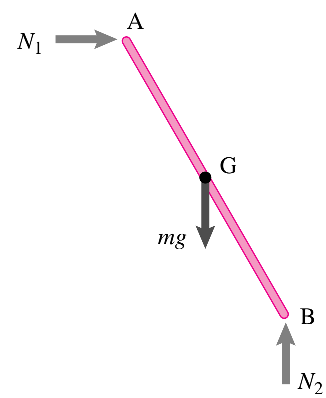



Figure 17.66

Let $E_1$ and $E_2$ be the total energy of the ladder at $\theta_0$ and $\theta_f$, respectively. Then

$$E_1 = E_2 \quad \text{(conservation of energy).}$$

$$\begin{aligned}
\text{Now} \quad E_1 &= \underbrace{E_{K_1}}_{\text{K.E.}} + \underbrace{E_{P_1}}_{\text{P.E.}} \\
&= 0 + mgh_1 \\
&= mg\frac{\ell}{2}\sin\theta_0 \\
\text{and} \quad E_2 &= E_{K_2} + E_{P_2} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^{G}\omega^2}_{E_{K_2}} + mgh_2.
\end{aligned}$$

Equating $E_1$ and $E_2$ we get

$$\begin{aligned}
\cancel{m}g\frac{\ell}{\cancel{2}}(\sin\theta_0 - \sin\theta_f) &= \frac{1}{\cancel{2}}(\cancel{m}v_G^2 + \underbrace{\frac{1}{12}\cancel{m}\ell^2}_{I_{zz}^{G}}\omega^2) \\
\text{or} \quad g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2\omega^2. \qquad (17.51)
\end{aligned}$$

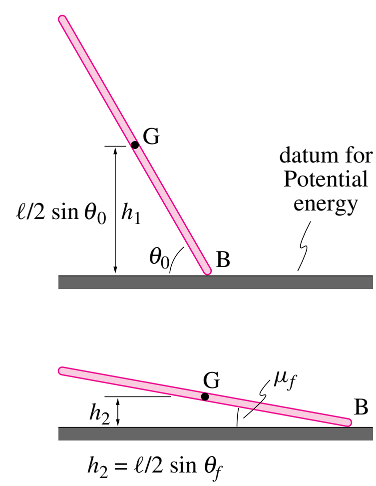



Figure 17.67

Clearly, we cannot find $v_G$ from this equation alone because the equation contains another unknown, $\omega$. So we need to find another equation which relates $v_G$ and $\omega$. To find this equation we turn to kinematics. Note that

$$\begin{aligned}
\vec{r}_G &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
\Rightarrow \quad \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2}(-\sin\theta\cdot\dot{\theta}\hat{\imath} + \cos\theta\cdot\dot{\theta}\hat{\jmath}) \\
\Rightarrow \quad v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4}(\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2} \\
&= \frac{\ell}{2}\dot{\theta} = \frac{\ell}{2}\omega \\
\Rightarrow \quad \omega &= \frac{2v_G}{\ell}.
\end{aligned}$$

Substituting the expression for $\omega$ in eqn. (17.51) we get

$$\begin{aligned}
g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\cancel{\ell^2}\cdot\frac{4v_G^2}{\cancel{\ell^2}} \\
&= \frac{4}{3}v_G^2 \\
\Rightarrow \quad v_G &= \sqrt{\frac{3g\ell}{4}(\sin\theta_0 - \sin\theta_f)} \\
&= 0.46\sqrt{g\ell}.
\end{aligned}$$

$$\boxed{v_G = 0.46\sqrt{g\ell}}$$

2. Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K.$$

For the ladder, the only force that does work is $mg$. This force acts at the center-of-mass G. Therefore,

$$\begin{aligned}
P &= \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} = -mg\hat{\boldsymbol{\jmath}}\cdot\vec{\boldsymbol{v}}_G \\
&= -mg\hat{\boldsymbol{\jmath}}\cdot\left[\frac{\ell}{2}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\,\dot\theta\right] \\
&= -mg\frac{\ell}{2}\dot\theta\cos\theta.
\end{aligned}$$

Now, the rate of change of kinetic energy is

$$\begin{aligned}
\dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^{G}\omega^2\right) \\
&= \frac{d}{dt}\left(\frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2\right) \\
&= \frac{m\ell^2}{4}\omega\dot\omega + \frac{m\ell^2}{12}\omega\dot\omega \\
&= \frac{m\ell^2}{3}\omega\dot\omega \equiv \frac{m\ell^2}{3}\dot\theta\ddot\theta \quad (\text{since } \omega = \dot\theta \text{ and } \dot\omega = \ddot\theta).
\end{aligned}$$

