

17.4 Mechanics of contacting bodies: rolling and sliding

A typical machine part has forces that come from contact with other parts. In fact, with the major exceptions of

- Gravity,
- Electromagnetic forces inside motors, and
- Magnetic attraction/repulsion

most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free-body diagrams have been contact forces: The force of the ground on a wheel, of an axle on a bearing, of any part on any other part it touches.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid objects in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book):

$$\dot{\vec{L}} = m_{\text{tot}} \vec{a}_{\text{cm}}, \quad (17.40)$$

$$\dot{\vec{H}}_{/C} = \vec{r}_{\text{cm}/C} \times (m_{\text{tot}} \vec{a}_{\text{cm}}) + I \dot{\omega} \hat{k} \quad (\text{where } I = I_{zz}^{\text{cm}}) \quad (17.41)$$

$$E_K = m_{\text{tot}} v_{\text{cm}}^2 / 2 + I \omega^2 / 2 \quad (17.42)$$

The key to success, as usual, is the drawing of appropriate free-body diagrams (see Chapter 3). For rolling and sliding the two cases one needs to consider as possible are, self-evidently, rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than μN , and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of μN but oppose the relative motion.

Work-energy relations and impulse-momentum relations are useful to solve some problems both with and without slip. In pure rolling contact the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated. The rate of loss of kinetic and potential energy is

$$\text{Rate of frictional dissipation} = P_{\text{diss}} = F_{\text{friction}} \cdot v_{\text{slip}} \quad (17.43)$$

where v_{slip} is the relative velocity of the contacting slipping points. If either the friction force (ideal lubrication) or sliding velocity (no slip) is zero there is no dissipation.

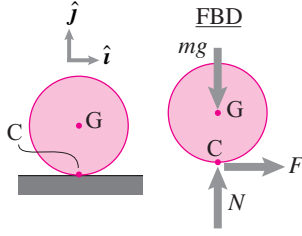


Figure 17.49: A ball rolls or slides on level ground.

As for various problems throughout the text, it is often a savings of calculation to use angular momentum balance (or moment balance in statics) relative to a point where there are unknown reaction forces. For rolling and slipping problems this often means making use of contact points.

Example: Pure rolling on level ground

A ball or wheel rolling on level ground, with no air friction etc, rolls at constant speed (see fig. 17.49). This is most directly deduced from angular momentum balance about the contact point C:

$$\begin{aligned}\vec{M}_C = \dot{\vec{H}}_C &\Rightarrow \vec{r}_{G/C} \times -mg\hat{j} = \vec{r}_{G/C} \times m\vec{a}_G + \dot{\omega} I_{zz}^{\text{cm}} \hat{k} \\ &\Rightarrow \vec{0} = R\hat{j} \times (-m\omega R\hat{i}) + \dot{\omega} I_{zz}^{\text{cm}} \hat{k} \\ \text{dotting with } \hat{k} &\Rightarrow \omega = 0 \Rightarrow \omega = \text{constant}.\end{aligned}$$

Because for rolling $v_G = -\omega R$ we thus have that v_G is a constant. [The result can also be obtained by combining angular momentum balance about the center-of-mass with linear momentum balance.]

Finally, linear momentum balance gives the reaction force at C to be $\vec{F}_{\text{tot}} = F\hat{i} + n\hat{j} = mg\hat{j}$. So,

assuming point contact, there is no rolling resistance.

Example: Bowling ball with initial sliding

A bowling ball is released with an initial speed of v_0 and no rotation rate. What is its subsequent motion? To start with, the motion is incompatible with rolling, the bottom of the ball is sliding to the right. So there is a frictional force which opposes motion and $F = -\mu N$ (see fig. 17.49). Linear and angular momentum balance give:

$$\begin{aligned}\text{LMB:} &\Rightarrow \{-F\hat{i} + N\hat{j} - mg\hat{j} = m\vec{a}\} \\ &\{\} \cdot \hat{j} \Rightarrow N = mg \\ &\{\} \cdot \hat{i} \Rightarrow a = -\mu g \\ \text{AMB}_{/G}: &\Rightarrow -R\mu mg = I_{zz}^{\text{cm}} \dot{\omega} \\ &\Rightarrow v = v_0 - \mu gt \quad \text{and} \quad \omega = -\mu R mgt / I_{zz}^{\text{cm}}\end{aligned}$$

Thus the forward speed of the ball decreases linearly with time while the counter-clockwise angular velocity decreases linearly with time.

This solution is only appropriate so long as there is rightward slip, $v_G > -\omega R$. Just like for a sliding block, there is no impetus for reversal, and the ball switches to pure rolling when

$$v = -\omega R \Rightarrow v_0 - \mu gt = -(-\mu R mgt / I_{zz}^{\text{cm}}) R \Rightarrow t = \frac{v_0}{\mu g \left(1 + \frac{mR^2}{I_{zz}^{\text{cm}}}\right)}.$$

Note that the energy lost during sliding is less than μmg times the distance the center of the ball moves during slip.

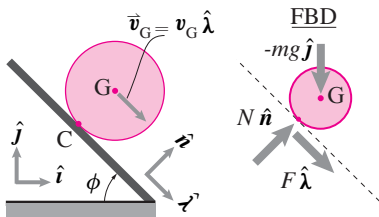


Figure 17.50: A ball rolls or slides down a slope.

Example: Ball rolling down hill.

Assuming rolling we can find the acceleration of a ball as it rolls downhill (see fig. 17.50). We start out with the kinematic observations that $\vec{a}_G = a_G \hat{\lambda}$, that

$R\omega = -v_G$ and that $R\dot{\omega} = -a_G$. Angular momentum balance about the stationary point on the ground instantaneously coinciding with the contact point gives

$$\begin{aligned} \text{AMB}_{/C} &\Rightarrow \vec{r}_{G/C} \times (-mg\hat{j}) = \vec{r}_{G/C} \times m\vec{a}_G + I_{zz}^{\text{cm}} \dot{\omega} \hat{k} \\ &\quad \{-R \sin \phi mg \hat{k} = (R\hat{n}) \times (ma_G \hat{\lambda}) + I_{zz}^{\text{cm}} \dot{\omega} \hat{k}\} \\ \{\} \cdot \hat{k} &\Rightarrow -Rmg \sin \phi = -Rma_G - I_{zz}^{\text{cm}} a_G / R \\ &\Rightarrow a_G = \frac{g \sin \phi}{1 + I_{zz}^{\text{cm}} / (mR^2)}. \end{aligned}$$

Which is less than the acceleration of a block sliding on a ramp without friction: $a = g \sin \phi$ (unless the mass of the rolling ball is concentrated at the center with $I_{zz}^{\text{cm}} = 0$). Note that a very small ball rolls just as slowly. In the limit as the ball radius goes to zero the behavior does not approach that of a point mass that slides; the rolling remains significant.

Example: Ball rolling down hill: energy approach

We can find the acceleration of the rolling ball using power balance or conservation of energy. For example

$$\begin{aligned} 0 = \frac{d}{dt} E_T &\Rightarrow 0 = \dot{E}_K + \dot{E}_P \\ &= \frac{d}{dt} (mv^2/2 + I_{zz}^{\text{cm}} \omega^2/2) + \frac{d}{dt} (mgy) \\ &= mv\dot{v} + I_{zz}^{\text{cm}} \omega \dot{\omega} + mg\dot{y} \\ &= mv\dot{v} + I_{zz}^{\text{cm}} (v/R)\dot{v}/R - mg(\sin \phi)v \\ \text{assuming } v \neq 0 &\Rightarrow 0 = (m + I_{zz}^{\text{cm}}/R^2)\dot{v} - mg \sin \phi \\ &\Rightarrow \dot{v} = \frac{g \sin \phi}{1 + I_{zz}^{\text{cm}} / (mR^2)} \end{aligned}$$

as before.

Example: Does the ball slide?

How big is the coefficient of friction μ needed to prevent slip for a ball rolling down a hill? Use angular momentum balance to find the normal and frictional components of the contact force, using the rolling example above.

$$\begin{aligned} \text{LMB } (\vec{F}_{\text{tot}} = m\vec{a}_G) &\Rightarrow \{N\hat{n} + F\hat{\lambda} - mg\hat{j} = ma_G \hat{\lambda}\} \\ \{\} \cdot \hat{n} &\Rightarrow N = mg \cos \phi \\ \{\} \cdot \hat{\lambda} &\Rightarrow F + mg \sin \phi = m \frac{g \sin \phi}{1 + I_{zz}^{\text{cm}} / (mR^2)} \\ F &= \frac{-mg \sin \phi}{1 + mR^2 / I_{zz}^{\text{cm}}} \\ \text{Critical condition:} &\Rightarrow \mu = \frac{|F|}{N} = \frac{\tan \phi}{1 + mR^2 / I_{zz}^{\text{cm}}} \end{aligned}$$

If I_{zz}^{cm} is very small (the mass concentrated near the center of the ball) then small friction is needed to prevent rolling. For a uniform rubber ball on pavement (with $\mu \approx 1$ and $I_{zz}^{\text{cm}} \approx 2mR^2/5$) the steepest slope for rolling without slip is a steep $\phi = \tan^{-1}(7/2) \approx 74^\circ$. A metal hoop on the other hand (with $\mu \approx .3$ and $I_{zz}^{\text{cm}} \approx mR^2$) will only roll without slip for slopes less than about $\phi = \tan^{-1}(.6) \approx 31^\circ$.

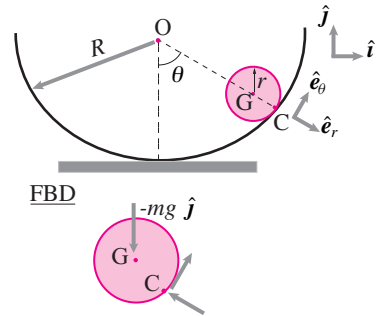


Figure 17.51: A ball rolls in a round cross-section bowl.

Example: Oscillations of a ball in a bowl.

A round ball can oscillate back and forth in the bottom of a circular cross section bowl or pipe (see fig. 17.51). Similarly, a cylindrical object can roll inside a pipe. What is the period of oscillation? Start with angular momentum balance about the contact point

$$\begin{aligned}\vec{r}_{G/C} \times (-mg\hat{j}) &= \vec{r}_{G/C} \times m\vec{a}_G + I_{zz}^{\text{cm}}\dot{\omega}\hat{k} \\ rmg \sin \theta \hat{k} &= -r\hat{e}_r \times \left(m \left((R-r)\ddot{\theta}\hat{e}_\theta - (R-r)\dot{\theta}^2\hat{e}_r \right) \right) \\ &\quad + I_{zz}^{\text{cm}}\dot{\omega}\hat{k}.\end{aligned}$$

Evaluating the cross products (using that $\hat{e}_r \times \hat{e}_\theta = \hat{k}$) and using the kinematics from the previous section (that $(R-r)\dot{\theta} = -r\omega$) and dotting the left and right sides with $\hat{k}$ gives

$$(R-r)\ddot{\theta} = \frac{-g \sin \theta}{1 + I_{zz}^{\text{cm}}/mr^2},$$

the tangential acceleration is the same as would have been predicted by putting the ball on a constant slope of $-\theta$. Using the small angle approximation that $\sin \theta = \theta$ the equation can be rearranged as a standard harmonic oscillator equation

$$\ddot{\theta} + \left(\frac{g}{(R-r)(1 + I_{zz}^{\text{cm}}/mr^2)} \right) \theta = 0.$$

If all the ball's mass were concentrated in its middle, keeping r a fixed non-zero size, (so $I_{zz}^{\text{cm}} = 0$, like a lead pellet inside a styrofoam ball) this is the same as for a simple pendulum with length $R-r$. For any parameter values the period of small oscillation is

$$T = 2\pi \sqrt{\frac{(R-r)(1 + I_{zz}^{\text{cm}}/mr^2)}{g}}.$$

For a marble or ball bearing in a sideways glass (with $R-r \approx 4 \text{ cm} = .04 \text{ m}$, $I_{zz}^{\text{cm}}/mr^2 \approx 2/5$ and $g \approx 10 \text{ m/s}^2$) this gives about one oscillation every half second. See page 1029 for the energy approach to this problem.

SAMPLE 17.16 Equation of motion of a driven wheel Consider a wheel of mass m and radius R driven by an axle force $\vec{F}$ as shown in the figure. The wheel rolls to the left without slipping. Write the equation of motion of the wheel.

Solution The free-body diagram of the wheel is shown in figure 17.53. We can write the equation of motion of the wheel in terms of either the center-of-mass position x or the angular displacement of the wheel θ . Since in pure rolling, these two variables share a simple relationship ($x = R\theta$), we can easily get the equation of motion in terms of x if we have the equation in terms of θ and vice versa. Let $\vec{\omega} = \omega \hat{k}$ and $\dot{\vec{\omega}} = \dot{\omega} \hat{k}$.

