SAMPLE 17.16 **Equation of motion of a driven wheel** Consider a wheel of mass m and radius R driven by an axle force $\vec{\boldsymbol{F}}$ as shown in the figure. The wheel rolls to the left without slipping. Write the equation of motion of the wheel.

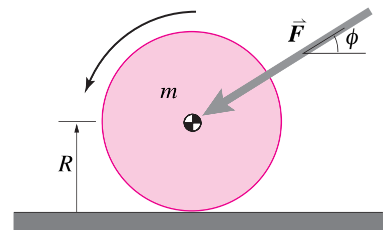

Figure 17.52

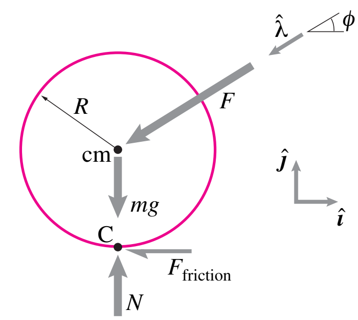

Figure 17.53: FBD of a wheel with mass m. Force F is applied by the axle.

Solution The free-body diagram of the wheel is shown in figure 17.53. We can write the equation of motion of the wheel in terms of either the center-of-mass position x or the angular displacement of the wheel θ. Since in pure rolling, these two variables share a simple relationship ($x = R\theta$), we can easily get the equation of motion in terms of x if we have the equation in terms of θ and vice versa. Let $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{k}}$.

Since all the forces are shown in the free-body diagram, we can readily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force, $-mg\hat{\boldsymbol{\jmath}}$, the friction force $-F_{\text{friction}}\hat{\boldsymbol{\imath}}$, and the normal reaction of the ground $N\hat{\boldsymbol{\jmath}}$, all pass through the contact point C and therefore, produce no moment about this point). We have

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{/\text{C}}$$

where

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\text{C}} &= \overbrace{\vec{\boldsymbol{r}}_{cm/C}}^{R\hat{\jmath}} \times (F\hat{\boldsymbol{\lambda}}) \\
&= R\hat{\boldsymbol{\jmath}} \times F(-\cos\phi\hat{\boldsymbol{\imath}} - \sin\phi\hat{\boldsymbol{\jmath}}) \\
&= FR\cos\phi\hat{\boldsymbol{k}}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\text{C}} &= \vec{\boldsymbol{r}}_{\text{cm/C}} \times m\vec{\boldsymbol{a}}_{\text{cm}} + I_{zz}^{\text{cm}}\dot{\vec{\boldsymbol{\omega}}} \\
&= R\hat{\boldsymbol{\jmath}} \times m \underbrace{\ddot{x}}_{-\dot{\omega}R} \hat{\boldsymbol{\imath}} + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
&= m\dot{\omega}R^2\hat{\boldsymbol{k}} + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
&= (I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
FR\cos\phi\hat{\boldsymbol{k}} &= (I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}
\end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute $\ddot{\theta} = -\ddot{x}/R$ in the equation of motion above to get the equation of motion in terms of the center-of-mass displacement x as

$$\ddot{x} = -\frac{FR^2\cos\phi}{I_{zz}^{\text{cm}} + mR^2}.$$

$$\boxed{\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}}$$

Comments: We could have, of course, used linear momentum balance with angular momentum balance about the center-of-mass to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the x and y directions involving all forces shown in the free-body diagram . The angular momentum balance , on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

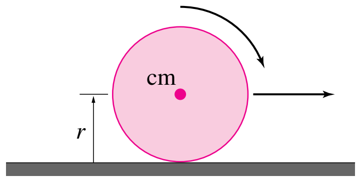

Figure 17.54

SAMPLE 17.17 **Energy and power of a rolling wheel.** A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft·lbf. Assume the wheel to be a thin, uniform disk.

1. Find the rate of rotation of the wheel.
2. Find the average power required to bring the wheel to a complete stop in 5 s.

Solution

1. Let ω be the rate of rotation of the wheel. Since the wheel rotates without slip, its center-of-mass moves with speed $v_{cm} = \omega r$. The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$
\begin{aligned}
E_K &= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I^{cm} \omega^2 \\
&= \frac{1}{2} m \omega^2 r^2 + \frac{1}{2} I^{cm} \omega^2 \\
&= \frac{1}{2} (m r^2 + \underbrace{I^{cm}}_{\frac{1}{2} m r^2}) \omega^2 \\
&= \frac{3}{4} m r^2 \omega^2 \\
\Rightarrow \quad \omega^2 &= \frac{4 E_K}{3 m r^2} \\
&= \frac{4 \times 1700\,\text{ft}\cdot\text{lbf}}{3 \times 20\,\text{lbm}\cdot 1\,\text{ft}^2} \\
&= \frac{4 \times 1700 \times 32.2\,\cancel{\text{lb}}\text{m}\cdot\cancel{\text{f}}\text{t}/\text{s}^2}{3 \times 20\,\cancel{\text{lb}}\text{m}\cdot\cancel{\text{f}}\text{t}} \\
&= 3649.33 \frac{1}{\text{s}^2} \\
\Rightarrow \quad \omega &= 60.4\,\text{rad/s}.
\end{aligned}
$$

$$\boxed{\omega = 60.4\,\text{rad/s}}$$

Note: This rotational speed, by the way, is extremely high. At this speed the center-of-mass moves at 60.4 ft/s!

2. Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the average power is,

$$
\begin{aligned}
P &= \frac{E_{K_1} - E_{K_2}}{\Delta t} \\
&= \frac{1700\,\text{ft}\cdot\text{lbf} - 0}{5\,\text{s}} = 340\,\text{ft}\cdot\text{lbf}/\text{s} \\
&= 340\,\text{ft}\cdot\text{lbf}/\text{s} \cdot \frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf}/\text{s}} \\
&= 0.62\,\text{hp}
\end{aligned}
$$

$$\boxed{P = 0.62\,\text{hp}}$$

SAMPLE 17.18 Equation of motion of a rolling wheel from energy balance. Consider the wheel with mass m from figure 17.55. The free-body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance. Assume CCW (counter clock-wise) rotation is positive.

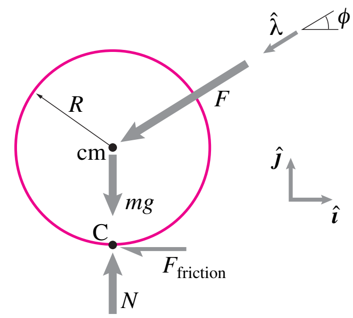

Figure 17.55: FBD of a rolling wheel.

Solution From energy balance, we have

$$P = \dot{E}_{\text{K}}$$

where

$$\begin{aligned}
P &= \sum \vec{F}_i \cdot \vec{v}_i \\
&= -F_{\text{friction}}\hat{\imath} \cdot \overbrace{\vec{v}_C}^{\vec{0}} + N\hat{\jmath} \cdot \overbrace{\vec{v}_C}^{\vec{0}} - mg\hat{\jmath} \cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} + F\hat{\lambda} \cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} \\
&= -mgv(\underbrace{\hat{\imath} \cdot \hat{\jmath}}_{0}) + Fv(\underbrace{\hat{\lambda} \cdot \hat{\imath}}_{-\cos\phi}) \\
&= -Fv\cos\phi
\end{aligned}$$

and

$$\begin{aligned}
\dot{E}_{\text{K}} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{zz}^{\text{cm}}\overbrace{\omega^2}^{(\dot{x}/R)^2}\right) \\
&= \frac{1}{2}\frac{d}{dt}\left[\left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}^2\right] \\
&= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}\ddot{x}.
\end{aligned}$$

Thus,

$$\begin{aligned}
-Fv\cos\phi &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}\ddot{x} \\
\text{or} \quad -F\cancel{\dot{x}}\cos\phi &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\cancel{\dot{x}}\ddot{x} \\
\Rightarrow \quad \ddot{x} &= -\frac{F\cos\phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.
\end{aligned}$$

We can also write the equation of motion in terms of θ by replacing $\ddot{x}$ with $-\ddot{\theta}R$ giving,

$$\ddot{\theta} = \frac{(F/R)\cos\phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.$$

$$\ddot{x} = -\frac{F\cos\phi}{m + I_{zz}^{\text{cm}}/R^2}$$

Comments: In the equations above (for calculating P), we have set $\vec{v}_C = \vec{0}$ because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center-of-mass. So, the only power supplied to the wheel is due to the force $F\hat{\lambda}$ acting at the center-of-mass.

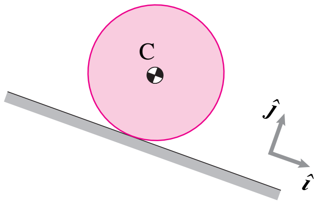

Figure 17.56

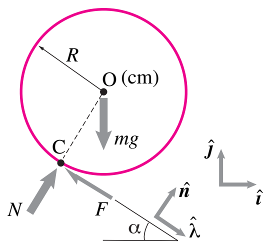

Figure 17.57

SAMPLE 17.19 Equation of motion of a rolling disk on an incline. A uniform circular disk of mass $m = 1$ kg and radius $R = 0.4$ m rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center-of-mass in 2 s.

Solution The free-body diagram of the disk is shown in fig. 17.57. In addition to the base unit vectors $\hat{\imath}$ and $\hat{\jmath}$, let us use unit vectors $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the other hand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

$$\sum \vec{\boldsymbol{M}}_C = \dot{\vec{\boldsymbol{H}}}_{/C}$$

where

$$\begin{aligned}\sum \vec{\boldsymbol{M}}_C &= \vec{\boldsymbol{r}}_{O/C} \times m\vec{\boldsymbol{g}} = R\hat{\boldsymbol{n}} \times (-mg\hat{\boldsymbol{\jmath}}) \\ &= -Rmg\sin\alpha\hat{\boldsymbol{k}}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/C} &= -I^{cm}_{zz}\dot{\omega}\hat{\boldsymbol{k}} + \overbrace{\vec{\boldsymbol{r}}_{O/C}}^{R\hat{\boldsymbol{n}}} \times m\, \overbrace{\vec{\boldsymbol{a}}_{cm}}^{R\dot{\omega}\hat{\boldsymbol{\lambda}}} \\ &= -I^{cm}_{zz}\dot{\omega}\hat{\boldsymbol{k}} + mR^2\dot{\omega}(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\lambda}}) \\ &= -(I^{cm}_{zz} + mR^2)\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\begin{aligned}-Rmg\sin\alpha\hat{\boldsymbol{k}} &= -(I^{cm}_{zz} + mR^2)\dot{\omega}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \dot{\omega} &= \frac{g\sin\alpha}{R[1 + I^{cm}_{zz}/(mR^2)]}.\end{aligned}$$

$$\boxed{\dot{\omega} = \frac{g\sin\alpha}{R[1+I^{cm}_{zz}/(mR^2)]}}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for ω and θ by simply integrating this equation and substituting the initial conditions $\omega(t=0) = 0$ and $\theta(t=0) = 0$. Let us write the equation of motion as $\dot{\omega} = \beta$ where $\beta = g\sin\alpha/R(1 + I^{cm}_{zz}/mR^2)$. Then,

$$\begin{aligned}\omega &\equiv \dot{\theta} = \beta t + C_1 \\ \theta &= \frac{1}{2}\beta t^2 + C_1 t + C_2.\end{aligned}$$