Now equating $P$ and $\dot{E}_K$ we get

$$\begin{aligned}
\frac{\cancel{m}\ell^2}{3}\cancel{\dot\theta}\ddot\theta &= -\cancel{m}g\frac{\ell}{2}\cancel{\dot\theta}\cos\theta \\
\Rightarrow \quad \ddot\theta &= -\frac{3g}{2\ell}\cos\theta
\end{aligned}$$

which is the same expression as obtained in Sample 17.22 (b).

$$\boxed{\ddot\theta = -\frac{3g}{2\ell}\cos\theta}$$

**Note:** Just like in the last problem, we have assumed that the upper end of the ladder stays in contact with the vertical wall. This assumption is used implicitly in the kinematics of the ladder. In energy calculations, the normal reactions of the two walls make no contribution as the work done by them is zero and hence the assumption of wall contact does not figure in the calculations above.

**SAMPLE 17.24** **Rolling on an inclined plane.** A wheel is made up of three uniform disks— the center disk of mass $m = 1\,\text{kg}$, radius $r = 10\,\text{cm}$ and two identical outer disks of mass $M = 2\,\text{kg}$ each and radius $R$. The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is $\theta = 30^\circ$. The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed 0.2g. Find the radius $R$ of the bigger disks.

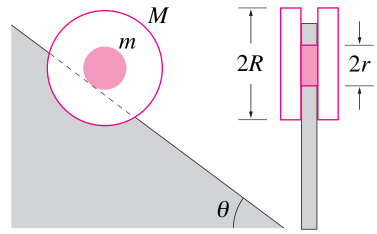



Figure 17.68: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

**Solution** Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius $R$.

The free-body diagram of the wheel is shown in Fig. 17.69. In addition to the weight $(m+2M)g$ of the wheel and the normal reaction $N$ of the wedge surface there is an unknown force of friction $F_f$ acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that $F_f \neq \mu N$ because there is neither slipping nor a condition of impending slipping. Thus the magnitude of $F_f$ is not known yet.

Let the acceleration of the center-of-mass of the wheel be

$$\vec{a}_G = a_G \hat{\boldsymbol{\lambda}}$$

and the angular acceleration of the wheel be

$$\dot{\vec{\omega}} = -\dot{\omega}\hat{\boldsymbol{k}}.$$

We assumed $\dot{\vec{\omega}}$ to be in the *negative* $\hat{\boldsymbol{k}}$ direction. But, if this assumption is wrong, we will get a negative value for $\dot{\omega}$.

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned}\sum \vec{F} &= m_{\text{total}}\,\vec{a}_{\text{cm}}\\ -(m+2M)g\hat{\boldsymbol{j}} + N\hat{\boldsymbol{n}} - F_f\hat{\boldsymbol{\lambda}} &= (m+2M)a_G\hat{\boldsymbol{\lambda}}\end{aligned}$$

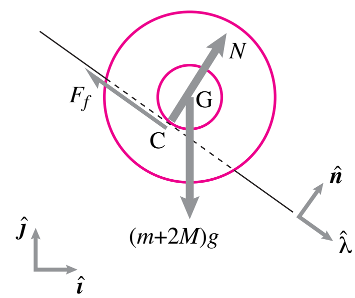



Figure 17.69: Free-body diagram of the wheel.

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns: $N$, $F_f$, and $a_G$. Thus we do not have enough equations to solve for the unknowns including the quantity of interest, $a_G$. So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using $\vec{r}_{G/C} = r\hat{\boldsymbol{n}}$):

$$\sum \vec{M}_{\text{C}} = \dot{\vec{H}}_{/\text{C}}$$

where

$$\begin{aligned}\vec{M}_{\text{C}} &= \vec{r}_{G/C} \times (m+2M)g(-\hat{\boldsymbol{j}})\\ &= r\hat{\boldsymbol{n}} \times (m+2M)g(-\hat{\boldsymbol{j}})\\ &= -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} \quad \text{(see Fig. 17.70)}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{H}}_{/\text{C}} &= I_{zz}^{G}\,\dot{\vec{\omega}} + \vec{r}_{G/C} \times m_{\text{total}}\,\vec{a}_G\\ &= I_{zz}^{G}(-\dot{\omega}\hat{\boldsymbol{k}}) - m_{\text{total}}\,\dot{\omega}r^2\hat{\boldsymbol{k}}\\ &= (I_{zz}^{G} + m_{\text{total}}r^2)(-\dot{\omega}\hat{\boldsymbol{k}})\\ &= \left[(\frac{1}{2}mr^2 + 2\cdot\frac{1}{2}MR^2) + \overbrace{(m+2M)}^{m_{\text{total}}}r^2\right](-\dot{\omega}\hat{\boldsymbol{k}})\\ &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

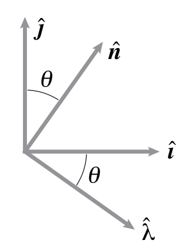



Figure 17.70: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example, $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{j}} = \sin\theta\hat{\boldsymbol{k}}$.