Since all the forces are shown in the free-body diagram, we can readily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force, $-mg\hat{j}$, the friction force $-F_{\text{friction}}\hat{i}$, and the normal reaction of the ground $N\hat{j}$, all pass through the contact point C and therefore, produce no moment about this point). We have

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned} \sum \vec{M}_C &= \overbrace{\vec{r}_{cm/C} \times (F\hat{\lambda})}^{R\hat{j}} \\ &= R\hat{j} \times F(-\cos\phi\hat{i} - \sin\phi\hat{j}) \\ &= FR\cos\phi\hat{k} \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_C &= \vec{r}_{cm/C} \times m\vec{a}_{cm} + I_{zz}^{cm}\dot{\vec{\omega}} \\ &= R\hat{j} \times m \underbrace{\ddot{x}}_{-\dot{\omega}R} \hat{i} + I_{zz}^{cm}\dot{\omega}\hat{k} \\ &= m\dot{\omega}R^2\hat{k} + I_{zz}^{cm}\dot{\omega}\hat{k} \\ &= (I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k}. \end{aligned}$$

Thus,

$$\begin{aligned} FR\cos\phi\hat{k} &= (I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k} \\ \Rightarrow \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{cm} + mR^2} \end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute $\ddot{\theta} = -\ddot{x}/R$ in the equation of motion above to get the equation of motion in terms of the center-of-mass displacement x as

$$\ddot{x} = -\frac{FR^2\cos\phi}{I_{zz}^{cm} + mR^2}.$$

$$\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{cm} + mR^2}$$

Comments: We could have, of course, used linear momentum balance with angular momentum balance about the center-of-mass to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the x and y directions involving all forces shown in the free-body diagram. The angular momentum balance, on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

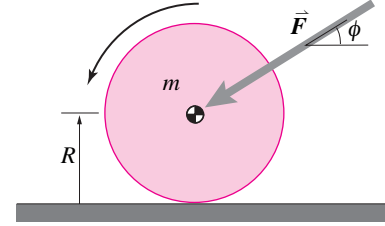


Figure 17.52

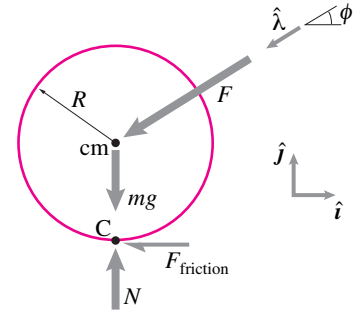


Figure 17.53: FBD of a wheel with mass m . Force F is applied by the axle.

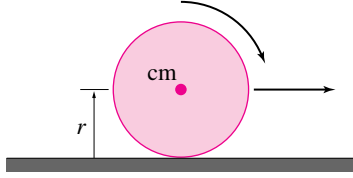


Figure 17.54

SAMPLE 17.17 Energy and power of a rolling wheel. A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft·lbf. Assume the wheel to be a thin, uniform disk.

1. Find the rate of rotation of the wheel.
2. Find the average power required to bring the wheel to a complete stop in 5 s.

Solution

1. Let ω be the rate of rotation of the wheel. Since the wheel rotates without slip, its center-of-mass moves with speed $v_{\text{cm}} = \omega r$. The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$\begin{aligned}
 E_K &= \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I^{\text{cm}} \omega^2 \\
 &= \frac{1}{2} m \omega^2 r^2 + \frac{1}{2} I^{\text{cm}} \omega^2 \\
 &= \frac{1}{2} (m r^2 + \underbrace{I^{\text{cm}}}_{\frac{1}{2} m r^2}) \omega^2 \\
 &= \frac{3}{4} m r^2 \omega^2 \\
 \Rightarrow \omega^2 &= \frac{4 E_K}{3 m r^2} \\
 &= \frac{4 \times 1700 \text{ ft} \cdot \text{lbf}}{3 \times 20 \text{ lbm} \cdot 1 \text{ ft}^2} \\
 &= \frac{4 \times 1700 \times 32.2 \text{ lbm} \cdot \text{ft} / \text{s}^2}{3 \times 20 \text{ lbm} \cdot \text{ft}} \\
 &= 3649.33 \frac{1}{\text{s}^2} \\
 \Rightarrow \omega &= 60.4 \text{ rad/s.}
 \end{aligned}$$

$$\omega = 60.4 \text{ rad/s}$$

Note: This rotational speed, by the way, is extremely high. At this speed the center-of-mass moves at 60.4 ft/s!

2. Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the average power is,

$$\begin{aligned}
 P &= \frac{E_{K_1} - E_{K_2}}{\Delta t} \\
 &= \frac{1700 \text{ ft} \cdot \text{lbf} - 0}{5 \text{ s}} = 340 \text{ ft} \cdot \text{lbf} / \text{s} \\
 &= 340 \text{ ft} \cdot \text{lbf} / \text{s} \cdot \frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lbf} / \text{s}} \\
 &= 0.62 \text{ hp}
 \end{aligned}$$

$$P = 0.62 \text{ hp}$$

SAMPLE 17.18 Equation of motion of a rolling wheel from energy balance. Consider the wheel with mass m from figure 17.55. The free-body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance. Assume CCW (counter clockwise) rotation is positive.

Solution From energy balance, we have

$$P = \dot{E}_K$$

where

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= -F_{\text{friction}} \hat{i} \cdot \vec{v}_C + N \hat{j} \cdot \vec{v}_C - mg \hat{j} \cdot \vec{v}_{cm} + F \hat{\lambda} \cdot \vec{v}_{cm} \\ &= -mgv \underbrace{(\hat{i} \cdot \hat{j})}_0 + Fv \underbrace{(\hat{\lambda} \cdot \hat{i})}_{-\cos \phi} \\ &= -Fv \cos \phi \end{aligned}$$

and

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{zz}^{\text{cm}} \omega^2 \right) \\ &= \frac{1}{2} \frac{d}{dt} \left[\left(m + \frac{I_{zz}^{\text{cm}}}{R^2} \right) \dot{x}^2 \right] \\ &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2} \right) \dot{x} \ddot{x}. \end{aligned}$$

Thus,

$$-Fv \cos \phi = \left(m + \frac{I_{zz}^{\text{cm}}}{R^2} \right) \dot{x} \ddot{x}$$

$$\text{or } -F \dot{x} \cos \phi = \left(m + \frac{I_{zz}^{\text{cm}}}{R^2} \right) \dot{x} \ddot{x}$$

$$\Rightarrow \ddot{x} = -\frac{F \cos \phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.$$

We can also write the equation of motion in terms of θ by replacing $\dot{x}$ with $-\dot{\theta}R$ giving,

$$\ddot{\theta} = \frac{(F/R) \cos \phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.$$

$$\ddot{x} = -\frac{F \cos \phi}{m + I_{zz}^{\text{cm}}/R^2}$$

Comments: In the equations above (for calculating P), we have set $\vec{v}_C = \vec{0}$ because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center-of-mass. So, the only power supplied to the wheel is due to the force $F\hat{\lambda}$ acting at the center-of-mass.

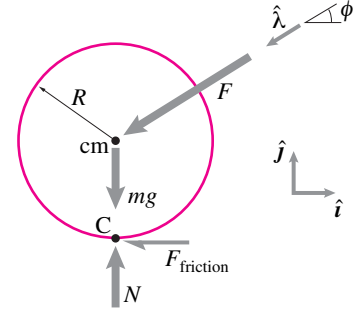


Figure 17.55: FBD of a rolling wheel.

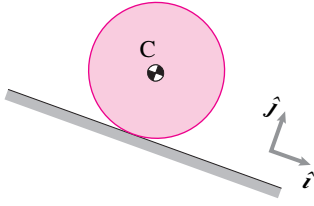


Figure 17.56

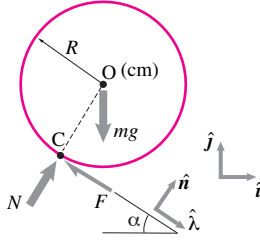


Figure 17.57

SAMPLE 17.19 Equation of motion of a rolling disk on an incline. A uniform circular disk of mass $m = 1$ kg and radius $R = 0.4$ m rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center-of-mass in 2 s.

Solution The free-body diagram of the disk is shown in fig. 17.57. In addition to the base unit vectors $\hat{i}$ and $\hat{j}$, let us use unit vectors $\hat{\lambda}$ and $\hat{n}$ along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the other hand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned} \sum \vec{M}_C &= \vec{r}_{O/C} \times m\vec{g} = R\hat{n} \times (-mg\hat{j}) \\ &= -Rmg \sin \alpha \hat{k} \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_C &= -I_{zz}^{\text{cm}} \dot{\omega} \hat{k} + \overbrace{\vec{r}_{O/C} \times m}^{R\hat{n}} \times \overbrace{\vec{a}_{\text{cm}}}^{R\dot{\omega}\hat{\lambda}} \\ &= -I_{zz}^{\text{cm}} \dot{\omega} \hat{k} + mR^2 \dot{\omega} (\hat{n} \times \hat{\lambda}) \\ &= -(I_{zz}^{\text{cm}} + mR^2) \dot{\omega} \hat{k}. \end{aligned}$$

Thus,

$$\begin{aligned} -Rmg \sin \alpha \hat{k} &= -(I_{zz}^{\text{cm}} + mR^2) \dot{\omega} \hat{k} \\ \Rightarrow \dot{\omega} &= \frac{g \sin \alpha}{R[1 + I_{zz}^{\text{cm}}/(mR^2)]}. \end{aligned}$$

$$\dot{\omega} = \frac{g \sin \alpha}{R[1 + I_{zz}^{\text{cm}}/(mR^2)]}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for ω and θ by simply integrating this equation and substituting the initial conditions $\omega(t=0) = 0$ and $\theta(t=0) = 0$. Let us write the equation of motion as $\dot{\omega} = \beta$ where $\beta = g \sin \alpha / R(1 + I_{zz}^{\text{cm}}/mR^2)$. Then,

$$\begin{aligned} \omega &\equiv \dot{\theta} = \beta t + C_1 \\ \theta &= \frac{1}{2} \beta t^2 + C_1 t + C_2. \end{aligned}$$

Substituting the given initial conditions $\dot{\theta}(0) = 0$ and $\theta(0) = 0$, we get $C_1 = 0$ and $C_2 = 0$, which implies that $\theta = \frac{1}{2} \beta t^2$. Now, in pure rolling, $x = R\theta$. Therefore,

$$\begin{aligned} x(t) &= R\theta(t) = \frac{1}{2} \beta t^2 = R \cdot \frac{1}{2} \frac{g \sin \alpha}{R(1 + I_{zz}^{\text{cm}}/mR^2)} t^2 \\ &= \frac{1}{2} \frac{g \sin \alpha}{1 + \frac{I_{zz}^{\text{cm}}}{mR^2}} t^2 = \frac{1}{3} (g \sin \alpha) t^2 \\ x(2\text{ s}) &= \frac{1}{3} \cdot 9.8 \text{ m/s}^2 \cdot \sin(30^\circ) \cdot (2\text{ s})^2 = 6.53 \text{ m}. \end{aligned}$$

$$x(2\text{ s}) = 6.53 \text{ m}$$

SAMPLE 17.20 Using Work and energy in pure rolling. Consider the disk of Sample 17.19 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center-of-mass when the disk is 2 m down the inclined plane.

Solution We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center-of-mass after it has travelled 2 m along the incline. We can, of course, solve this problem using equation of motion, by first solving for the time t the disk takes to travel the given distance and then evaluating the expression for speed $\omega(t)$ or $x(t)$ at that t . However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let ω_1 and ω_2 be the initial and final (after rolling down by $d = 2$ m) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

$$E_K = \frac{1}{2} m \overbrace{v_{cm}^2}^{(\omega R)^2} + \frac{1}{2} I_{zz}^{cm} \omega^2 = \frac{1}{2} (mR^2 + I_{zz}^{cm}) \omega^2.$$

Therefore,

$$\Delta E_K = E_{K_2} - E_{K_1} = \frac{1}{2} (mR^2 + I_{zz}^{cm}) (\omega_2^2 - \omega_1^2). \quad (17.44)$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of the mass-center by d along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg \hat{j}) \cdot (d \hat{\lambda}) = -mgd (\hat{j} \cdot \hat{\lambda}) = mgd \sin \alpha. \quad (17.45)$$

From work-energy principle (integral form of power balance, $P = \dot{E}_K$), we know that $W = \Delta E_K$. Therefore, from eqn. (17.44) and eqn. (17.45), we get

$$\begin{aligned} mgd \sin \alpha &= \frac{1}{2} (mR^2 + I_{zz}^{cm}) (\omega_2^2 - \omega_1^2) \\ \Rightarrow \omega_2^2 &= \omega_1^2 + \frac{2mgd \sin \alpha}{mR^2 + I_{zz}^{cm}} = \omega_1^2 + \frac{2gd \sin \alpha}{R^2 \left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)} \\ &= \omega_1^2 + \frac{4gd \sin \alpha}{3R^2}. \end{aligned}$$

Substituting the values of g, d, α, R , etc., and setting $\omega_1 = 0$, we get

$$\begin{aligned} \omega_2^2 &= \frac{4 \cdot (9.8 \text{ m/s}^2) \cdot (2 \text{ m}) \cdot (\sin 30^\circ)}{3 \cdot (0.4 \text{ m})^2} = 81.67/\text{s}^2 \\ \Rightarrow \omega_2 &= 9.04 \text{ rad/s}. \end{aligned}$$

The corresponding speed of the center-of-mass is

$$v_{cm} = \omega_2 R = 9.04 \text{ rad/s} \cdot 0.4 \text{ m} = 3.61 \text{ m/s}.$$

$$v_{cm} = 3.61 \text{ m/s}$$

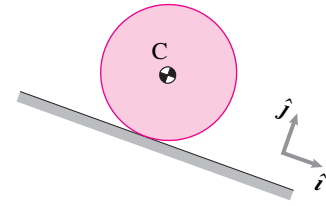


Figure 17.58

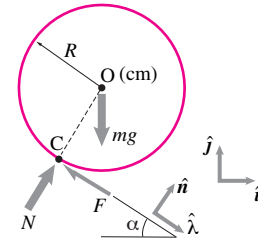


Figure 17.59

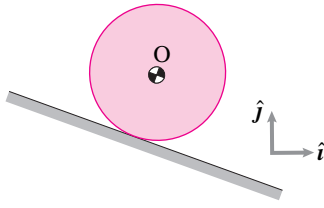


Figure 17.60

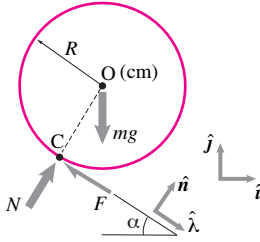


Figure 17.61

SAMPLE 17.21 Impulse and momentum calculations in pure rolling. Consider the disk of Sample 17.19 rolling down the incline again. Find an expression for the rolling speed (ω) of the disk after a finite time Δt , given the rolling speed ω_1 at some instant t_1 .