Substituting the given initial conditions $\dot{\theta}(0) = 0$ and $\theta(0) = 0$, we get $C_1 = 0$ and $C_2 = 0$, which implies that $\theta = \frac{1}{2}\beta t^2$. Now, in pure rolling, $x = R\theta$. Therefore,

$$\begin{aligned}x(t) &= R\theta(t) = \frac{1}{2}\beta t^2 = \cancel{R} \cdot \frac{1}{2}\frac{g\sin\alpha}{\cancel{R}(1 + I^{cm}_{zz}/mR^2)}t^2 \\ &= \frac{1}{2}\frac{g\sin\alpha}{1 + \frac{\frac{1}{2}mR^2}{mR^2}}t^2 = \frac{1}{3}(g\sin\alpha)t^2 \\ x(2\,\text{s}) &= \frac{1}{3} \cdot 9.8\,\text{m/s}^2 \cdot \sin(30^\circ) \cdot (2\,\text{s})^2 = 6.53\,\text{m}.\end{aligned}$$

$$\boxed{x(2\,\text{s}) = 6.53\,\text{m}}$$

SAMPLE 17.20 **Using Work and energy in pure rolling.** Consider the disk of Sample 17.19 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center-of-mass when the disk is 2 m down the inclined plane.

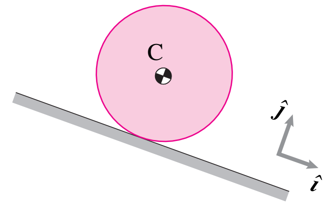

Figure 17.58

Solution We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center-of-mass after it has travelled 2 malong the incline. We can, of course, solve this problem using equation of motion, by first solving for the time t the disk takes to travel the given distance and then evaluating the expression for speed $\omega(t)$ or $x(t)$ at that t. However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let ω_1 and ω_2 be the initial and final (after rolling down by $d = 2\,\mathrm{m}$) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

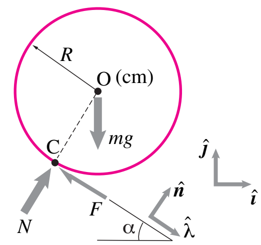

Figure 17.59

$$E_K = \frac{1}{2}m\overbrace{v_{cm}^2}^{(\omega R)^2} + \frac{1}{2}I_{zz}^{cm}\omega^2 = \frac{1}{2}(mR^2 + I_{zz}^{cm})\omega^2.$$

Therefore,

$$\Delta E_K = E_{K_2} - E_{K_1} = \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2). \tag{17.44}$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of the mass-center by d along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{\jmath})\cdot(d\hat{\lambda}) = -mgd(\overbrace{\hat{\jmath}\cdot\hat{\lambda}}^{-\sin\alpha}) = mgd\sin\alpha. \tag{17.45}$$

From work-energy principle (integral form of power balance, $P = \dot{E}_K$), we know that $W = \Delta E_K$. Therefore, from eqn. (17.44) and eqn. (17.45), we get

$$\begin{aligned}
mgd\sin\alpha &= \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2)\\
\Rightarrow \quad \omega_2^2 &= \omega_1^2 + \frac{2mgd\sin\alpha}{mR^2 + I_{zz}^{cm}} = \omega_1^2 + \frac{2gd\sin\alpha}{R^2\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)}\\
&= \omega_1^2 + \frac{4gd\sin\alpha}{3R^2}.
\end{aligned}$$

Substituting the values of g, d, α, R, etc., and setting $\omega_1 = 0$, we get

$$\begin{aligned}
\omega_2^2 &= \frac{4\cdot(9.8\,\mathrm{m/s^2})\cdot(2\,\mathrm{m})\cdot(\sin(30^\circ))}{3\cdot(0.4\,\mathrm{m})^2} = 81.67/s^2\\
\Rightarrow \quad \omega_2 &= 9.04\,\mathrm{rad/s}.
\end{aligned}$$

The corresponding speed of the center-of-mass is

$$v_{cm} = \omega_2 R = 9.04\,\mathrm{rad/s}\cdot 0.4\,\mathrm{m} = 3.61\,\mathrm{m/s}.$$

$$\boxed{v_{cm} = 3.61\,\mathrm{m/s}}$$

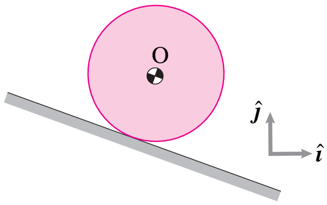

Figure 17.60

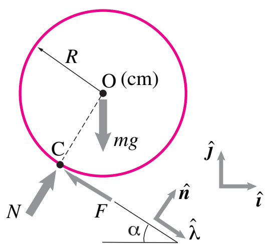

Figure 17.61

SAMPLE 17.21 Impulse and momentum calculations in pure rolling. Consider the disk of Sample 17.19 rolling down the incline again. Find an expression for the rolling speed (ω) of the disk after a finite time Δt, given the rolling speed ω_1 at some instant t_1.