Thus,

$$\begin{aligned}
-(m+2M)gr\sin\theta\hat{\boldsymbol{k}} &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \dot{\omega} &= \frac{(m+2M)gr\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}. \qquad (17.52)
\end{aligned}$$

Now we need to relate $\dot{\omega}$ to $a_G$. From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (17.52) we get

$$a_G = \frac{(m+2M)gr^2\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}.$$

Now we can solve for $R$ in terms of $a_G$:

$$\begin{aligned}
\frac{3}{2}mr^2 + M(R^2+2r^2) &= \frac{(m+2M)gr^2\sin\theta}{a_G} \\
\Rightarrow \quad M(R^2+2r^2) &= \frac{(m+2M)g}{a_G}r^2\sin\theta - \frac{3}{2}mr^2 \\
\Rightarrow \quad R^2 &= \frac{(m+2M)g}{M\,a_G}r^2\sin\theta - \frac{3m}{2M}r^2 - 2r^2.
\end{aligned}$$

Since we require $a_G \leq 0.2g$ we get

$$\begin{aligned}
R^2 &\geq \left(\frac{(m+2M)g}{M\cdot 0.2g}\sin\theta - \frac{3m}{2M} - 2\right)r^2 \\
&\geq \left(\frac{5\,\text{kg}}{0.4\,\text{kg}}\cdot\frac{1}{2} - \frac{3\,\text{kg}}{4\,\text{kg}} - 2\right)(0.1\,\text{m})^2 \\
&\geq 0.035\,\text{m}^2 \\
\Rightarrow \quad R &\geq 0.187\,\text{m}.
\end{aligned}$$

Thus the outer disks of radius 20 cm will do the job.

$R \geq 18.7\,\text{cm}$

**SAMPLE 17.25 Which one starts rolling first — a marble or a bowling ball?** A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say $v_0$. The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

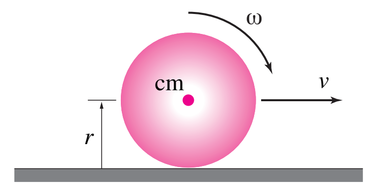



Figure 17.71

**Solution** Let us consider one ball, say the bowling ball, first. Let the radius of the ball be $r$ and mass $m$. The ball starts with center-of-mass velocity $\vec{\boldsymbol{v}}_o = v_0\hat{\boldsymbol{\imath}}$. The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal $\mu N$ (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition, $v = \omega r$. Let the instant of transition from the mixed motion to pure rolling be $t^*$. From linear momentum balance , we have

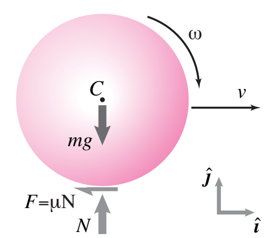



Figure 17.72: Free-body diagram of the ball during sliding.

$$\begin{aligned}
& & m\dot{v}\hat{\boldsymbol{\imath}} &= -\mu N\hat{\boldsymbol{\imath}} + (N - mg)\hat{\boldsymbol{\jmath}} & (17.53)\\
\text{eqn. (17.53)} \cdot \hat{\boldsymbol{\jmath}} & \Rightarrow & N &= mg & \\
\text{eqn. (17.53)} \cdot \hat{\boldsymbol{\imath}} & \Rightarrow & m\dot{v} &= -\mu N = -\mu mg & \\
& \Rightarrow & \dot{v} &= -\mu g & \\
& \Rightarrow & v &= v_0 - \mu g t. & (17.54)
\end{aligned}$$

Similarly, from angular momentum balance about the mass-center, we get

$$\begin{aligned}
-I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} &= -\mu N r\hat{\boldsymbol{k}} = -\mu m g r\hat{\boldsymbol{k}} \\
\Rightarrow \quad \dot{\omega} &= \frac{\mu m g r}{I_{zz}^{\text{cm}}} \\
\Rightarrow \quad \omega &= \underbrace{\omega_0}_{0} + \frac{\mu m g r}{I_{zz}^{\text{cm}}} t. \qquad (17.55)
\end{aligned}$$