Solution Once again, this problem can be solved by integrating the equation of motion (as done in Sample 17.19). However, we will solve this problem here using impulse-momentum relationship. Let the desired angular speed at $t_2 = t_1 + \Delta t$ be ω_2 . We need to find ω_2 , given the ω_1 at $t = t_1$. Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relationship, $\sum \vec{F} \cdot \Delta t = \vec{L}_2 - \vec{L}_1$, we have

$$(-F\hat{\lambda} + N\hat{i} - mg\hat{j})\Delta t = m(v_2 - v_1)\hat{\lambda}. \quad (17.46)$$

Dotting eqn. (17.46) with $\hat{\lambda}$ gives

$$\begin{aligned} (-F - mg(\hat{j} \cdot \hat{\lambda}))\Delta t &= m(v_2 - v_1) \\ (-F + mg \sin \alpha)\Delta t &= mR(\omega_2 - \omega_1). \end{aligned} \quad (17.47)$$

Similarly, the angular impulse-momentum relationship about the mass-center, $\vec{M}_O \Delta t = (\vec{H}_O)_2 - (\vec{H}_O)_1$, gives

$$\begin{aligned} (-FR\hat{k})\Delta t &= -I_{zz}^{\text{cm}}(\omega_2 - \omega_1)\hat{k} \\ \Rightarrow FR\Delta t &= I_{zz}^{\text{cm}}(\omega_2 - \omega_1). \end{aligned} \quad (17.48)$$

Note that the other forces (N and mg) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force F from eqn. (17.47) and eqn. (17.48) by multiplying eqn. (17.47) with R and adding to eqn. (17.48):

$$\begin{aligned} (-F + mg \sin \alpha)\Delta t \cdot R + FR\Delta t &= mR(\omega_2 - \omega_1) \cdot R + I_{zz}^{\text{cm}}(\omega_2 - \omega_1) \\ \text{or } mgR \sin \alpha \Delta t &= (I_{zz}^{\text{cm}} + mR^2)(\omega_2 - \omega_1) \\ \text{or } g \sin \alpha \Delta t &= R \left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2} \right) (\omega_2 - \omega_1) \\ \Rightarrow \omega_2 &= \omega_1 + \frac{g \sin \alpha}{R \left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2} \right)} \Delta t. \end{aligned}$$

$$\omega_2 = \omega_1 + \frac{g \sin \alpha}{R \left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2} \right)} \Delta t$$

Note: From the answer obtained, we clearly see that $\omega_2 = \omega_1 + \dot{\omega} \Delta t$ where $\dot{\omega}$ is the angular acceleration given by the expression $\dot{\omega} = \frac{g \sin \alpha}{R(1 + I_{zz}^{\text{cm}}/mR^2)}$. This is the same expression we obtained for the angular acceleration in Sample 17.19.

SAMPLE 17.22 Falling ladder. A ladder AB, modeled as a uniform rigid rod of mass m and length ℓ , rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at $\theta = \theta_o$ ($\theta_o < \pi/2$). Assume the motion to be planar (in the vertical plane).

1. As the ladder falls, what is the path of the center-of-mass of the ladder?
2. Find the equation of motion (e.g., a differential equation in terms of θ and its time derivatives) for the ladder.
3. How does the angular speed ω ($= \dot{\theta}$) depend on θ ?

Solution Since the ladder is modeled by a uniform rod AB, its center-of-mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle θ (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

1. **Path of the center-of-mass:** Let the origin of our x - y coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 17.63). The position vector of the center-of-mass G may be written as

$$\begin{aligned}\vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell \cos \theta \hat{i} + \frac{\ell}{2}(-\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= \frac{\ell}{2}(\cos \theta \hat{i} + \sin \theta \hat{j}).\end{aligned}\quad (17.49)$$

Thus the coordinates of the center-of-mass are

$$x_G = \frac{\ell}{2} \cos \theta \quad \text{and} \quad y_G = \frac{\ell}{2} \sin \theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius $\frac{\ell}{2}$. Therefore, the center-of-mass of the ladder follows a circular path of radius $\frac{\ell}{2}$ centered at the origin. Of course, the center-of-mass traverses only that part of the circle which lies between its initial position at $\theta = \theta_o$ and the final position at $\theta = 0$.

2. **Equation of motion:** The free-body diagram of the ladder is shown in Fig. 17.64. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the ladder we get

$$N_1 \hat{i} + (N_2 - mg) \hat{j} = m\vec{a}_G = m\ddot{\vec{r}}_G.$$

Differentiating eqn. (17.49) twice we get $\ddot{\vec{r}}_G$ as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta} \sin \theta - \dot{\theta}^2 \cos \theta) \hat{i} + (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \hat{j}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by $\hat{i}$ and then by $\hat{j}$ we get

$$\begin{aligned}N_1 &= -\frac{1}{2}m\ell(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) + mg.\end{aligned}$$

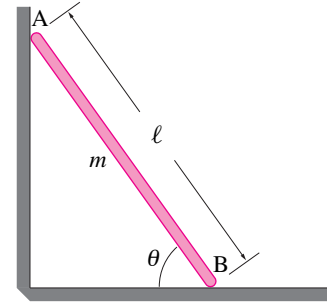


Figure 17.62: A ladder, modeled as a uniform rod of mass m and length ℓ , falls from a rest position at $\theta = \theta_o$ ($< \pi/2$) such that its ends slide along frictionless vertical and horizontal surfaces.

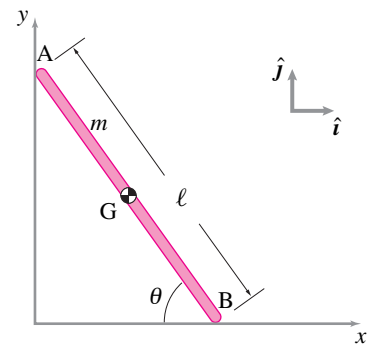


Figure 17.63:

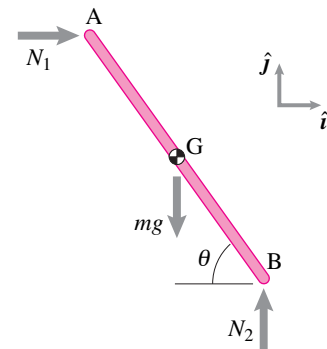


Figure 17.64: The free-body diagram of the ladder.

Next, we write the angular momentum balance for the ladder about its center-of-mass, $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$, where

$$\begin{aligned}\sum \vec{M}_{/G} &= \left(-N_1 \frac{\ell}{2} \sin \theta + N_2 \frac{\ell}{2} \cos \theta \right) \hat{k} \\ &= \frac{1}{2} m \ell (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \frac{\ell}{2} \sin \theta \hat{k} \\ &\quad + \left[\frac{1}{2} m \ell (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) + mg \right] \frac{\ell}{2} \cos \theta \hat{k} \\ &= \left(\frac{1}{4} m \ell^2 \ddot{\theta} + \frac{1}{2} mg \ell \cos \theta \right) \hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_{/G} = I_{zz/G} \dot{\vec{\omega}} = \frac{1}{12} m \ell^2 \ddot{\theta} (-\hat{k}),$$

where $\dot{\vec{\omega}} = \ddot{\theta}(-\hat{k})$ because θ is measured positive in the clockwise direction $(-\hat{k})$. Now, equating the two quantities $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ and dotting both sides with $\hat{k}$ we get

$$\begin{aligned}\frac{1}{4} m \ell^2 \ddot{\theta} + \frac{1}{2} mg \ell \cos \theta &= -\frac{1}{12} m \ell^2 \ddot{\theta} \\ \text{or } \left(\frac{1}{12} + \frac{1}{4} \right) \ell^2 \ddot{\theta} &= -\frac{1}{2} g \ell \cos \theta \\ \text{or } \ddot{\theta} &= -\frac{3g}{2\ell} \cos \theta\end{aligned}\quad (17.50)$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for $\theta(t)$.

3. **Angular Speed of the ladder:** To solve for the angular speed $\omega (= \dot{\theta})$ as a function of θ we need to express eqn. (17.50) in terms of ω , θ , and derivatives of ω with respect to θ . Now,

$$\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta}.$$

Substituting in eqn. (17.50) and integrating both sides from the initial rest position to an arbitrary position θ we get

$$\begin{aligned}\int_0^\omega \omega d\omega &= -\int_{\theta_0}^\theta \frac{3g}{2\ell} \cos \theta d\theta \\ \Rightarrow \frac{1}{2} \omega^2 &= -\frac{3g}{2\ell} (\sin \theta - \sin \theta_0) \\ \Rightarrow \omega &= \pm \sqrt{\frac{3g}{\ell} (\sin \theta_0 - \sin \theta)}.\end{aligned}$$

Since end B is sliding to the right, θ is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, i.e.,

$$\vec{\omega} = \dot{\theta}(-\hat{k}) = -\sqrt{\frac{3g}{\ell} (\sin \theta_0 - \sin \theta)} \hat{k}.$$

Note: To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. If a real ladder were sliding against a slippery wall and floor, would it not lose contact? The answer is yes, it would. One way of finding when the contact would be lost is to calculate the normal reaction N_1 and finding out at what value of θ it passes through zero. The answer depends on θ_0 . For example, if $\theta_0 = 80^\circ$, then N_1 is zero at about $\theta = 41^\circ$. You can verify this result by substituting the expressions for $\ddot{\theta}$ and $\dot{\theta}$ in the expression for N_1 and solving for θ when $N_1 = 0$.

SAMPLE 17.23 The falling ladder again. Consider the falling ladder of Sample 17.10 again. The mass of the ladder is m and the length is ℓ . The ladder is released from rest at $\theta = 80^\circ$.

1. At the instant when $\theta = 45^\circ$, find the speed of the center-of-mass of the ladder using energy.
2. Derive the equation of motion of the ladder using work-energy balance.

Solution

1. Since there is no friction, there is no loss of energy between the two states: $\theta_0 = 80^\circ$ and $\theta_f = 45^\circ$. The only external forces on the ladder are N_1 , N_2 , and mg as shown in the free-body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls, N_1 and N_2 , respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

Let E_1 and E_2 be the total energy of the ladder at θ_0 and θ_f , respectively. Then

$$E_1 = E_2 \quad (\text{conservation of energy}).$$

$$\begin{aligned} \text{Now } E_1 &= \underbrace{E_{K_1}}_{\text{K.E.}} + \underbrace{E_{P_1}}_{\text{P.E.}} \\ &= 0 + mgh_1 \\ &= mg \frac{\ell}{2} \sin \theta_0 \\ \text{and } E_2 &= E_{K_2} + E_{P_2} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2}_{E_{K_2}} + mgh_2. \end{aligned}$$

Equating E_1 and E_2 we get

$$\begin{aligned} mg \frac{\ell}{2} (\sin \theta_0 - \sin \theta_f) &= \frac{1}{2} (mv_G^2 + \underbrace{\frac{1}{12}m\ell^2\omega^2}_{I_{zz}^G}) \\ \text{or } g\ell (\sin \theta_0 - \sin \theta_f) &= v_G^2 + \frac{1}{12}\ell^2\omega^2. \end{aligned} \quad (17.51)$$

Clearly, we cannot find v_G from this equation alone because the equation contains another unknown, ω . So we need to find another equation which relates v_G and ω . To find this equation we turn to kinematics. Note that

$$\begin{aligned} \vec{r}_G &= \frac{\ell}{2}(\cos \theta \hat{i} + \sin \theta \hat{j}) \\ \Rightarrow \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2}(-\sin \theta \dot{\theta} \hat{i} + \cos \theta \dot{\theta} \hat{j}) \\ \Rightarrow v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4}(\cos^2 \theta + \sin^2 \theta) \dot{\theta}^2} \\ &= \frac{\ell}{2} \dot{\theta} = \frac{\ell}{2} \omega \\ \Rightarrow \omega &= \frac{2v_G}{\ell}. \end{aligned}$$

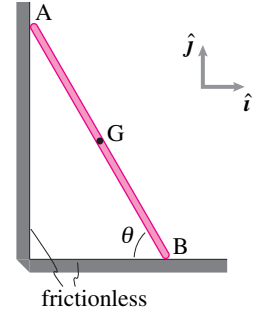


Figure 17.65

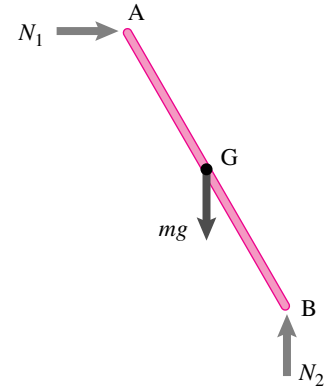


Figure 17.66

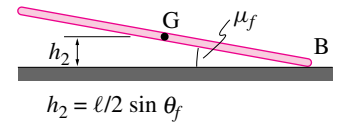
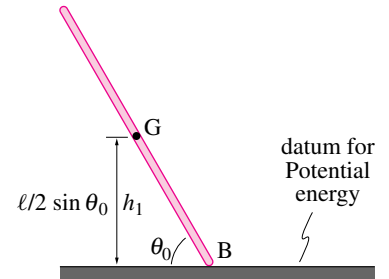