Solution Once again, this problem can be solved by integrating the equation of motion (as done in Sample 17.19). However, we will solve this problem here using impulse-momentum relationship. Let the desired angular speed at $t_2 = t_1 + \Delta t$ be ω_2. We need to find ω_2, given the ω_1 at $t = t_1$. Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relationship, $\sum \vec{F} \cdot \Delta t = \vec{L}_2 - \vec{L}_1$, we have

$$(-F\hat{\lambda} + N\hat{\imath} - mg\hat{\jmath})\Delta t = m(v_2 - v_1)\hat{\lambda}. \tag{17.46}$$

Dotting eqn. (17.46) with $\hat{\lambda}$ gives

$$\begin{aligned}(-F - mg(\underbrace{\hat{\jmath}\cdot\hat{\lambda}}_{-\sin\alpha}))\Delta t &= m(v_2 - v_1)\\ (-F + mg\sin\alpha)\Delta t &= mR(\omega_2 - \omega_1). \qquad (17.47)\end{aligned}$$

Similarly, the angular impulse-momentum relationship about the mass-center, $\vec{M}_{\text{O}}\Delta t = (\vec{H}_{/\text{O}})_2 - (\vec{H}_{/\text{O}})_1$, gives

$$\begin{aligned}(-FR\hat{k})\Delta t &= -I_{zz}^{\text{cm}}(\omega_2 - \omega_1)\hat{k}\\ \Rightarrow \quad FR\Delta t &= I_{zz}^{\text{cm}}(\omega_2 - \omega_1). \qquad (17.48)\end{aligned}$$

Note that the other forces (N and mg) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force F from eqn. (17.47) and eqn. (17.48) by multiplying eqn. (17.47) with R and adding to eqn. (17.48):

$$\begin{aligned}(-F + mg\sin\alpha)\Delta t\cdot R + FR\Delta t &= mR(\omega_2 - \omega_1)\cdot R + I_{zz}^{\text{cm}}(\omega_2 - \omega_1)\\ \text{or} \quad mgR\sin\alpha\Delta t &= (I_{zz}^{\text{cm}} + mR^2)(\omega_2 - \omega_1)\\ \text{or} \quad g\sin\alpha\Delta t &= R\left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)(\omega_2 - \omega_1)\\ \Rightarrow \quad \omega_2 &= \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)}\Delta t.\end{aligned}$$

$$\boxed{\omega_2 = \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)}\Delta t}$$

Note: From the answer obtained, we clearly see that $\omega_2 = \omega_1 + \dot{\omega}\Delta t$ where $\dot{\omega}$ is the angular acceleration given by the expression $\dot{\omega} = \frac{g\sin\alpha}{R(1+I_{zz}^{\text{cm}}/mR^2)}$. This is the same expression we obtained for the angular acceleration in Sample 17.19.

SAMPLE 17.22 Falling ladder. A ladder AB, modeled as a uniform rigid rod of mass m and length ℓ, rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at $\theta = \theta_o$ ($\theta_o < \pi/2$). Assume the motion to be planar (in the vertical plane).

1. As the ladder falls, what is the path of the center-of-mass of the ladder?
2. Find the equation of motion (e.g., a differential equation in terms of θ and its time derivatives) for the ladder.
3. How does the angular speed ω ($= \dot{\theta}$) depend on θ?

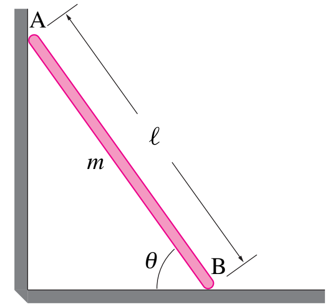

Figure 17.62: A ladder, modeled as a uniform rod of mass m and length ℓ, falls from a rest position at $\theta = \theta_o$ ($< \pi/2$) such that its ends slide along frictionless vertical and horizontal surfaces.

Solution Since the ladder is modeled by a uniform rod AB, its center-of-mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle θ (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

1. **Path of the center-of-mass:** Let the origin of our x-y coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 17.63). The position vector of the center-of-mass G may be written as

$$\begin{aligned} \vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell\cos\theta\hat{\imath} + \frac{\ell}{2}(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}). \end{aligned} \tag{17.49}$$

Thus the coordinates of the center-of-mass are

$$x_G = \frac{\ell}{2}\cos\theta \quad \text{and} \quad y_G = \frac{\ell}{2}\sin\theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius $\frac{\ell}{2}$. Therefore, the center-of-mass of the ladder follows a circular path of radius $\frac{\ell}{2}$ centered at the origin. Of course, the center-of-mass traverses only that part of the circle which lies between its initial position at $\theta = \theta_o$ and the final position at $\theta = 0$.

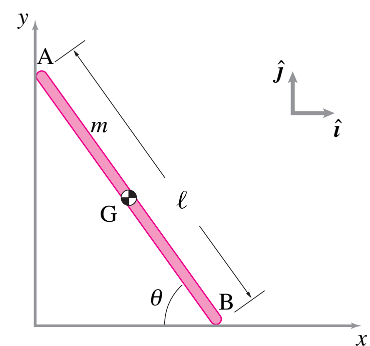

Figure 17.63:

2. **Equation of motion:** The free-body diagram of the ladder is shown in Fig. 17.64. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the ladder we get

$$N_1\hat{\imath} + (N_2 - mg)\hat{\jmath} = m\vec{a}_G = m\ddot{\vec{r}}_G.$$

Differentiating eqn. (17.49) twice we get $\ddot{\vec{r}}_G$ as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\hat{\imath} + (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\jmath}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by $\hat{\imath}$ and then by $\hat{\jmath}$ we get

$$\begin{aligned} N_1 &= -\frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg. \end{aligned}$$

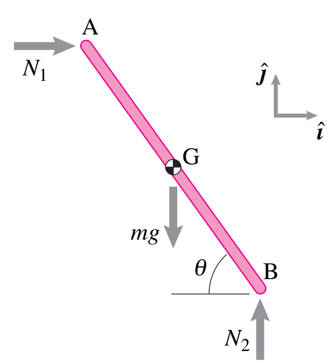

Figure 17.64: The free-body diagram of the ladder.