At the instant of transition from mixed rolling and sliding to pure rolling, *i.e.*, at $t = t^*$, $v = \omega r$. Therefore, from eqn. (17.54) and eqn. (17.55), we get

$$\begin{aligned}
v_0 - \mu g t^* &= \frac{\mu m g r^2}{I_{zz}^{\text{cm}}} t^* \\
\Rightarrow \quad v_0 &= \mu g t^* \left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}}\right) \\
\Rightarrow \quad t^* &= \frac{v_0}{\mu g \left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}}\right)}.
\end{aligned}$$

Now, for a sphere, $I_{zz}^{\text{cm}} = \frac{2}{5}mr^2$. Therefore,

$$t^* = \frac{v_0}{\mu g \left(1 + \frac{mr^2}{\frac{2}{5}mr^2}\right)} = \frac{2v_0}{7\mu g}.$$

Note that the expression for $t^*$ is independent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$t^* = \frac{2v_0}{7\mu g}$ for both.

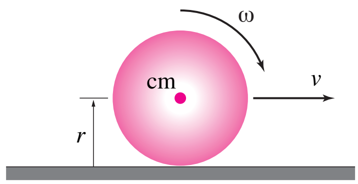



Figure 17.73

**SAMPLE 17.26 Transition from a mix of sliding and rolling to pure rolling, using impulse-momentum.** Consider the problem in Sample 17.25 again: A ball of radius $r = 10\,\text{cm}$ and mass $m = 1\,\text{kg}$ is launched horizontally with initial velocity $v_0 = 5\,\text{m/s}$ on a surface with coefficient of friction $\mu = 0.2$. The ball starts sliding, rolls and slides simultaneously for a while, and then rolls without sliding. Find the time it takes to start pure rolling.

**Solution** Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by $t^*$. At $t = 0$, we know that $v_{\text{cm}} = v_0 = 5\,\text{m/s}$, and $\omega_0 = 0$. We also know that at $t = t^*$, $v_{\text{cm}} = v_{t^*} = \omega_{t^*} r$, where $r$ is the radius of the ball. We do not know $t^*$ and $v_{t^*}$. However, we are considering a finite time event (during $t^*$) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

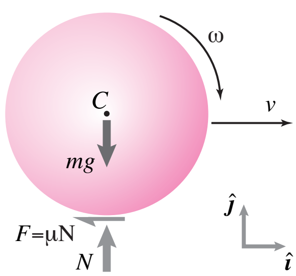



Figure 17.74: Free-body diagram of the ball during sliding.

$$
\begin{aligned}
\left(\sum \vec{F}\right) t^* &= m v_{t^*}\hat{\imath} - m v_0 \hat{\imath} \\
(-\mu N \hat{\imath} + (N - mg)\hat{\jmath})\, t^* &= m(v_{t^*} - v_0)\hat{\imath}.
\end{aligned}
$$

Dotting the above equation with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get

$$
\begin{aligned}
N &= mg \\
-\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0) \\
\Rightarrow \quad -\mu g t^* &= v_{t^*} - v_0. \qquad (17.56)
\end{aligned}
$$

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$
\begin{aligned}
\sum \vec{M}_{\text{cm}} t^* &= (\vec{H}_{\text{cm}})_{t^*} - (\vec{H}_{\text{cm}})_0 \\
(-\mu N r \hat{k}) t^* &= (I_{zz}^{\text{cm}} \omega_{t^*} - I_{zz}^{\text{cm}} \underbrace{\omega_0}_{0})(-\hat{k}) \\
\text{or} \qquad -\mu m g r t^* &= -I_{zz}^{\text{cm}} \omega_{t^*} \\
\Rightarrow \quad \omega_{t^*} &= \mu m g r t^* / I_{zz}^{\text{cm}} \\
\Rightarrow \quad v_{t^*} &\equiv \omega_{t^*} r = \mu m g r^2 t^* / I_{zz}^{\text{cm}}.
\end{aligned}
$$

Substituting this expression for $v_{t^*}$ in eqn. (17.56), we get

$$
\begin{aligned}
-\mu g t^* &= \mu m g r^2 t^* / I_{zz}^{\text{cm}} - v_0 \\
\Rightarrow \quad t^* &= \frac{v_0}{\mu g \left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}}\right)}
\end{aligned}
$$

which is, of course, the same expression we obtained for $t^*$ in Sample 17.25. Again, noting that $I_{zz}^{\text{cm}} = \frac{2}{5} m r^2$ for a sphere, we calculate the time of transition as

$$
t^* = \frac{2 v_0}{7 \mu g} = \frac{2 \cdot (5\,\text{m/s})}{7 \cdot (0.2) \cdot (9.8\,\text{m/s}^2)} = 0.73\,\text{s}.
$$

$t^* = 0.73\,\text{s}$

## 17.4 Mechanics of contact

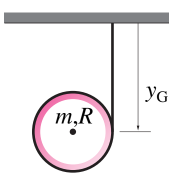
**17.4.1** A uniform disc of mass $m$ and radius $r$ rolls without slip at constant rate. What is the total kinetic energy of the disk?