Figure 17.67

Substituting the expression for ω in eqn. (17.51) we get

$$\begin{aligned} g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2 \cdot \frac{4v_G^2}{\ell^2} \\ &= \frac{4}{3}v_G^2 \\ \Rightarrow v_G &= \sqrt{\frac{3g\ell}{4}(\sin\theta_0 - \sin\theta_f)} \\ &= 0.46\sqrt{g\ell}. \end{aligned}$$

$$v_G = 0.46\sqrt{g\ell}$$

2. Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K.$$

For the ladder, the only force that does work is mg . This force acts at the center-of-mass G. Therefore,

$$\begin{aligned} P &= \vec{F} \cdot \vec{v} = -mg\mathbf{j} \cdot \vec{v}_G \\ &= -mg\mathbf{j} \cdot \left[\frac{\ell}{2}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\dot{\theta} \right] \\ &= -mg\frac{\ell}{2}\dot{\theta}\cos\theta. \end{aligned}$$

Now, the rate of change of kinetic energy is

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt} \left(\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2 \right) \\ &= \frac{d}{dt} \left(\frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2 \right) \\ &= \frac{m\ell^2}{4}\omega\dot{\omega} + \frac{m\ell^2}{12}\omega\dot{\omega} \\ &= \frac{m\ell^2}{3}\omega\dot{\omega} \equiv \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} \quad (\text{since } \omega = \dot{\theta} \text{ and } \dot{\omega} = \ddot{\theta}). \end{aligned}$$

Now equating P and $\dot{E}_K$ we get

$$\begin{aligned} \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} &= -mg\frac{\ell}{2}\dot{\theta}\cos\theta \\ \Rightarrow \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta \end{aligned}$$

which is the same expression as obtained in Sample 17.22 (b).

$$\ddot{\theta} = -\frac{3g}{2\ell}\cos\theta$$

Note: Just like in the last problem, we have assumed that the upper end of the ladder stays in contact with the vertical wall. This assumption is used implicitly in the kinematics of the ladder. In energy calculations, the normal reactions of the two walls make no contribution as the work done by them is zero and hence the assumption of wall contact does not figure in the calculations above.

SAMPLE 17.24 Rolling on an inclined plane. A wheel is made up of three uniform disks—the center disk of mass $m = 1$ kg, radius $r = 10$ cm and two identical outer disks of mass $M = 2$ kg each and radius R . The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is $\theta = 30^\circ$. The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed $0.2g$. Find the radius R of the bigger disks.

Solution Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius R .

The free-body diagram of the wheel is shown in Fig. 17.69. In addition to the weight $(m + 2M)g$ of the wheel and the normal reaction N of the wedge surface there is an unknown force of friction F_f acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that $F_f \neq \mu N$ because there is neither slipping nor a condition of impending slipping. Thus the magnitude of F_f is not known yet.

Let the acceleration of the center-of-mass of the wheel be

$$\vec{a}_G = a_G \hat{\lambda}$$

and the angular acceleration of the wheel be

$$\vec{\omega} = -\dot{\omega} \hat{k}.$$

We assumed $\vec{\omega}$ to be in the *negative* $\hat{k}$ direction. But, if this assumption is wrong, we will get a negative value for $\dot{\omega}$.

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned} \sum \vec{F} &= m_{\text{total}} \vec{a}_{\text{cm}} \\ -(m + 2M)g\hat{j} + N\hat{n} - F_f\hat{\lambda} &= (m + 2M)a_G\hat{\lambda} \end{aligned}$$

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns: N , F_f , and a_G . Thus we do not have enough equations to solve for the unknowns including the quantity of interest, a_G . So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using $\vec{r}_{G/C} = r\hat{n}$):

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned} \vec{M}_C &= \vec{r}_{G/C} \times (m + 2M)g(-\hat{j}) \\ &= r\hat{n} \times (m + 2M)g(-\hat{j}) \\ &= -(m + 2M)gr \sin \theta \hat{k} \quad (\text{see Fig. 17.70}) \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_C &= I_{zz}^G \dot{\vec{\omega}} + \vec{r}_{G/C} \times m_{\text{total}} \vec{a}_G \\ &= I_{zz}^G (-\dot{\omega} \hat{k}) - m_{\text{total}} \dot{\omega} r^2 \hat{k} \\ &= (I_{zz}^G + m_{\text{total}} r^2) (-\dot{\omega} \hat{k}) \\ &= \left[\left(\frac{1}{2} m r^2 + 2 \cdot \frac{1}{2} M R^2 \right) + \overbrace{(m + 2M)}^{m_{\text{total}}} r^2 \right] (-\dot{\omega} \hat{k}) \\ &= - \left[\frac{3}{2} m r^2 + M(R^2 + 2r^2) \right] \dot{\omega} \hat{k}. \end{aligned}$$

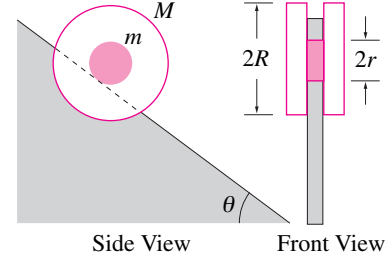


Figure 17.68: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

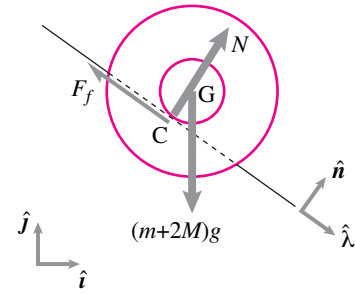


Figure 17.69: Free-body diagram of the wheel.

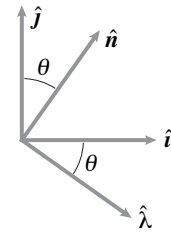


Figure 17.70: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example, $\hat{n} \times \hat{j} = \sin \theta \hat{k}$.

Thus,

$$\begin{aligned} -(m + 2M)gr \sin \theta \hat{\mathbf{k}} &= -\left[\frac{3}{2}mr^2 + M(R^2 + 2r^2)\right] \dot{\omega} \hat{\mathbf{k}} \\ \Rightarrow \dot{\omega} &= \frac{(m + 2M)gr \sin \theta}{\frac{3}{2}mr^2 + M(R^2 + 2r^2)}. \end{aligned} \quad (17.52)$$

Now we need to relate $\dot{\omega}$ to a_G . From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (17.52) we get

$$a_G = \frac{(m + 2M)gr^2 \sin \theta}{\frac{3}{2}mr^2 + M(R^2 + 2r^2)}.$$

Now we can solve for R in terms of a_G :

$$\begin{aligned} \frac{3}{2}mr^2 + M(R^2 + 2r^2) &= \frac{(m + 2M)gr^2 \sin \theta}{a_G} \\ \Rightarrow M(R^2 + 2r^2) &= \frac{(m + 2M)g}{a_G} r^2 \sin \theta - \frac{3}{2}mr^2 \\ \Rightarrow R^2 &= \frac{(m + 2M)g}{Ma_G} r^2 \sin \theta - \frac{3m}{2M} r^2 - 2r^2. \end{aligned}$$

Since we require $a_G \leq 0.2g$ we get

$$\begin{aligned} R^2 &\geq \left(\frac{(m + 2M)g}{M \cdot 0.2g} \sin \theta - \frac{3m}{2M} - 2 \right) r^2 \\ &\geq \left(\frac{5 \text{ kg}}{0.4 \text{ kg}} \cdot \frac{1}{2} - \frac{3 \text{ kg}}{4 \text{ kg}} - 2 \right) (0.1 \text{ m})^2 \\ &\geq 0.035 \text{ m}^2 \\ \Rightarrow R &\geq 0.187 \text{ m}. \end{aligned}$$

Thus the outer disks of radius 20 cm will do the job.

$$R \geq 18.7 \text{ cm}$$

SAMPLE 17.25 Which one starts rolling first — a marble or a bowling ball? A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say v_0 . The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

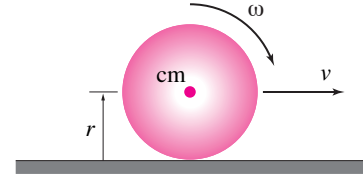


Figure 17.71

Solution Let us consider one ball, say the bowling ball, first. Let the radius of the ball be r and mass m . The ball starts with center-of-mass velocity $\vec{v}_0 = v_0 \hat{i}$. The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal μN (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition, $v = \omega r$. Let the instant of transition from the mixed motion to pure rolling be t^* . From linear momentum balance, we have

$$\begin{aligned} m\dot{v}\hat{i} &= -\mu N\hat{i} + (N - mg)\hat{j} & (17.53) \\ \text{eqn. (17.53)} \cdot \hat{j} &\Rightarrow N = mg \\ \text{eqn. (17.53)} \cdot \hat{i} &\Rightarrow m\dot{v} = -\mu N = -\mu mg \\ &\Rightarrow \dot{v} = -\mu g \\ &\Rightarrow v = v_0 - \mu gt. & (17.54) \end{aligned}$$

Similarly, from angular momentum balance about the mass-center, we get

$$\begin{aligned} -I_{zz}^{\text{cm}} \dot{\omega} \hat{k} &= -\mu N r \hat{k} = -\mu mgr \hat{k} \\ \Rightarrow \dot{\omega} &= \frac{\mu mgr}{I_{zz}^{\text{cm}}} \\ \Rightarrow \omega &= \underbrace{\omega_0}_0 + \frac{\mu mgr}{I_{zz}^{\text{cm}}} t. & (17.55) \end{aligned}$$

At the instant of transition from mixed rolling and sliding to pure rolling, *i.e.*, at $t = t^*$, $v = \omega r$. Therefore, from eqn. (17.54) and eqn. (17.55), we get

$$\begin{aligned} v_0 - \mu gt^* &= \frac{\mu mgr^2}{I_{zz}^{\text{cm}}} t^* \\ \Rightarrow v_0 &= \mu gt^* \left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}} \right) \\ \Rightarrow t^* &= \frac{v_0}{\mu g \left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}} \right)}. \end{aligned}$$

Now, for a sphere, $I_{zz}^{\text{cm}} = \frac{2}{5}mr^2$. Therefore,

$$t^* = \frac{v_0}{\mu g \left(1 + \frac{mr^2}{\frac{2}{5}mr^2} \right)} = \frac{2v_0}{7\mu g}.$$

Note that the expression for t^* is independent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$$t^* = \frac{2v_0}{7\mu g} \text{ for both.}$$

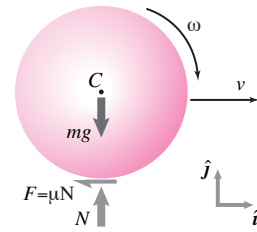


Figure 17.72: Free-body diagram of the ball during sliding.

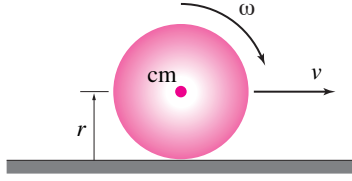


Figure 17.73

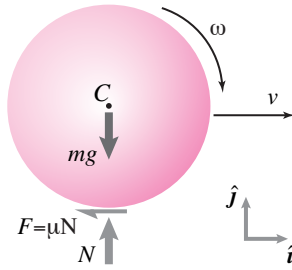


Figure 17.74: Free-body diagram of the ball during sliding.

SAMPLE 17.26 Transition from a mix of sliding and rolling to pure rolling, using impulse-momentum. Consider the problem in Sample 17.25 again: A ball of radius $r = 10$ cm and mass $m = 1$ kg is launched horizontally with initial velocity $v_0 = 5$ m/s on a surface with coefficient of friction $\mu = 0.2$. The ball starts sliding, rolls and slides simultaneously for a while, and then rolls without sliding. Find the time it takes to start pure rolling.