Next, we write the angular momentum balance for the ladder about its center-of-mass, $\sum \vec{\boldsymbol{M}}_{/G} = \dot{\vec{\boldsymbol{H}}}_{/G}$, where

$$\begin{aligned}\sum \vec{\boldsymbol{M}}_{/G} &= \left(-N_1 \frac{\ell}{2}\sin\theta + N_2 \frac{\ell}{2}\cos\theta\right)\hat{\boldsymbol{k}} \\ &= \frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\frac{\ell}{2}\sin\theta\hat{\boldsymbol{k}} \\ &\quad + \left[\frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg\right]\frac{\ell}{2}\cos\theta\hat{\boldsymbol{k}} \\ &= \left(\frac{1}{4}m\ell^2\ddot{\theta} + \frac{1}{2}mg\ell\cos\theta\right)\hat{\boldsymbol{k}}\end{aligned}$$

and

$$\dot{\vec{\boldsymbol{H}}}_{/G} = I_{zz/G}\dot{\vec{\boldsymbol{\omega}}} = \frac{1}{12}m\ell^2\ddot{\theta}(-\hat{\boldsymbol{k}}),$$

where $\dot{\vec{\boldsymbol{\omega}}} = \ddot{\theta}(-\hat{\boldsymbol{k}})$ because θ is measured positive in the clockwise direction $(-\hat{\boldsymbol{k}})$. Now, equating the two quantities $\sum \vec{\boldsymbol{M}}_{/G} = \dot{\vec{\boldsymbol{H}}}_{/G}$ and dotting both sides with $\hat{\boldsymbol{k}}$ we get

$$\begin{aligned}\frac{1}{4}\cancel{m}\ell^2\ddot{\theta} + \frac{1}{2}\cancel{m}g\ell\cos\theta &= -\frac{1}{12}\cancel{m}\ell^2\ddot{\theta} \\ \text{or} \quad (\frac{1}{12} + \frac{1}{4})\ell^2\ddot{\theta} &= -\frac{1}{2}g\ell\cos\theta \\ \text{or} \quad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta \qquad (17.50)\end{aligned}$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for $\theta(t)$.

3. **Angular Speed of the ladder:** To solve for the angular speed ω $(= \dot{\theta})$ as a function of θ we need to express eqn. (17.50) in terms of ω, θ, and derivatives of ω with respect to θ. Now,
$$\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}.$$
Substituting in eqn. (17.50) and integrating both sides from the initial rest position to an arbitrary position θ we get

$$\begin{aligned}\int_0^{\omega}\omega\, d\omega &= -\int_{\theta_0}^{\theta}\frac{3g}{2\ell}\cos\theta\, d\theta \\ \Rightarrow \quad \frac{1}{2}\omega^2 &= -\frac{3g}{2\ell}(\sin\theta - \sin\theta_0) \\ \Rightarrow \quad \omega &= \pm\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}.\end{aligned}$$

Since end B is sliding to the right, θ is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, *i.e.,*

$$\vec{\boldsymbol{\omega}} = \dot{\theta}(-\hat{\boldsymbol{k}}) = -\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}\,\hat{\boldsymbol{k}}.$$

Note: To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. If a real ladder were sliding against a slippery wall and floor, would it not lose contact? The answer is yes, it would. One way of finding when the contact would be lost is to calculate the normal reaction N_1 and finding out at what value of θ it passes through zero. The answer depends on θ_0. For example, if $\theta_0 = 80^\circ$, then N_1 is zero at about $\theta = 41°$. You can verify this result by substituting the expressions for $\ddot{\theta}$ and $\dot{\theta}$ in the expression for N_1 and solving for θ when $N_1 = 0$.

SAMPLE 17.23 **The falling ladder again.** Consider the falling ladder of Sample 17.10 again. The mass of the ladder is m and the length is ℓ. The ladder is released from rest at $\theta = 80^\circ$.

1. At the instant when $\theta = 45^\circ$, find the speed of the center-of-mass of the ladder using energy.
2. Derive the equation of motion of the ladder using work-energy balance.

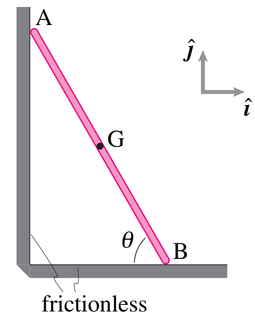

Figure 17.65

Solution

1. Since there is no friction, there is no loss of energy between the two states: $\theta_0 = 80^\circ$ and $\theta_f = 45^\circ$. The only external forces on the ladder are N_1, N_2, and mg as shown in the free-body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls, N_1 and N_2, respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

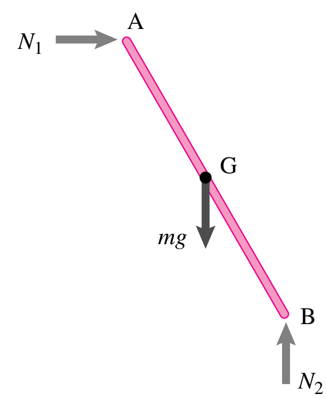

Figure 17.66

Let E_1 and E_2 be the total energy of the ladder at θ_0 and θ_f, respectively. Then

$$E_1 = E_2 \quad \text{(conservation of energy).}$$

$$\begin{aligned}
\text{Now}\quad E_1 &= \underbrace{E_{K_1}}_{\text{K.E.}} + \underbrace{E_{P_1}}_{\text{P.E.}} \\
&= 0 + mgh_1 \\
&= mg\frac{\ell}{2}\sin\theta_0 \\
\text{and}\quad E_2 &= E_{K_2} + E_{P_2} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2}_{E_{K_2}} + mgh_2.
\end{aligned}$$