**17.4.2** A non-uniform disc of mass $m$ and radius $r$ rolls without slip at constant rate. The center of mass is located at a distance $\frac{r}{2}$ from the center of the disc. What is the total kinetic energy of the disc when the center of mass is directly above the center of the disc?

**17.4.3 Falling hoop.** A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass $m$ and radius $R$. $G$ is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find $y_G$ at a later time $t$ in terms of any or all of $m$, $R$, $g$, and $t$.

b) Does $G$ move sideways as the hoop falls and unrolls?

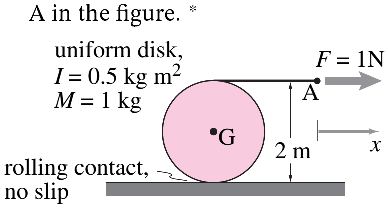



Problem 17.4.3

**17.4.4** A uniform disk with radius $R$ and mass $m$ has a string wrapped around it. The string is pulled with a force $F$. The disk rolls without slipping.

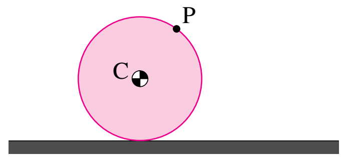
a) What is the angular acceleration of the disk, $\vec{\alpha}_{Disk} = -\ddot{\theta}\hat{k}$? Make any reasonable assumptions you need that are consistent with the figure information and the laws of mechanics. State your assumptions. *

b) Find the acceleration of the point A in the figure. *




Problem 17.4.4

**17.4.5** If a pebble is stuck to the edge of the wheel in problem 17.3.3, what is the maximum speed of the pebble during the motion? When is the force on the pebble from the wheel maximum? Draw a good FBD including the force due to gravity.

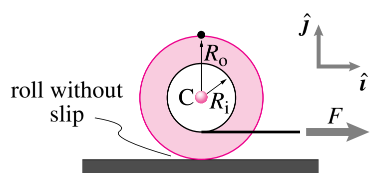



Problem 17.4.5

**17.4.6 Spool Rolling without Slip and Pulled by a Cord.** The light-weight spool is nearly empty but a lead ball with mass $m$ has been placed at its center. A force $F$ is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

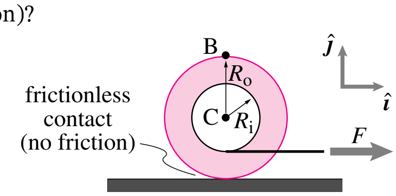
a) What is the acceleration of the center of the spool? *

b) What is the horizontal force of the ground on the spool? *




Problem 17.4.6

**17.4.7 A film spool** is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius $R_i$. What, in terms of $R_i$, $R_o$, $m$, $g$, $\hat{\imath}$, $\hat{\jmath}$, and $F$ are the accelerations of points C and B at the instant shown (the start of motion)?




Problem 17.4.7

**17.4.8 Again, Spool Rolling without Slip and Pulled by a Cord.** Reconsider the spool from problem 17.4.6. This time, a force $F$ is applied vertically to the cord wound around the wheel. In this case, what is the acceleration of the center of the spool? Is it possible to pull the cord at some angle between horizontal and vertical so that the angular acceleration of the spool or the acceleration of the center of mass is zero? If so, find the angle in terms of $R_i$, $R_0$, $m$, and $F$.

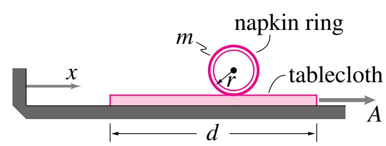
**17.4.9** A napkin ring lies on a black velvet tablecloth. The thin ring (of mass $m$, radius $r$) rolls without slip as a mischievous child pulls the tablecloth (mass $M$) out with acceleration $\vec{a}_A$. The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

a) What is the ring's acceleration as the tablecloth is being withdrawn?

b) How far has the tablecloth moved to the right from its starting point $x = 0$ when the ring rolls off its left-hand end?

c) Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?

d) Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?