Solution Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by t^* . At $t = 0$, we know that $v_{\text{cm}} = v_0 = 5$ m/s, and $\omega_0 = 0$. We also know that at $t = t^*$, $v_{\text{cm}} = v_{t^*} = \omega_{t^*} r$, where r is the radius of the ball. We do not know t^* and v_{t^*} . However, we are considering a finite time event (during t^*) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

$$\begin{aligned} (\sum \vec{F}) t^* &= m v_{t^*} \hat{i} - m v_0 \hat{i} \\ (-\mu N \hat{i} + (N - mg) \hat{j}) t^* &= m(v_{t^*} - v_0) \hat{i}. \end{aligned}$$

Dotting the above equation with $\hat{j}$ and $\hat{i}$, respectively, we get

$$\begin{aligned} N &= mg \\ -\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0) \\ \Rightarrow -\mu g t^* &= v_{t^*} - v_0. \end{aligned} \quad (17.56)$$

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$\begin{aligned} \sum \vec{M}_{\text{cm}} t^* &= (\vec{H}_{\text{cm}})_{t^*} - (\vec{H}_{\text{cm}})_0 \\ (-\mu N r \hat{k}) t^* &= (I_{zz}^{\text{cm}} \omega_{t^*} - I_{zz}^{\text{cm}} \underbrace{\omega_0}_0)(-\hat{k}) \end{aligned}$$

or

$$\begin{aligned} -\mu m g r t^* &= -I_{zz}^{\text{cm}} \omega_{t^*} \\ \Rightarrow \omega_{t^*} &= \mu m g r t^* / I_{zz}^{\text{cm}} \\ \Rightarrow v_{t^*} &\equiv \omega_{t^*} r = \mu m g r^2 t^* / I_{zz}^{\text{cm}}. \end{aligned}$$

Substituting this expression for v_{t^*} in eqn. (17.56), we get

$$\begin{aligned} -\mu g t^* &= \mu m g r^2 t^* / I_{zz}^{\text{cm}} - v_0 \\ \Rightarrow t^* &= \frac{v_0}{\mu g (1 + \frac{m r^2}{I_{zz}^{\text{cm}}})} \end{aligned}$$

which is, of course, the same expression we obtained for t^* in Sample 17.25. Again, noting that $I_{zz}^{\text{cm}} = \frac{2}{5} m r^2$ for a sphere, we calculate the time of transition as

$$t^* = \frac{2v_0}{7\mu g} = \frac{2 \cdot (5 \text{ m/s})}{7 \cdot (0.2) \cdot (9.8 \text{ m/s}^2)} = 0.73 \text{ s}.$$

$$t^* = 0.73 \text{ s}$$

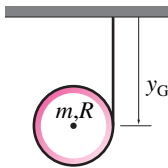
17.4 Mechanics of contact

17.4.1 A uniform disc of mass m and radius r rolls without slip at constant rate. What is the total kinetic energy of the disc?

17.4.2 A non-uniform disc of mass m and radius r rolls without slip at constant rate. The center of mass is located at a distance $\frac{r}{2}$ from the center of the disc. What is the total kinetic energy of the disc when the center of mass is directly above the center of the disc?

17.4.3 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R . G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

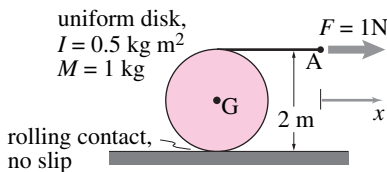
- Find y_G at a later time t in terms of any or all of m , R , g , and t .
- Does G move sideways as the hoop falls and unrolls?



Problem 17.4.3

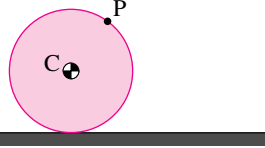
17.4.4 A uniform disk with radius R and mass m has a string wrapped around it. The string is pulled with a force F . The disk rolls without slipping.

- What is the angular acceleration of the disk, $\ddot{\alpha}_{\text{Disk}} = -\ddot{\theta}\hat{k}$? Make any reasonable assumptions you need that are consistent with the figure information and the laws of mechanics. State your assumptions. *
- Find the acceleration of the point A in the figure. *



Problem 17.4.4

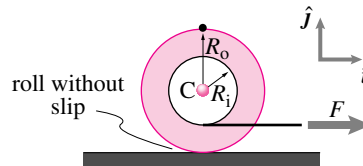
17.4.5 If a pebble is stuck to the edge of the wheel in problem 17.3.3, what is the maximum speed of the pebble during the motion? When is the force on the pebble from the wheel maximum? Draw a good FBD including the force due to gravity.



Problem 17.4.5

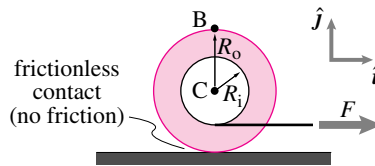
17.4.6 Spool Rolling without Slip and Pulled by a Cord. The light-weight spool is nearly empty but a lead ball with mass m has been placed at its center. A force F is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

- What is the acceleration of the center of the spool? *
- What is the horizontal force of the ground on the spool? *



Problem 17.4.6

17.4.7 A film spool is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i . What, in terms of R_i , R_o , m , g , $\hat{i}$, $\hat{j}$, and F are the accelerations of points C and B at the instant shown (the start of motion)?



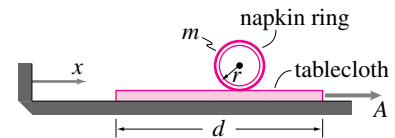
Problem 17.4.7

17.4.8 Again, Spool Rolling without Slip and Pulled by a Cord. Reconsider the spool from problem 17.4.6. This time, a force F is applied vertically to the cord wound around the wheel. In this case, what is the acceleration of the center of

the spool? Is it possible to pull the cord at some angle between horizontal and vertical so that the angular acceleration of the spool or the acceleration of the center of mass is zero? If so, find the angle in terms of R_i , R_o , m , and F .

17.4.9 A napkin ring lies on a black velvet tablecloth. The thin ring (of mass m , radius r) rolls without slip as a mischievous child pulls the tablecloth (mass M) out with acceleration $\ddot{a}_A$. The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

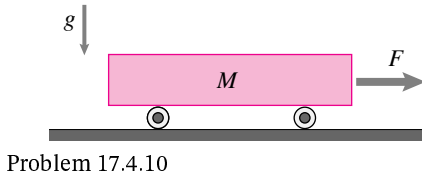
- What is the ring's acceleration as the tablecloth is being withdrawn?
- How far has the tablecloth moved to the right from its starting point $x = 0$ when the ring rolls off its left-hand end?
- Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?
- Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?



Problem 17.4.9

17.4.10 A block of mass M is supported by two rollers (uniform cylinders) each of mass m and radius r . They roll without slip on the block and the ground. A force F is applied in the horizontal direction to the right, as shown in the figure. Given F , m , r , and M , find:

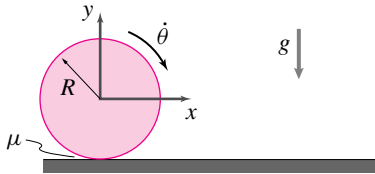
- the acceleration of the block,
- the acceleration of the center of mass of this block/roller system,
- the reaction at the wheel bases,
- the force of the right wheel on the block,
- the acceleration of the wheel centers, and
- the angular acceleration of the wheels.



Problem 17.4.10

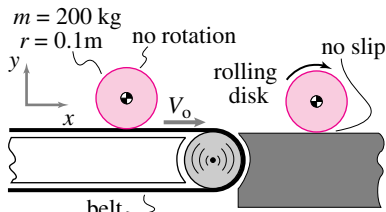
17.4.11 Dropped spinning disk. 2-D . A uniform disk of radius R and mass m is gently dropped onto a surface and doesn't bounce. When it is released it is spinning clockwise at the rate $\dot{\theta}_0$. The disk skids for a while and then is eventually rolling.

- What is the speed of the center of the disk when the disk is eventually rolling (answer in terms of g , μ , R , $\dot{\theta}_0$, and m)? *
- In the transition from slipping to rolling, energy is lost to friction. How does the amount lost depend on the coefficient of friction (and other parameters)? How does this loss make or not make sense in the limit as $\mu \rightarrow 0$ and the dissipation rate $\rightarrow$ zero? *



Problem 17.4.11

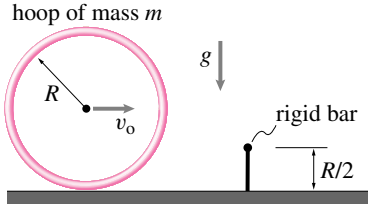
17.4.12 Disk on a conveyor belt. A uniform metal cylinder with mass of 200 kg is carried on a conveyor belt which moves at $V_0 = 3$ m/s. The disk is not rotating when on the belt. The disk is delivered to a flat hard platform where it slides for a while and ends up rolling. How fast is it moving (i.e. what is the speed of the center of mass) when it eventually rolls? *



Problem 17.4.12

17.4.13 A rigid hoop with radius R and mass m is rolling without slip so that its

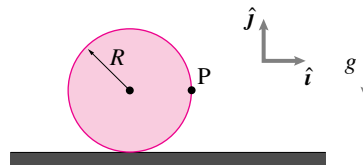
center has translational speed v_0 . It then hits a narrow bar with height $R/2$. When the hoop hits the bar suddenly it sticks and doesn't slide. It does hinge freely about the bar, however. The gravitational constant is g . How big is v_0 if the hoop just barely rolls over the bar? *



Problem 17.4.13

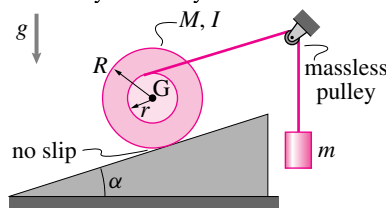
17.4.14 2-D rolling of an unbalanced wheel. A wheel, modeled as massless, has a point mass (mass = m) at its perimeter. The wheel is released from rest at the position shown. The wheel makes contact with coefficient of friction μ .

- What is the acceleration of the point P just after the wheel is released if $\mu = 0$?
- What is the acceleration of the point P just after the wheel is released if $\mu = 2$?
- Assuming the wheel rolls without slip (no-slip requires, by the way, that the friction be high: $\mu = \infty$) what is the velocity of the point P just before it touches the ground?



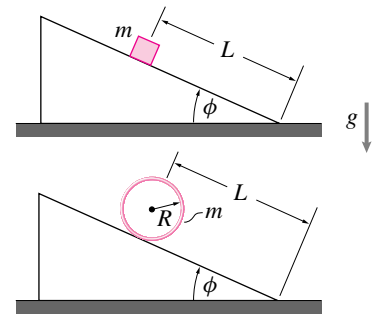
Problem 17.4.14

17.4.15 Spool and mass. A reel of mass M and moment of inertia $I_{zz}^m = I$ rolls without slipping upwards on an incline with slope-angle α . It is pulled up by a string attached to mass m as shown. Find the acceleration of point G in terms of some or all of M, m, I, R, r, α, g and any base vectors you clearly define.



Problem 17.4.15

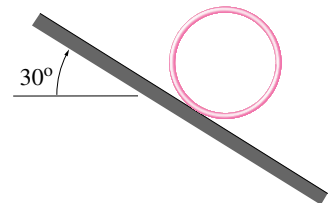
17.4.16 Two objects are released on two identical ramps. One is a sliding block (no friction), the other a rolling hoop (no slip). Both have the same mass, m , are in the same gravity field and have the same distance to travel. It takes the sliding mass 1 s to reach the bottom of the ramp. How long does it take the hoop? [Useful formula: " $s = \frac{1}{2}at^2$ "]



Problem 17.4.16

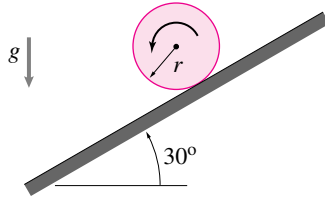
17.4.17 The hoop is rolled down an incline that is 30° from horizontal. It does not slip. It does not fall over sideways. It is let go from rest at $t = 0$.

- At $t = 0^+$ what is the acceleration of the hoop center of mass?
- At $t = 0^+$ what is the acceleration of the point on the hoop that is on the incline?
- At $t = 0^+$ what is the acceleration of the point on the hoop that is furthest from the incline?
- After the hoop has descended 2 vertical meters (and traveled an appropriate distance down the incline) what is the acceleration of the point on the hoop that is (at that instant) furthest from the incline?



Problem 17.4.17

17.4.18 A uniform cylinder of mass m and radius r rolls down an incline without slip, as shown below. Determine: (a) the angular acceleration α of the disk; (b) the minimum value of the coefficient of friction μ that will insure no slip.

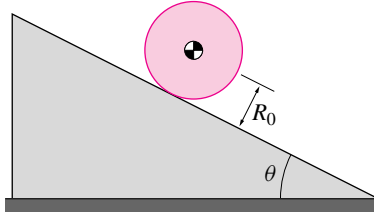


Problem 17.4.18

17.4.19 Race of rollers. A uniform disk with mass M_0 and radius R_0 is allowed to roll down the quite slip-resistant ($\mu = 1$) 30° ramp shown. It is raced against four other objects (A, B, C and D), one at a time. Who wins the races, or are there ties? First try to construct any plausible reasoning. Good answers will be based, at least in part, on careful use of equations of mechanics. *

- Block A has the same mass and has center of mass a distance R_0 from the ramp. It rolls on massless wheels with frictionless bearings.
- Uniform disk B has the same mass ($M_B = M_0$) but twice the radius ($R_B = 2R_0$).
- Hollow pipe C has the same mass ($M_C = M_0$) and the same radius ($R_C = R_0$).
- Uniform disk D has the same radius ($R_D = R_0$) but twice the mass ($M_D = 2M_0$).

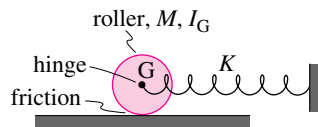
Can you find a round object which will roll as fast as the block slides? How about a massless cylinder with a point mass in its center? Can you find an object which will go slower than the slowest or faster than the fastest of these objects? What would they be and why? (This problem is harder.)



Problem 17.4.19

17.4.20 A roller of mass M and polar moment of inertia about the center of mass I_G is connected to a spring of stiffness K by a frictionless hinge as shown in the figure. Consider two kinds of friction between the roller and the surface it moves on:

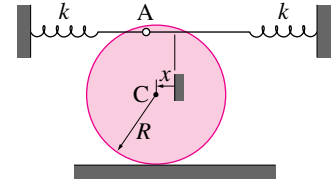
- Perfect slipping (no friction), and
 - Perfect rolling (infinite friction).
- What is the period of oscillation in the first case?
 - What is the period of oscillation in the second case?