Equating E_1 and E_2 we get

$$\begin{aligned}
\cancel{m}g\frac{\ell}{\cancel{2}}(\sin\theta_0 - \sin\theta_f) &= \frac{1}{\cancel{2}}(\cancel{m}v_G^2 + \underbrace{\frac{1}{12}\cancel{m}\ell^2}_{I_{zz}^G}\omega^2) \\
\text{or}\quad g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2\omega^2. \qquad (17.51)
\end{aligned}$$

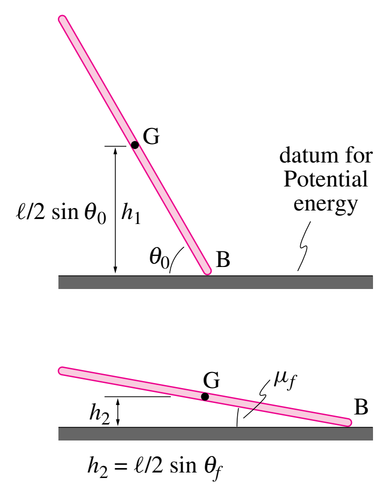

Figure 17.67

Clearly, we cannot find v_G from this equation alone because the equation contains another unknown, ω. So we need to find another equation which relates v_G and ω. To find this equation we turn to kinematics. Note that

$$\begin{aligned}
\vec{r}_G &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
\Rightarrow\quad \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2}(-\sin\theta\cdot\dot{\theta}\hat{\imath} + \cos\theta\cdot\dot{\theta}\hat{\jmath}) \\
\Rightarrow\quad v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4}(\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2} \\
&= \frac{\ell}{2}\dot{\theta} = \frac{\ell}{2}\omega \\
\Rightarrow\quad \omega &= \frac{2v_G}{\ell}.
\end{aligned}$$

Substituting the expression for ω in eqn. (17.51) we get

$$\begin{aligned} g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\cancel{\ell^2}\cdot\frac{4v_G^2}{\cancel{\ell^2}} \\ &= \frac{4}{3}v_G^2 \\ \Rightarrow \quad v_G &= \sqrt{\frac{3g\ell}{4}(\sin\theta_0 - \sin\theta_f)} \\ &= 0.46\sqrt{g\ell}. \end{aligned}$$

$$\boxed{v_G = 0.46\sqrt{g\ell}}$$

2. Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K.$$

For the ladder, the only force that does work is mg. This force acts at the center-of-mass G. Therefore,

$$\begin{aligned} P &= \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} = -mg\hat{\boldsymbol{\jmath}}\cdot\vec{\boldsymbol{v}}_G \\ &= -mg\hat{\boldsymbol{\jmath}}\cdot\left[\frac{\ell}{2}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\,\dot{\theta}\right] \\ &= -mg\frac{\ell}{2}\dot{\theta}\cos\theta. \end{aligned}$$

Now, the rate of change of kinetic energy is

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^{G}\omega^2\right) \\ &= \frac{d}{dt}\left(\frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2\right) \\ &= \frac{m\ell^2}{4}\omega\dot{\omega} + \frac{m\ell^2}{12}\omega\dot{\omega} \\ &= \frac{m\ell^2}{3}\omega\dot{\omega} \equiv \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} \quad (\text{since } \omega = \dot{\theta} \text{ and } \dot{\omega} = \ddot{\theta}). \end{aligned}$$

Now equating P and $\dot{E}_K$ we get

$$\begin{aligned} \frac{\cancel{m}\ell^2}{3}\cancel{\dot{\theta}}\ddot{\theta} &= -\cancel{m}g\frac{\ell}{2}\cancel{\dot{\theta}}\cos\theta \\ \Rightarrow \quad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta \end{aligned}$$

which is the same expression as obtained in Sample 17.22 (b).

$$\boxed{\ddot{\theta} = -\frac{3g}{2\ell}\cos\theta}$$

Note: Just like in the last problem, we have assumed that the upper end of the ladder stays in contact with the vertical wall. This assumption is used implicitly in the kinematics of the ladder. In energy calculations, the normal reactions of the two walls make no contribution as the work done by them is zero and hence the assumption of wall contact does not figure in the calculations above.

SAMPLE 17.24 Rolling on an inclined plane. A wheel is made up of three uniform disks— the center disk of mass $m = 1\,\text{kg}$, radius $r = 10\,\text{cm}$ and two identical outer disks of mass $M = 2\,\text{kg}$ each and radius R. The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is $\theta = 30^\circ$. The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed 0.2g. Find the radius R of the bigger disks.

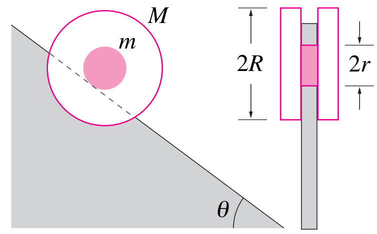

Figure 17.68: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

Solution Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius R.

The free-body diagram of the wheel is shown in Fig. 17.69. In addition to the weight $(m + 2M)g$ of the wheel and the normal reaction N of the wedge surface there is an unknown force of friction F_f acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that $F_f \neq \mu N$ because there is neither slipping nor a condition of impending slipping. Thus the magnitude of F_f is not known yet.

Let the acceleration of the center-of-mass of the wheel be

$$\vec{\boldsymbol{a}}_G = a_G \hat{\boldsymbol{\lambda}}$$

and the angular acceleration of the wheel be

$$\dot{\vec{\boldsymbol{\omega}}} = -\dot{\omega}\hat{\boldsymbol{k}}.$$

We assumed $\dot{\vec{\boldsymbol{\omega}}}$ to be in the *negative* $\hat{\boldsymbol{k}}$ direction. But, if this assumption is wrong, we will get a negative value for $\dot{\omega}$.