Problem 17.4.9

**17.4.10** A block of mass $M$ is supported by two rollers (uniform cylinders) each of mass $m$ and radius $r$. They roll without slip on the block and the ground. A force $F$ is applied in the horizontal direction to the right, as shown in the figure. Given $F$, $m$, $r$, and $M$, find:

a) the acceleration of the block,

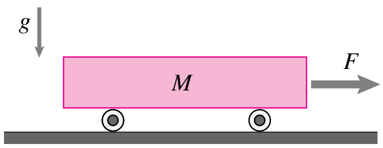
b) the acceleration of the center of mass of this block/roller system,

c) the reaction at the wheel bases,

d) the force of the right wheel on the block,

e) the acceleration of the wheel centers, and

f) the angular acceleration of the wheels.

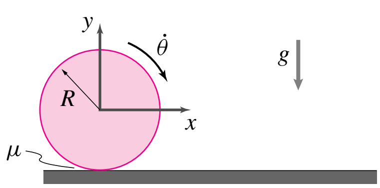



Problem 17.4.10

**17.4.11 Dropped spinning disk. 2-D**. A uniform disk of radius $R$ and mass $m$ is gently dropped onto a surface and doesn't bounce. When it is released it is spinning clockwise at the rate $\dot{\theta}_0$. The disk skids for a while and then is eventually rolling.

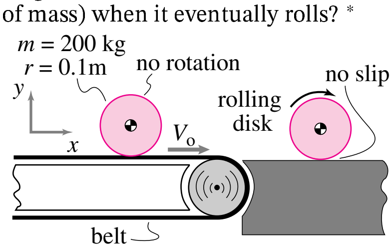
a) What is the speed of the center of the disk when the disk is eventually rolling (answer in terms of $g$, $\mu, R, \dot{\theta}_0$, and $m$)? *

b) In the transition from slipping to rolling, energy is lost to friction. How does the amount lost depend on the coefficient of friction (and other parameters)? How does this loss make or not make sense in the limit as $\mu \to 0$ and the dissipation rate $\to$ zero? *



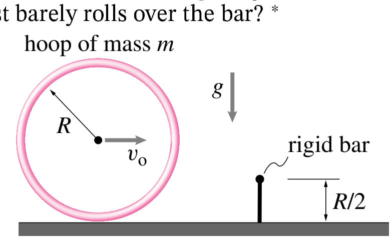

Problem 17.4.11

**17.4.12 Disk on a conveyor belt.** A uniform metal cylinder with mass of 200 kg is carried on a conveyer belt which moves at $V_0 = 3\,\text{m/s}$. The disk is not rotating when on the belt. The disk is delivered to a flat hard platform where it slides for a while and ends up rolling. How fast is it moving (i.e. what is the speed of the center of mass) when it eventually rolls? *




Problem 17.4.12

**17.4.13** A rigid hoop with radius $R$ and mass $m$ is rolling without slip so that its center has translational speed $v_o$. It then hits a narrow bar with height $R/2$. When the hoop hits the bar suddenly it sticks and doesn't slide. It does hinge freely about the bar, however. The gravitational constant is g. How big is $v_o$ if the hoop just barely rolls over the bar? *



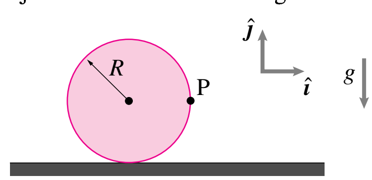

Problem 17.4.13

**17.4.14 2-D rolling of an unbalanced wheel.** A wheel, modeled as massless, has a point mass (mass $= m$) at its perimeter. The wheel is released from rest at the position shown. The wheel makes contact with coefficient of friction $\mu$.

a) What is the acceleration of the point P just after the wheel is released if $\mu = 0$?

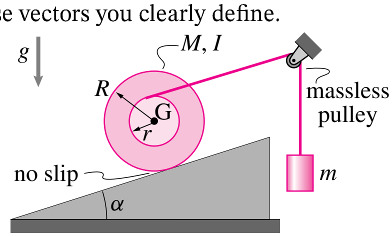
b) What is the acceleration of the point P just after the wheel is released if $\mu = 2$?

c) Assuming the wheel rolls without slip (no-slip requires, by the way, that the friction be high: $\mu = \infty$) what is the velocity of the point P just before it touches the ground?



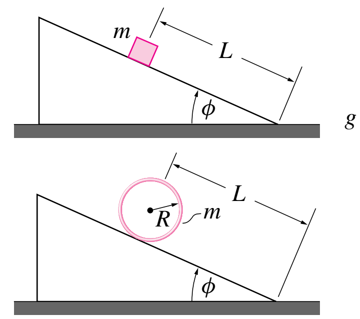

Problem 17.4.14

**17.4.15 Spool and mass.** A reel of mass $M$ and moment of inertia $I_{zz}^{cm} = I$ rolls without slipping upwards on an incline with slope-angle $\alpha$. It is pulled up by a string attached to mass $m$ as shown. Find the acceleration of point G in terms of some or all of $M, m, I, R, r, \alpha, g$ and any base vectors you clearly define.