Problem 17.4.20

17.4.21 A uniform cylinder of mass m and radius R rolls back and forth without slipping through small amplitudes (i.e., the springs attached at point A on the rim act linearly and the vertical change in the height of point A is negligible). The springs, which act both in compression and tension, are unextended when A is directly over C.

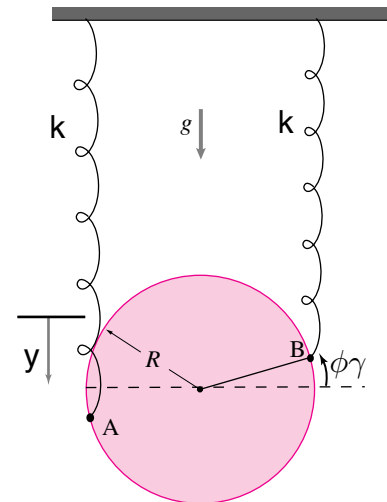
- Determine the differential equation of motion for the cylinder's center.
- Calculate the natural frequency of the system for small oscillations.



Problem 17.4.21

17.4.22 Hanging disk, 2-D. A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k . In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating their stretch ($\sin \phi \approx \phi$ and $\cos \phi \approx 1$).

- Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of $\phi, \dot{\phi}, y, \dot{y}, k, m, R$, and g .
- Find the natural frequencies of vibration in terms of some or all of k, m, R , and g .

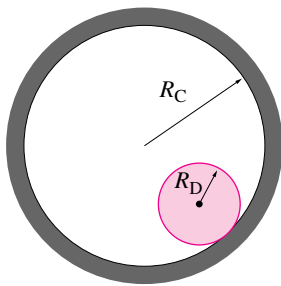


Problem 17.4.22

17.4.23 A disk rolls in a cylinder. For all of the problems below, the disk rolls without slip and rocks back and forth due to gravity.

- Sketch.** Draw a neat sketch of the disk in the cylinder. The sketch should show all variables, coordinates and dimension used in the problem.

- b) **FBD.** Draw a free-body diagram of the disk.
- c) **Momentum balance.** Write the equations of linear and angular momentum balance for the disk. Use the point on the cylinder which touches the disk for the angular momentum balance equation. Leave as unknown in these equations variables which you do not know.
- d) **Kinematics.** The disk rolling in the cylinder is a *one-degree-of-freedom* system. That is, the values of only *one* coordinate and its derivatives are enough to determine the positions, velocities and accelerations of all points. The angle that the line from the center of the cylinder to the center of the disk makes from the vertical can be used as such a variable. Find all of the velocities and accelerations needed in the momentum balance equation in terms of this variable and its derivative. [Hint: you'll need to think about the rolling contact in order to do this part.]
- e) **Equation of motion.** Write the angular momentum balance equation as a single second order differential equation.
- f) **Simple pendulum?** Does this equation reduce to the equation for a pendulum with a point mass and length equal to the radius of the cylinder, when the disk radius gets arbitrarily small? Why, or why not?

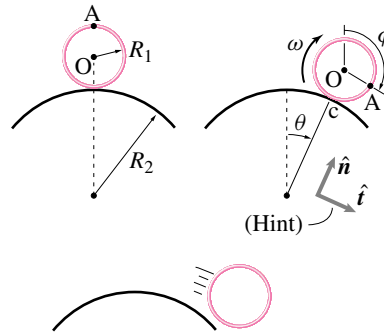


Problem 17.4.23: A disk rolls without slip inside a bigger cylinder.

17.4.24 A uniform hoop of radius R_1 and mass m rolls from rest down a semi-circular track of radius R_2 as shown. Assume that no slipping occurs. At what angle θ does the hoop leave the track and what is its angular velocity ω and the linear velocity $\bar{v}$ of its center of mass at that

instant? If the hoop slides down the track without friction, so that it does not rotate, will it leave at a smaller or larger angle θ than if it rolls without slip (as above)? Give a qualitative argument to justify your answer.

HINT: Here is a geometric relationship between angle ϕ hoop turns through and angle θ subtended by its center when no slipping occurs: $\phi = [(R_1 + R_2)/R_2]\theta$. (You may or may not need to use this hint.)



Problem 17.4.24

17.4.1 A uniform disc of mass m and radius r rolls without slip at constant rate. What is the total kinetic energy of the disk?

16.4.1



Rolling rate = constant = ω

$$\vec{V}_C = \vec{\omega} \times \vec{r} \Rightarrow \vec{V}_C = -\omega \hat{k} \times r \hat{j} \Rightarrow V_C = r\omega \hat{i}$$

$$K.E = \frac{1}{2} m \vec{V}_C \cdot \vec{V}_C + \frac{1}{2} I (\vec{\omega} \cdot \vec{\omega})$$

$$= \frac{1}{2} m (r\omega)^2 + \frac{1}{2} \frac{mr^2}{2} \omega^2$$

$$K.E = \frac{3}{4} mr^2 \omega^2$$

17.4.2 A non-uniform disc of mass m and radius r rolls without slip at constant rate. The center of mass is located at a distance $\frac{r}{2}$ from the center of the disc. What is the total kinetic energy of the disc when the center of mass is directly above the center of the disc?

$$I = m \times \frac{r^2}{4}$$

$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A} = -\omega \hat{k} \times \frac{3r}{2} \hat{j} = \frac{3\omega r}{2} \hat{i}$$

$$T_B = \frac{1}{2} m (\vec{v}_{B/A} \cdot \vec{v}_{B/A}) + \frac{1}{2} I \omega^2$$

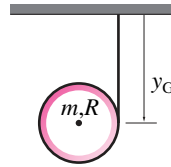
$$= \frac{1}{2} m \left(\frac{3r}{2} \omega \right)^2 + \frac{1}{2} m \left(\frac{r}{2} \right)^2 \omega^2$$

$$T_B = \frac{5}{4} m r^2 \omega^2$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

17.4.3 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R . G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

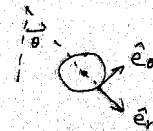
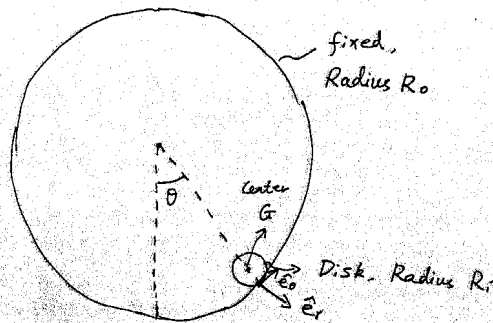
- b) Does G move sideways as the hoop falls and unrolls?



Problem 17.3

- a) Find y_G at a later time t in terms of any or all of m , R , g , and t .

14
17.56



G is doing a circular motion with a Radius $(R_0 - R_i)$

$$\vec{V}_G = \dot{\theta}(R_0 - R_i) \hat{e}_\theta$$

$$\vec{A}_G = \ddot{\theta}(R_0 - R_i) \hat{e}_\theta - \dot{\theta}^2(R_0 - R_i) \hat{e}_r$$

No slip condition

$\Rightarrow$ Angular velocity of the disk

$$\vec{V}_G = -\omega R_i \hat{e}_\theta$$

($\omega > 0$ if counter-clockwise)

$$\omega = -\frac{R_0 - R_i}{R_i} \dot{\theta}$$

Angular acceleration

$$\dot{\omega} = -\frac{R_0 - R_i}{R_i} \ddot{\theta}$$

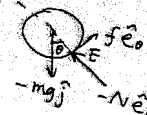
$$\vec{\omega} = \dot{\omega} \hat{k} = -\frac{R_0 - R_i}{R_i} \ddot{\theta} \hat{k}$$

Take A.M.B. about the contact point E .

$$\dot{H}_{/E} = \vec{r}_{G/E} \times m \vec{A}_G + I_G \dot{\omega}$$

$$= (-R_i \hat{e}_r) \times m (\ddot{\theta}(R_0 - R_i) \hat{e}_\theta - \dot{\theta}^2(R_0 - R_i) \hat{e}_r) + \frac{1}{2} m R_i^2 \left(-\frac{R_0 - R_i}{R_i} \ddot{\theta}\right) \hat{k}$$

$$= -m \ddot{\theta} R_i (R_0 - R_i) \hat{k} - \frac{1}{2} m (R_0 - R_i) R_i \ddot{\theta} \hat{k}$$



$$= -\frac{3}{2}mR_i(R_o - R_i)\ddot{\theta}\hat{k}$$

$$\vec{I}\vec{M}_{/E} = \vec{r}_{G/E} \times (-mg\hat{j})$$

$$= (-R_i\hat{e}_r) \times (-mg\hat{j}) = mgR_i \sin\theta \hat{k}$$

$$\text{AMB: } \vec{I}\vec{M}_{/E} = \vec{H}_{/E} \Rightarrow mgR_i \sin\theta \hat{k} = -\frac{3}{2}mR_i(R_o - R_i)\ddot{\theta}\hat{k}$$

$$\Rightarrow \frac{3}{2}mR_i(R_o - R_i)\ddot{\theta} + mgR_i \sin\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{2g}{3(R_o - R_i)} \sin\theta = 0$$

If the disk radius gets arbitrarily small, then $R_i = 0$

The equation of motion becomes

$$\ddot{\theta} + \frac{2g}{3R_o} \sin\theta = 0$$

This equation does not reduce to the simple pendulum equation,

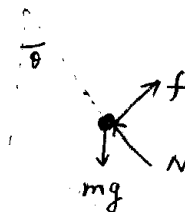
$$\text{which should be } \ddot{\theta} + \frac{g}{R_o} \sin\theta = 0$$

Reason: As the radius of the disk, R_i , goes to 0, our problem is NOT the same as a simple pendulum.

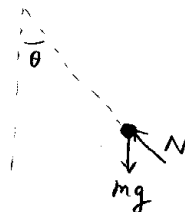
To enforce no slip condition, there must be friction acting on the object. However, for a simple pendulum, the reaction force should be in normal direction.

The difference can be illustrated in the FBD's below

Our problem when $R_i \rightarrow 0$



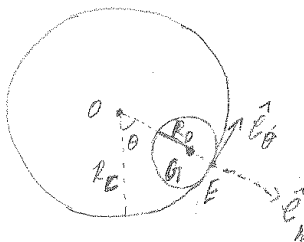
Simple pendulum



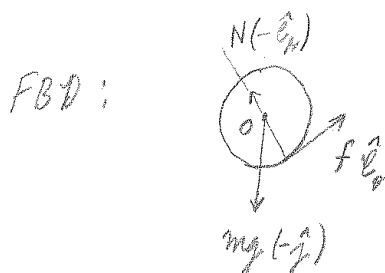
The existence of friction force in our problem makes it different from simple pendulum problem

14.56

(a)



(b) During rolling, there is no kinetic friction but there is static friction.



(c) LMB: $mg(-\hat{j}) + f(\hat{e}_\theta) + N(-\hat{e}_n) = m\vec{a}_G$

AMB: $\sum \vec{M}_E = \vec{H}_{/E} = \vec{r}_{G/E} \times m\vec{a}_G + I\dot{\omega}\hat{k}$

(d) Kinematics: G is moving in a circle of radius $(R_C - R_D)$

$$\vec{v}_G = \dot{\theta} (R_C - R_D) \hat{e}_\theta \quad \text{--- (i)}$$

$$\Rightarrow \vec{a}_G = \ddot{\theta} (R_C - R_D) \hat{e}_\theta - \dot{\theta}^2 (R_C - R_D) \hat{e}_n \quad \text{--- (ii)}$$

(Remember: $\dot{\theta} \neq \omega$. Here, ω refers to rotation of the disc.)

Rolling w/o slip :

$$\vec{V}_G = -\omega(\hat{k}) \times R_D(\hat{e}_\theta) = -\omega R_D \hat{e}_\theta$$

Using (i) : $\omega = -\frac{R_C - R_D}{R_D} \dot{\theta}$

(e) Now substitute $\vec{a}_G$ from (ii) in AMB,

$$\& \vec{r}_{G/E} = R_D(-\hat{e}_\theta)$$

$$\& \sum \vec{M}_{/E} = \vec{r}_{G/E} \times mg(-\hat{j}) = mg R_D \sin \theta \hat{k}$$

(after some algebra ...)

$$\ddot{\theta} + \frac{2g}{3(R_C - R_D)} \sin \theta = 0$$

(f) If $R_D \rightarrow 0$, $\ddot{\theta} = -\frac{2}{3} \frac{g}{R_C} \sin \theta$

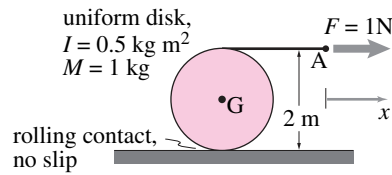
which is not the same as $\ddot{\theta} = -\frac{g}{R_C} \sin \theta$,
the equation for simple pendulum.

This is because even at small R_C , there is rolling! The function for $\ddot{\theta}$ will not abruptly jump by continuously varying R_D .

17.4.4 A uniform disk with radius R and mass m has a string wrapped around it. The string is pulled with a force F . The disk rolls without slipping.

- What is the angular acceleration of the disk, $\vec{\alpha}_{Disk} = -\ddot{\theta}\hat{k}$? Make any reasonable assumptions you need that are consistent with the figure information and the laws of mechanics. State your assumptions.
- Find the acceleration of the point

A in the figure.