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m_{\text{total}}\, \vec{\boldsymbol{a}}_{\text{cm}} \\ -(m+2M)g\hat{\boldsymbol{j}} + N\hat{\boldsymbol{n}} - F_f\hat{\boldsymbol{\lambda}} &= (m+2M)a_G\hat{\boldsymbol{\lambda}} \end{aligned}$$

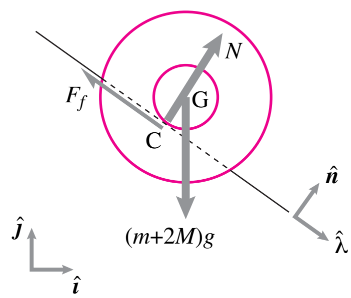

Figure 17.69: Free-body diagram of the wheel.

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns: N, F_f, and a_G. Thus we do not have enough equations to solve for the unknowns including the quantity of interest, a_G. So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using $\vec{\boldsymbol{r}}_{G/C} = r\hat{\boldsymbol{n}}$):

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{/\text{C}}$$

where

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{C}} &= \vec{\boldsymbol{r}}_{G/C} \times (m+2M)g(-\hat{\boldsymbol{j}}) \\ &= r\hat{\boldsymbol{n}} \times (m+2M)g(-\hat{\boldsymbol{j}}) \\ &= -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} \quad \text{(see Fig. 17.70)} \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{/\text{C}} &= I_{zz}^G\,\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{r}}_{G/C} \times m_{\text{total}}\,\vec{\boldsymbol{a}}_G \\ &= I_{zz}^G(-\dot{\omega}\hat{\boldsymbol{k}}) - m_{\text{total}}\,\dot{\omega}r^2\hat{\boldsymbol{k}} \\ &= (I_{zz}^G + m_{\text{total}}r^2)(-\dot{\omega}\hat{\boldsymbol{k}}) \\ &= \left[(\frac{1}{2}mr^2 + 2\cdot\frac{1}{2}MR^2) + \overbrace{(m+2M)}^{m_{\text{total}}}r^2 \right](-\dot{\omega}\hat{\boldsymbol{k}}) \\ &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}}. \end{aligned}$$

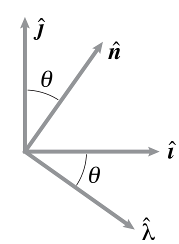

Figure 17.70: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example, $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{j}} = \sin\theta\hat{\boldsymbol{k}}$.

Thus,

$$
\begin{aligned}
-(m+2M)gr\sin\theta\hat{\boldsymbol{k}} &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \dot{\omega} &= \frac{(m+2M)gr\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}.
\end{aligned} \tag{17.52}
$$

Now we need to relate $\dot{\omega}$ to a_G. From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (17.52) we get

$$a_G = \frac{(m+2M)gr^2\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}.$$

Now we can solve for R in terms of a_G:

$$
\begin{aligned}
\frac{3}{2}mr^2 + M(R^2+2r^2) &= \frac{(m+2M)gr^2\sin\theta}{a_G} \\
\Rightarrow \quad M(R^2+2r^2) &= \frac{(m+2M)g}{a_G}r^2\sin\theta - \frac{3}{2}mr^2 \\
\Rightarrow \quad R^2 &= \frac{(m+2M)g}{M\,a_G}r^2\sin\theta - \frac{3m}{2M}r^2 - 2r^2.
\end{aligned}
$$

Since we require $a_G \leq 0.2g$ we get

$$
\begin{aligned}
R^2 &\geq \left(\frac{(m+2M)g}{M\cdot 0.2g}\sin\theta - \frac{3m}{2M} - 2\right)r^2 \\
&\geq \left(\frac{5\,\text{kg}}{0.4\,\text{kg}}\cdot\frac{1}{2} - \frac{3\,\text{kg}}{4\,\text{kg}} - 2\right)(0.1\,\text{m})^2 \\
&\geq 0.035\,\text{m}^2 \\
\Rightarrow \quad R &\geq 0.187\,\text{m}.
\end{aligned}
$$

Thus the outer disks of radius 20 cm will do the job.

$R \geq 18.7\,\text{cm}$

SAMPLE 17.25 Which one starts rolling first — a marble or a bowling ball? A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say v_0. The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

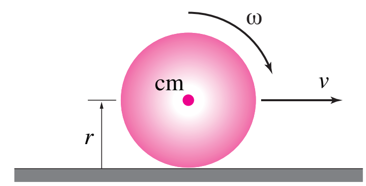

Figure 17.71

Solution Let us consider one ball, say the bowling ball, first. Let the radius of the ball be r and mass m. The ball starts with center-of-mass velocity $\vec{v}_o = v_0\hat{\imath}$. The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal μN (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition, $v = \omega r$. Let the instant of transition from the mixed motion to pure rolling be t^*. From linear momentum balance , we have

$$
\begin{aligned}
&& m\dot{v}\hat{\imath} &= -\mu N\hat{\imath} + (N - mg)\hat{\jmath} && (17.53)\\
\text{eqn. (17.53)} \cdot \hat{\jmath} &\Rightarrow & N &= mg &&\\
\text{eqn. (17.53)} \cdot \hat{\imath} &\Rightarrow & m\dot{v} &= -\mu N = -\mu mg &&\\
&\Rightarrow & \dot{v} &= -\mu g &&\\
&\Rightarrow & v &= v_0 - \mu g t. && (17.54)
\end{aligned}
$$

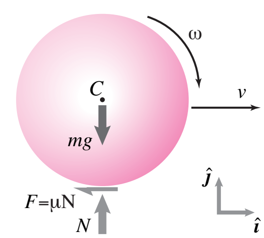

Figure 17.72: Free-body diagram of the ball during sliding.