Problem 17.4.15

**17.4.16** Two objects are released on two identical ramps. One is a sliding block (no friction), the other a rolling hoop (no slip). Both have the same mass, $m$, are in the same gravity field and have the same distance to travel. It takes the sliding mass 1 s to reach the bottom of the ramp. How long does it take the hoop? [Useful formula: "$s = \frac{1}{2}at^2$"]




Problem 17.4.16

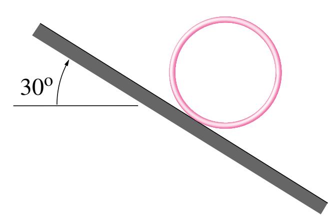
**17.4.17** The hoop is rolled down an incline that is 30° from horizontal. It does not slip. It does not fall over sideways. It is let go from rest at t = 0.

a) At $t = 0^+$ what is the acceleration of the hoop center of mass?

b) At $t = 0^+$ what is the acceleration of the point on the hoop that is on the incline?

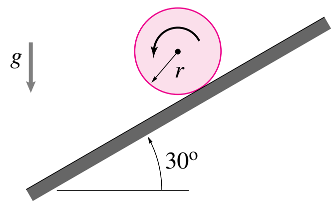
c) At $t = 0^+$ what is the acceleration of the point on the hoop that is furthest from the incline?

d) After the hoop has descended 2 vertical meters (and traveled an appropriate distance down the incline) what is the acceleration of the point on the hoop that is (at that instant) furthest from the incline?




Problem 17.4.17

**17.4.18** A uniform cylinder of mass $m$ and radius $r$ rolls down an incline *without slip*, as shown below. Determine: (a) the angular acceleration $\alpha$ of the disk; (b) the minimum value of the coefficient of friction $\mu$ that will insure no slip.




Problem 17.4.18

**17.4.19 Race of rollers.** A uniform disk with mass $M_0$ and radius $R_0$ is allowed to roll down the quite slip-resistant ($\mu = 1$) 30° ramp shown. It is raced against four other objects ($A$, $B$, $C$ and $D$), one at a time. Who wins the races, or are there ties? First try to construct any plausible reasoning. Good answers will be based, at least in part, on careful use of equations of mechanics. *

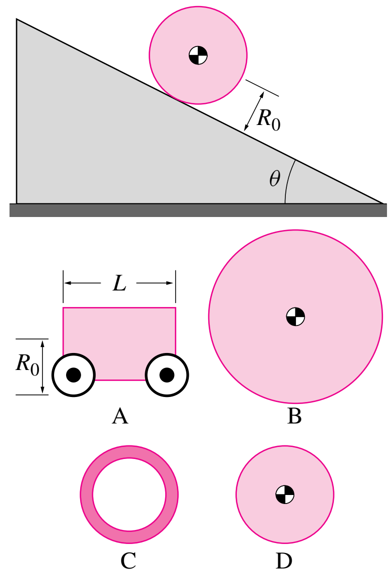
a) Block $A$ has the same mass and has center of mass a distance $R_0$ from the ramp. It rolls on massless wheels with frictionless bearings.

b) Uniform disk $B$ has the same mass ($M_B = M_0$) but twice the radius ($R_B = 2R_0$).

c) Hollow pipe $C$ has the same mass ($M_C = M_0$) and the same radius ($R_C = R_0$).

d) Uniform disk $D$ has the same radius ($R_D = R_0$) but twice the mass ($M_D = 2M_0$).

Can you find a round object which will roll as fast as the block slides? How about a massless cylinder with a point mass in its center? Can you find an object which will go slower than the slowest or faster than the fastest of these objects? What would they be and why? (This problem is harder.)



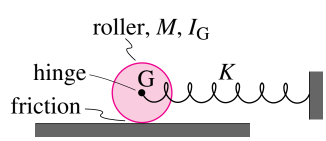

Problem 17.4.19

**17.4.20** A roller of mass $M$ and polar moment of inertia about the center of mass $I_G$ is connected to a spring of stiffness $K$ by a frictionless hinge as shown in the figure. Consider two kinds of friction between the roller and the surface it moves on:

1. Perfect slipping (no friction), and
2. Perfect rolling (infinite friction).

a) What is the period of oscillation in the first case?

b) What is the period of oscillation in the second case?