Problem 17.4

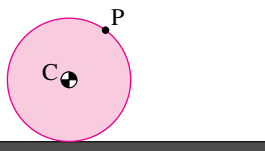
Answer:

(a) $\vec{\alpha}_{Disk} = -\frac{4}{3} \text{ rad/s}^2 \hat{k}$.

(b) $\vec{a}_A = \frac{8}{3} \text{ m/s}^2 \hat{i}$.

17.4.5 If a pebble is stuck to the edge of the wheel in problem ??, what is the maximum speed of the pebble during the motion? When is the force on the pebble from the wheel maximum? Draw a good FBD including the force due to gravity.

Problem 17.5



16.4.5

$$\vec{r}_C = R\theta \hat{i} + R\hat{j}$$

$$\vec{v}_C = R\dot{\theta} \hat{i}$$

$$\vec{\omega} = -\dot{\theta} \hat{k}$$

$$\vec{\alpha} = -\ddot{\theta} \hat{k}$$

$$\begin{aligned} \vec{r}_P &= \vec{r}_C + \vec{r}_{P/C} = R\theta \hat{i} + R\hat{j} + (-R\cos\theta \hat{j} - R\sin\theta \hat{i}) \\ &= R(\theta - \sin\theta) \hat{i} + R(1 - \cos\theta) \hat{j} \end{aligned}$$

$$\vec{v}_P = R[(\dot{\theta} - \cos\theta \dot{\theta}) \hat{i} + \sin\theta \dot{\theta} \hat{j}]$$

$$v_P = R\dot{\theta} [(1 - \cos\theta) \hat{i} + \sin\theta \hat{j}]$$

$$\text{If } \theta = 90^\circ \Rightarrow v_P = R\dot{\theta} (\hat{i} + \hat{j})$$

$$\theta = 180^\circ \rightarrow v_P = 2R\dot{\theta} \hat{i}$$

$$|\vec{v}_P|_{\max} = 2R\dot{\theta}$$

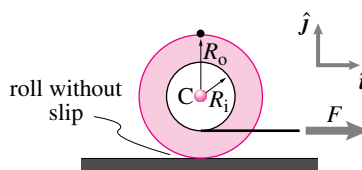
$$\vec{a}_P = \left. \frac{d\vec{v}_P}{dt} \right|_{\dot{\theta}=0} = R\ddot{\theta}^2 (\sin\theta \hat{i} + \cos\theta \hat{j}) \quad \text{when } \theta = \pi$$

$$\vec{a}_P = -R\ddot{\theta} \cos\theta \hat{j} - g \hat{j}$$

$$|a_P| = R\ddot{\theta}^2 \sqrt{\sin^2\theta + \cos^2\theta} = R\ddot{\theta}^2$$

17.4.6 Spool Rolling without Slip and Pulled by a Cord. The light-weight spool is nearly empty but a lead ball with mass m has been placed at its center. A force F is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

ground on the spool?



Problem 17.6

Answer:

(a) $\mathbf{a}_C = (F/m)(1 - R_i/R_o)\mathbf{i}$.

(b) $(-R_i/R_o)F\mathbf{i}$.

- What is the acceleration of the center of the spool?
- What is the horizontal force of the

7-22

Roll without slip

FBD

$\mathbf{v}_C = v_C \mathbf{i}$, $\mathbf{a}_C = a_C \mathbf{i}$, $\omega = -\omega \mathbf{k}$

$\mathbf{v}_A = \mathbf{v}_B = 0$, because point A coincides with point B at this instant, and point A is a stationary point on the ground the wheel doesn't slip

(a) (contd)

$\Rightarrow \mathbf{v}_B = \mathbf{v}_C + \omega \times \mathbf{r}_{B/C} = 0$

$v_C \mathbf{i} - R_o \omega \mathbf{k} = 0$

$\Rightarrow v_C = R_o \omega$

Taking the time derivative, $a_C = R_o \dot{\omega}$

(a) $\sum \mathbf{M}_B : \sum \mathbf{M}_B = \dot{\mathbf{H}}_B$

$\sum \mathbf{M}_B = \mathbf{r}_{B/A} \times \mathbf{F} = (R_o - R_i) \mathbf{j} \times F \mathbf{i}$

$= -F(R_o - R_i) \mathbf{k}$

$\dot{\mathbf{H}}_B = [\mathbf{I}_{CM}] \cdot \dot{\omega} + \mathbf{v}_B \times [\mathbf{I}_{CM}] \cdot \omega + \mathbf{r}_{B/C} \times m \mathbf{a}_C$

$= R_o \mathbf{j} \times m a_C \mathbf{i} = -R_o m a_C \mathbf{k}$

From (1) and (2):

$-F(R_o - R_i) \mathbf{k} = -R_o m a_C \mathbf{k}$

$\Rightarrow a_C = \frac{F}{m} \left(\frac{R_o - R_i}{R_o} \right)$ The spool moves to the right.

Horizontal force of ground on spool?

(b) $\sum \mathbf{F} = m \mathbf{a}_C$

$F \mathbf{i} + F_f (-\mathbf{i}) + N \mathbf{j} - mg \mathbf{j} = m a_C \mathbf{i}$

$\{ \} \cdot \mathbf{j} \Rightarrow N = mg$

$\{ \} \cdot \mathbf{i} \Rightarrow F - F_f = m a_C$

We know a_C from part (a), and $F_f = \mu N = \mu mg$:

$\Rightarrow F_f = F - m a_C = F - m \frac{F}{m} \left(\frac{R_o - R_i}{R_o} \right)$

$= F \left(\frac{R_i}{R_o} \right)$

$\mathbf{F}_f = F \frac{R_i}{R_o} (-\mathbf{i}) = -F \frac{R_i}{R_o} \mathbf{i}$

$\mathbf{F}_f = -F \frac{R_i}{R_o} \mathbf{i}$

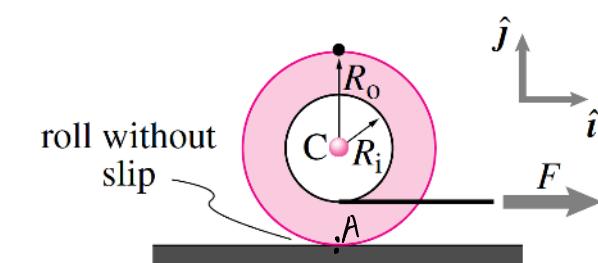
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16.4.6

Saturday, April 20, 2019

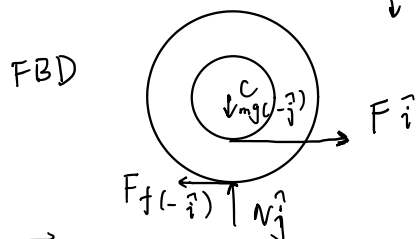
8:23 PM

Spool Rolling without slip and pulled by a cord



Filename: pfigure-s94h11p5

Problem 16.4.6

A light-weight spool with center C A lead ball with mass m at C A force F is in $\hat{i}$ direction
 $\curvearrowright \omega$ 

$$\vec{v}_C = v_C \hat{i}$$

$$\vec{a}_C = a_C \hat{i}$$

$$\omega = -\omega \hat{k}$$

$$\vec{v}_A = \vec{v}_{A'} = 0 \quad (\text{no slip})$$

$$\vec{v}_C = \vec{v}_A + \vec{\omega} \times \vec{r}_{C/A}$$

$$= \vec{0} + (-\omega \hat{k}) \times R_o \hat{j}$$

$$= R_o \omega \hat{i} \Rightarrow v_C = R_o \omega$$

Take derivative,

$$\vec{a}_C = R_0 \dot{\omega}$$

(a) Find the acceleration of C

$$A \text{ MB/A} : \Sigma M/A = \dot{H}/A$$

$$\Rightarrow (R_0 - R_1) \hat{j} \times F \hat{i} = I^{CM} \dot{\omega} + \vec{r}_{C/A} \times m \vec{a}_C$$

$$-(R_0 - R_1) F \hat{k} = R_0 \hat{j} \times m a_C \hat{i} = -R_0 m a_C \hat{k}$$

$$\therefore a_C = \frac{F}{m} \left(1 - \frac{R_1}{R_0}\right)$$

$$\vec{a}_C = \frac{F}{m} \left(1 - \frac{R_1}{R_0}\right) \hat{i}$$

(b) Find F_f

$$LMB : \Sigma \vec{F} = m \vec{a}_C$$

$$\left\{ F \hat{i} + F_f (-\hat{i}) + N \hat{j} - mg \hat{j} = m a_C \hat{i} \right\}$$

$$\left\{ \right\} \cdot \hat{i} \Rightarrow F - F_f = m a_C$$

We know a_C from part (a)

$$\Rightarrow F_f = F - m a_C = F \frac{R_1}{R_0}$$

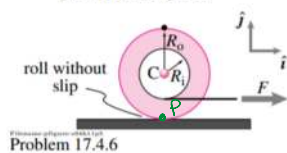
$$F_f = - F \frac{R_1}{R_0} \hat{i}$$

#58 Solution Michelle deSouza

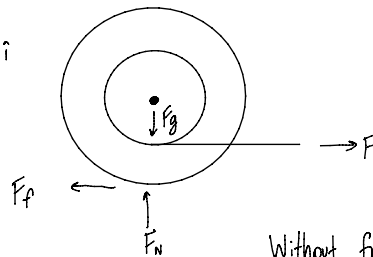
Saturday, April 24, 2021 10:07 PM

17.4.6 Spool Rolling without Slip and Pulled by a Cord. The light-weight spool is nearly empty but a lead ball with mass m has been placed at its center. A force F is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

- What is the acceleration of the center of the spool? *
- What is the horizontal force of the ground on the spool? *



FBD:

 $j \rightarrow i$ 

Without friction, the spool would be sliding in the $+i$ direction. Friction force opposes motion and acts in the $-i$ direction.

a) Find $\vec{a}_c$ (c is also the center of mass)

Since the spool is rolling, we know $\vec{a}_c = a_c \hat{i}$

To find $\vec{a}_c$ in terms of only known F , take AMB about point P

$$\sum \vec{M}_P = \vec{H}_P$$

0, massless spool

$$(\vec{R}_o - \vec{R}_i) \times F \hat{i} = \vec{R}_o \times m a_c \hat{i} + I_c \ddot{\theta} \hat{k}$$

$$-F(\vec{R}_o - \vec{R}_i) \hat{k} = -m a_c \vec{R}_o \hat{k}$$

$$\text{E3} \cdot \hat{k} \quad \frac{F}{m} \frac{R_o - R_i}{R_o} = a_c$$



b) Find F_f

$$\text{LMB: } \sum \vec{F} = -F_f \hat{i} + F \hat{i} - mg \hat{j} + F_n \hat{j} = m \vec{a}_c \quad \text{plug in from a)}$$

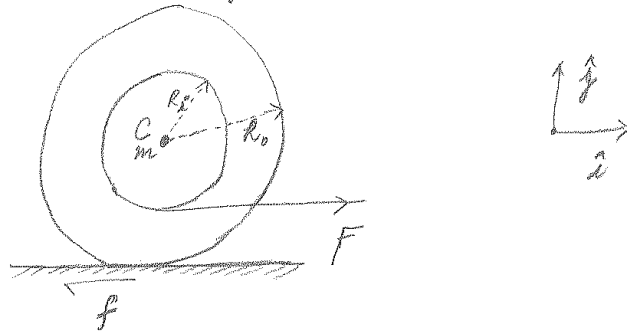
$$\text{E3} \cdot \hat{i} \quad -F_f + F = F \left(\frac{R_o - R_i}{R_o} \right)$$

$$F_f = F \left(1 - \frac{R_o - R_i}{R_o} \right)$$

$$F_f = \frac{R_i}{R_o} F$$

Note: This problem is made simpler by modeling a point mass instead of a rigid object with distributed mass $\rightarrow$ no rolling kinematics necessary.

14.39 Spool rolling w/o slip



* We need to determine the acceleration of the center of mass $\vec{a}_{cm}$, and the horizontal force $\vec{F}$ acting on the lowest pt. as shown.

* Newton's II Law immediately yields:

$$\text{LMB: } F \hat{i} - f \hat{i} = m \vec{a}_{cm} \quad \text{--- (i)}$$

$$\begin{aligned} \& \text{ AMB: } R_i (-\hat{j}) \times F \hat{i} + R_o (-\hat{j}) \times f (-\hat{i}) \\ &= I^c \vec{\alpha} \quad \text{--- (ii)} \end{aligned}$$

* Since the spool is "massless", $I^c \rightarrow 0$

$$\Rightarrow \text{(ii) yields } R_i F = R_o f$$

Substituting in (i)

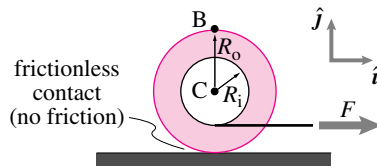
$$\vec{a}_{cm} = \frac{F}{m} \left(1 - \frac{R_i}{R_o} \right) \hat{i}$$

Also,

$$\left(\frac{R_i}{R_o} \right) F \hat{i} = \vec{f}$$

17.4.7 A film spool is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i . What, in terms of R_i , R_o , m , g , $\hat{i}$, $\hat{j}$, and F are the accelerations of points C and B at the instant shown (the start of mo-

tion)?