Similarly, from angular momentum balance about the mass-center, we get

$$
\begin{aligned}
-I_{zz}^{cm}\dot{\omega}\hat{k} &= -\mu N r\hat{k} = -\mu m g r\hat{k}\\
\Rightarrow \quad \dot{\omega} &= \frac{\mu m g r}{I_{zz}^{cm}}\\
\Rightarrow \quad \omega &= \underbrace{\omega_0}_{0} + \frac{\mu m g r}{I_{zz}^{cm}} t. \qquad (17.55)
\end{aligned}
$$

At the instant of transition from mixed rolling and sliding to pure rolling, *i.e.,* at $t = t^*$, $v = \omega r$. Therefore, from eqn. (17.54) and eqn. (17.55), we get

$$
\begin{aligned}
v_0 - \mu g t^* &= \frac{\mu m g r^2}{I_{zz}^{cm}} t^*\\
\Rightarrow \quad v_0 &= \mu g t^*\left(1 + \frac{mr^2}{I_{zz}^{cm}}\right)\\
\Rightarrow \quad t^* &= \frac{v_0}{\mu g\left(1 + \frac{mr^2}{I_{zz}^{cm}}\right)}.
\end{aligned}
$$

Now, for a sphere, $I_{zz}^{cm} = \frac{2}{5}mr^2$. Therefore,

$$
t^* = \frac{v_0}{\mu g\left(1 + \frac{mr^2}{\frac{2}{5}mr^2}\right)} = \frac{2v_0}{7\mu g}.
$$

Note that the expression for t^* is independent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$t^* = \frac{2v_0}{7\mu g}$ for both.

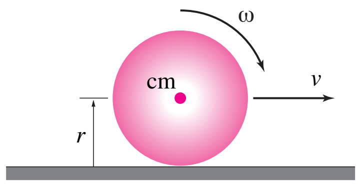

Figure 17.73

SAMPLE 17.26 Transition from a mix of sliding and rolling to pure rolling, using impulse-momentum. Consider the problem in Sample 17.25 again: A ball of radius $r = 10\,\mathrm{cm}$ and mass $m = 1\,\mathrm{kg}$ is launched horizontally with initial velocity $v_0 = 5\,\mathrm{m/s}$ on a surface with coefficient of friction $\mu = 0.2$. The ball starts sliding, rolls and slides simultaneously for a while, and then rolls without sliding. Find the time it takes to start pure rolling.

Solution Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by t^*. At $t = 0$, we know that $v_{\mathrm{cm}} = v_0 = 5\,\mathrm{m/s}$, and $\omega_0 = 0$. We also know that at $t = t^*$, $v_{\mathrm{cm}} = v_{t^*} = \omega_{t^*} r$, where r is the radius of the ball. We do not know t^* and v_{t^*}. However, we are considering a finite time event (during t^*) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

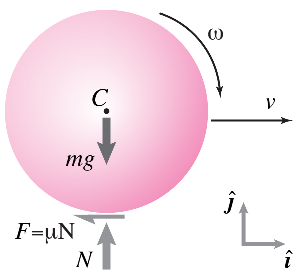

Figure 17.74: Free-body diagram of the ball during sliding.

$$\begin{aligned}\left(\sum \vec{\boldsymbol{F}}\right) t^* &= m v_{t^*}\hat{\boldsymbol{\imath}} - m v_0 \hat{\boldsymbol{\imath}} \\ (-\mu N \hat{\boldsymbol{\imath}} + (N - mg)\hat{\boldsymbol{\jmath}})\, t^* &= m(v_{t^*} - v_0)\hat{\boldsymbol{\imath}}.\end{aligned}$$

Dotting the above equation with $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}}$, respectively, we get

$$\begin{aligned}N &= mg \\ -\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0) \\ \Rightarrow \quad -\mu g t^* &= v_{t^*} - v_0. \qquad (17.56)\end{aligned}$$

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$\begin{aligned}\sum \vec{\boldsymbol{M}}_{\mathrm{cm}} t^* &= (\vec{\boldsymbol{H}}_{\mathrm{cm}})_{t^*} - (\vec{\boldsymbol{H}}_{\mathrm{cm}})_0 \\ (-\mu N r \hat{\boldsymbol{k}}) t^* &= (I_{zz}^{\mathrm{cm}}\omega_{t^*} - I_{zz}^{\mathrm{cm}} \underbrace{\omega_0}_{0})(-\hat{\boldsymbol{k}}) \\ \text{or} \qquad -\mu m g r t^* &= -I_{zz}^{\mathrm{cm}}\omega_{t^*} \\ \Rightarrow \quad \omega_{t^*} &= \mu m g r t^*/I_{zz}^{\mathrm{cm}} \\ \Rightarrow \quad v_{t^*} &\equiv \omega_{t^*} r = \mu m g r^2 t^*/I_{zz}^{\mathrm{cm}}.\end{aligned}$$

Substituting this expression for v_{t^*} in eqn. (17.56), we get

$$\begin{aligned}-\mu g t^* &= \mu m g r^2 t^*/I_{zz}^{\mathrm{cm}} - v_0 \\ \Rightarrow \quad t^* &= \frac{v_0}{\mu g(1 + \frac{mr^2}{I_{zz}^{\mathrm{cm}}})}\end{aligned}$$

which is, of course, the same expression we obtained for t^* in Sample 17.25. Again, noting that $I_{zz}^{\mathrm{cm}} = \frac{2}{5}mr^2$ for a sphere, we calculate the time of transition as

$$t^* = \frac{2v_0}{7\mu g} = \frac{2\cdot(5\,\mathrm{m/s})}{7\cdot(0.2)\cdot(9.8\,\mathrm{m/s^2})} = 0.73\,\mathrm{s}.$$

$t^* = 0.73\,\mathrm{s}$