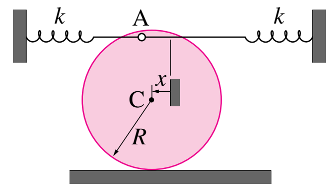



Problem 17.4.20

**17.4.21** A uniform cylinder of mass $m$ and radius $R$ rolls back and forth without slipping through small amplitudes (i.e., the springs attached at point A on the rim act linearly and the vertical change in the height of point A is negligible). The springs, which act both in compression and tension, are unextended when A is directly over C.

a) Determine the differential equation of motion for the cylinder's center.

b) Calculate the natural frequency of the system for small oscillations.

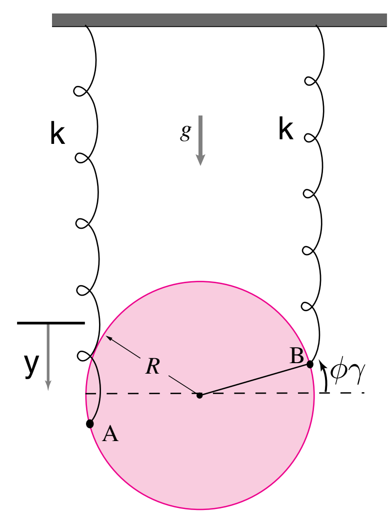



Problem 17.4.21

**17.4.22 Hanging disk, 2-D.** A uniform thin disk of radius $R$ and mass $m$ hangs in a gravity field $g$ from a pair of massless springs each with constant $k$. In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that $\phi$ is small so that the springs may be regarded as vertical when calculating their stretch ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$).

a) Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of $\phi, \dot{\phi}, y, \dot{y}, k, m, R$, and $g$.

b) Find the natural frequencies of vibration in terms of some or all of $k, m, R$, and $g$.




Problem 17.4.22

**17.4.23 A disk rolls in a cylinder.** For all of the problems below, the disk rolls without slip and rocks back and forth due to gravity.

a) **Sketch.** Draw a neat sketch of the disk in the cylinder. The sketch should show all variables, coordinates and dimension used in the problem.

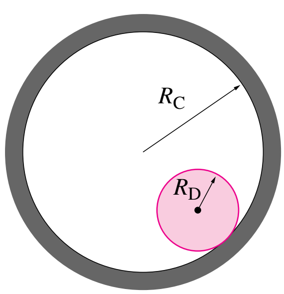
b) **FBD.** Draw a free-body diagram of the disk.

c) **Momentum balance.** Write the equations of linear and angular momentum balance for the disk. Use the point on the cylinder which touches the disk for the angular momentum balance equation. Leave as unknown in these equations variables which you do not know.

d) **Kinematics.** The disk rolling in the cylinder is a *one*-degree-of-freedom system. That is, the values of only *one* coordinate and its derivatives are enough to determine the positions, velocities and accelerations of all points. The angle that the line from the center of the cylinder to the center of the disk makes from the vertical can be used as such a variable. Find all of the velocities and accelerations needed in the momentum balance equation in terms of this variable and its derivative. [Hint: you'll need to think about the rolling contact in order to do this part.]

e) **Equation of motion.** Write the angular momentum balance equation as a single second order differential equation.

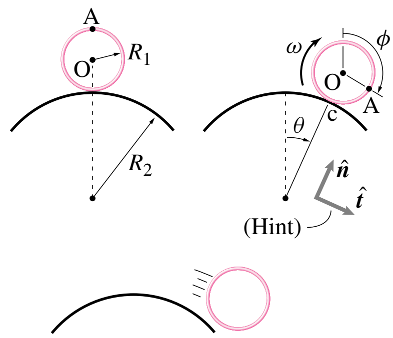
f) **Simple pendulum?** Does this equation reduce to the equation for a pendulum with a point mass and length equal to the radius of the cylinder, when the disk radius gets arbitrarily small? Why, or why not?




Problem 17.4.23: A disk rolls without slip inside a bigger cylinder.

**17.4.24** A uniform hoop of radius $R_1$ and mass $m$ rolls from rest down a semi-circular track of radius $R_2$ as shown. Assume that no slipping occurs. At what angle $\theta$ does the hoop leave the track and what is its angular velocity $\omega$ and the linear velocity $\bar{\boldsymbol{v}}$ of its center of mass at that instant? If the hoop slides down the track without friction, so that it does not rotate, will it leave at a smaller or larger angle $\theta$ than if it rolls without slip (as above)? Give a qualitative argument to justify your answer.

HINT: Here is a geometric relationship between angle $\phi$ hoop turns through and angle $\theta$ subtended by its center when no slipping occurs: $\phi = [(R_1 + R_2)/R_2]\theta$. (You may or may not need to use this hint.)




Problem 17.4.24