Problem 17.7

16.4.7

$$\sum \vec{F} = m \vec{a}_{cm}$$

$$\vec{a}_c = \vec{a}_A + \vec{a}_{c/A}$$

L M B

$$F \hat{i} = m \vec{a}_{cm}$$

$$= m \ddot{x} \hat{i} + m \ddot{y} \hat{j}$$

$$\ddot{x} = \frac{F}{m} \quad \therefore \vec{a}_{cm} = \frac{F}{m} \hat{i}$$

AMB /c

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$-R_o \hat{j} \times F \hat{i} = \vec{r}_{cm} \times m \vec{a}_{cm} + \vec{I}_{cm} \dot{\vec{\omega}} \hat{k} + \vec{\omega} \times \vec{I}_{cm} \vec{\omega}$$

$$R_o F \hat{k} = \vec{I}_{cm} \dot{\vec{\omega}} \hat{k} = \frac{m R_i^2}{2} \dot{\vec{\omega}} \hat{k}$$

$$\dot{\vec{\omega}} = \frac{2F}{m R_i} \hat{k}$$

$$F = \frac{m R_i \dot{\omega}}{2}$$

$$\Rightarrow \vec{a}_{cm} = \frac{m R_i \dot{\vec{\omega}}}{2m} = \frac{R_i \dot{\vec{\omega}}}{2}$$

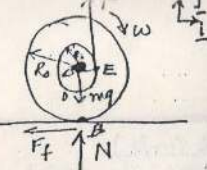
Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

$$\begin{aligned}
 \vec{a}_B &= \vec{a}_c + \vec{a}_{B/c} = \vec{a}_{cm} + [\vec{\dot{\omega}} \times \vec{R}_0] + \vec{\omega} \times (\vec{\omega} \times \vec{R}_0) \\
 &= \vec{a}_{cm} + \frac{2}{m R_i} F \hat{k} \times R_0 \hat{j} + \omega \hat{k} \times (\omega \hat{k} \times R_0 \hat{j}) \quad \left| \quad \begin{aligned} \dot{\omega} &= \frac{2F}{m R_i} \\ \omega &= \frac{2F}{m R_i} t + C_1 \\ \omega &= \frac{2F}{m R_i} t \\ t=0, \omega &= 0 \end{aligned} \right. \\
 &= \frac{F}{m} \hat{i} + \frac{2 R_0 F}{m R_i} (-\hat{i}) - \omega^2 R_0 \hat{j} \\
 \vec{a}_B &= \frac{F}{m} \left(1 - \frac{2 R_0}{R_i} \right) \hat{i} - \omega^2 R_0 \hat{j} \\
 \vec{a}_B &= \frac{F}{m} \left(1 - \frac{2 R_0}{R_i} \right) \hat{i}
 \end{aligned}$$

17.4.8 Again, Spool Rolling without Slip and Pulled by a Cord. Reconsider the spool from problem 17.4.6. This time, a force F is applied vertically to the cord wound around the wheel. In this case, what is the acceleration of the center of

the spool? Is it possible to pull the cord at some angle between horizontal and vertical so that the angular acceleration of the spool or the acceleration of the center of mass is zero? If so, find the angle in terms of R_i , R_o , m , and F .

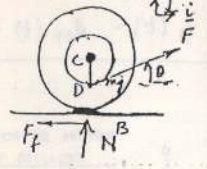
7.23 Same as 7.22 but F acts vertically as shown.



From 7.22, we obtain
 $v_c = R_o \omega$
 and $a_c = R_o \dot{\omega}$

(a) AMB: $\sum \underline{M}_B = \underline{\dot{H}}_B$
 $\sum \underline{M}_B = \underline{r}_{E/B} \times \underline{F} = (R_i \underline{i} + R_o \underline{j}) \times F \underline{j}$
 $= R_i F \underline{k}$ (1)
 $\underline{\dot{H}}_B = [\underline{I}_B] \cdot \underline{\dot{\omega}} + \underline{\omega} \times [\underline{I}_B] \cdot \underline{\omega} + \underline{r}_{c/B} \times m \underline{a}_c$
 $= R_o \underline{j} \times m a_c \underline{i} = -R_o m a_c \underline{k}$
 From (1) and (2):
 $a_c = -\frac{F}{m} \left(\frac{R_i}{R_o} \right)$
 $a_c = -\frac{F}{m} \left(\frac{R_i}{R_o} \right)$ The spool moves to the left!

(b)



AMB: $\sum \underline{M}_B = \underline{\dot{H}}_B$
 $\sum \underline{M}_B = \underline{r}_{E/B} \times \underline{F} = (R_o - R_i) \underline{j} \times (F \cos \theta) \underline{i}$
 $= -(R_o - R_i) (F \cos \theta) \underline{k}$ (1)
 $\underline{\dot{H}}_B = [\underline{I}_B] \cdot \underline{\dot{\omega}} + \underline{\omega} \times [\underline{I}_B] \cdot \underline{\omega} + \underline{r}_{c/B} \times m \underline{a}_c$
 $\underline{\dot{H}}_B = -R_o m a_c \underline{k}$ (2)
 $\therefore -(R_o - R_i) (F \cos \theta) \underline{k} = -R_o m a_c \underline{k}$
 $\Rightarrow \cos \theta = \frac{R_o}{(R_o - R_i)} \cdot \frac{1}{F} m a_c$
 $\Rightarrow \theta = \cos^{-1} \left[\left(\frac{R_o}{R_o - R_i} \right) \frac{1}{F} m a_c \right]$
 If $a_c = 0$ then $\theta = \cos^{-1}(0) \Rightarrow \theta = \pi/2, 3\pi/2, \dots$

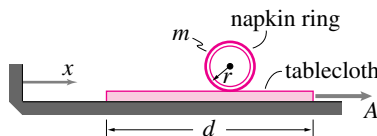
Q (contd.)
 It is not possible to pull the cord at some angle between horizontal and vertical in order for the accel. of center of mass to be zero.

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

17.4.9 A napkin ring lies on a black velvet tablecloth. The thin ring (of mass m , radius r) rolls without slip as a mischievous child pulls the tablecloth (mass M) out with acceleration $\bar{a}_A$. The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

- What is the ring's acceleration as the tablecloth is being withdrawn?
- How far has the tablecloth moved to the right from its starting point $x = 0$ when the ring rolls off its left-hand end?

- Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?
- Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?



Problem 17.9

14.42 "Napkin ring" problem.

(a) The lowest pt. on the ring will have the same acceleration as the cloth.

$$\bar{a}_p = a \hat{i} = a_{cm} \hat{i} + \alpha \hat{k} \times r(-\hat{j}) \quad (1)$$

Since friction acts at the lowest pt., then

$$\bar{M}_p = \bar{H}_p = r \hat{j} \times m a_{cm} \hat{i} + I_{zz} \alpha \hat{k} = \vec{0}$$

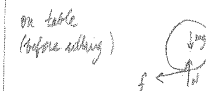
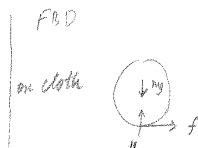
$$\Rightarrow m a_{cm} r = I_{zz} \alpha$$

$$m a_{cm} r = m r \alpha$$

$$\Rightarrow a_{cm} = r \alpha$$

Using (i)

$$a_{cm} = \frac{a}{2}$$



(b) The ring moves 'd' in time 't' relative to the cloth with acceleration $a/2$

$$\Rightarrow d = \frac{1}{2} \frac{a}{2} t^2 \Rightarrow t = 2\sqrt{d/a}$$

$$\text{In the same time, the cloth moves } \frac{1}{2} a (2\sqrt{d/a})^2 = \boxed{2d}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

(C) When the ring leaves the cloth, the lowest pt. has velocity $\vec{v}_b = 2\sqrt{ad} \hat{i}$ ($v = v_0 + at$)

Now two cases are possible :

(i) Table is smooth : In this all velocities (angular, linear) are conserved. The ring coats with the same velocity it has when it leaves the cloth.

(ii) Table has friction : In this case, friction will act until lowest pt. comes to rest.

Since lowest pt. is moving to the right — friction will be in the ~~to~~ $-\hat{i}$ direction.

Friction will cause a clockwise torque about CM.

Assuming friction (μ), one can calculate the time ' t_2 ' when lowest pt. will come to rest : $t_2 = \frac{2\sqrt{ad}}{\mu g}$ ($v = at$)

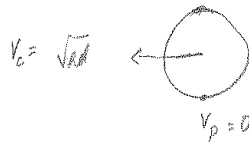
Now turn attention to the center-of-mass :

As it leaves the cloth, the center-of-mass

will have velocity $v_c = \sqrt{ad} \hat{i}$.

On the table, because of friction, it will decelerate. By the time the lowest pt. comes to rest (in time ' t_c '), the cm will have velocity $\sqrt{ad} - \frac{2\sqrt{ad} \mu g}{\mu g} = -\sqrt{ad} \hat{i}$

At t_c , we have a situation like this:



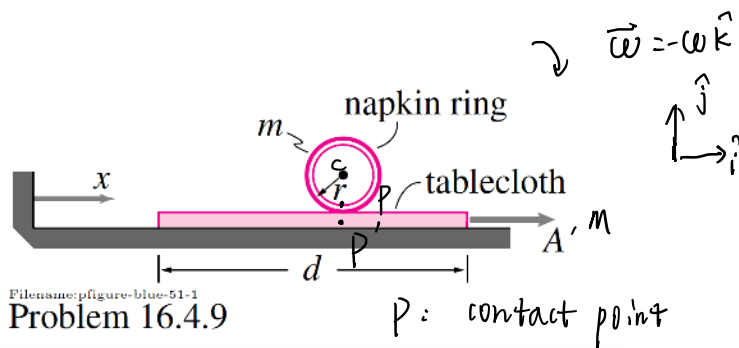
Now on there is no work from friction. So it stays rolling like above. (Obviously $v_c = \omega R$).

(d) Of course, if there is sliding initially on the cloth, we don't expect the same v_c as above. But it remains true the ring will roll in the same direction.

16.4.9

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Problem 16.4.9

 p : contact point

Given :

ring mass m , radius r tablecloth mass M ring starts at $x=d$ without slipas tablecloth moves with acceleration A .

$$\vec{\omega} = -\omega \hat{k}$$

$$\vec{a}_c = a_c \hat{i}$$

(a) Find the ring's acceleration $\vec{a} = \dot{\vec{\omega}} = -\dot{\omega} \hat{k} = -\alpha \hat{k}$

$$\vec{a}_p = \vec{a}_c + \dot{\vec{\omega}} \times \vec{r}_{p/c} - \omega^2 \vec{r}_{p/c}$$

$$\vec{a}_p' = \vec{A} = A \hat{i}$$

$$\text{no slip} \Rightarrow \{\vec{a}_p\} \cdot \hat{i} = \{\vec{a}_p'\} \cdot \hat{i} = A$$

$$\Rightarrow a_c - \alpha r = A \quad (1)$$

$$\text{AMB/p: } \vec{M}_p = \vec{H}_p$$

$$\vec{0} = r \hat{j} \times m a_c \hat{i} + I^c \alpha \hat{k} \quad (2)$$

$$I^c = mr^2 \quad (3)$$

$$(1) (2) (3) \Rightarrow a_c = \frac{A}{2}$$

- (b) How far has the tablecloth moved?
 (from $x=0$ to when the ring rolls off)
 Assume the whole process lasts for t .
 Relative distance of motion = d

$$x_{\text{cloth}} - x_{\text{ring}} = d$$

$$x_{\text{cloth}} = \frac{1}{2} A t^2$$

$$x_{\text{ring}} = \frac{1}{2} a_c t^2$$

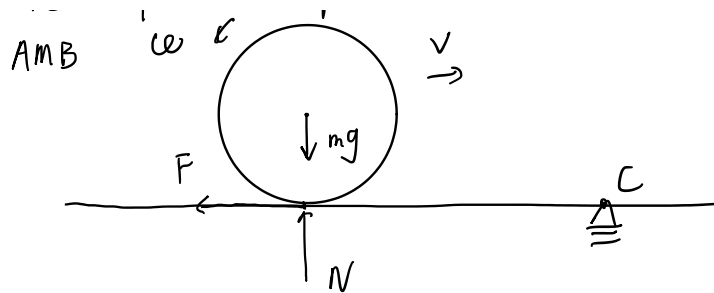
$$\Rightarrow x_{\text{ring}} = 2d$$

- (c) Clearly describe the subsequent motion of ring
 Which way does it end up rolling at what speed?
 When the ring leaves the cloth, the contact point
 has velocity $\vec{v}_p = 2\sqrt{Ad}$?

Now two cases are possible:

- (i) Table is frictionless: then all velocities (angular, linear) are conserved.
- (ii) Table has friction: friction will act until the ring stops moving at all. The linear velocity and angular velocity of the ring while rolling are not compatible with each other. The ring will slip at the very beginning. In fact, the angular momentum provided by the friction and its actual angular momentum $I\omega$ are in the opposite directions, and they are incompatible with the rolling condition. Thus the ring will stop in the end.

The equation explanations are as follows:



$$\vec{H}_C = \vec{0} \quad \text{at all } t.$$

$$\Rightarrow \vec{r} \times m \vec{v} - I \vec{\omega} = 0 \quad \text{at all } t. \quad (1)$$

$$\text{rolling condition: } v = -r\omega$$

$$v + r\omega = 0 \quad (2)$$

$$(1) \quad (2) \Rightarrow v = \omega = 0$$

(d) Answer to (c) different if there is slip?

No. the ring will still stop in the end.

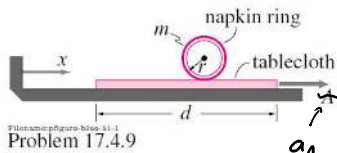
The reason is the same as above.

HW11-P59-17.4.9

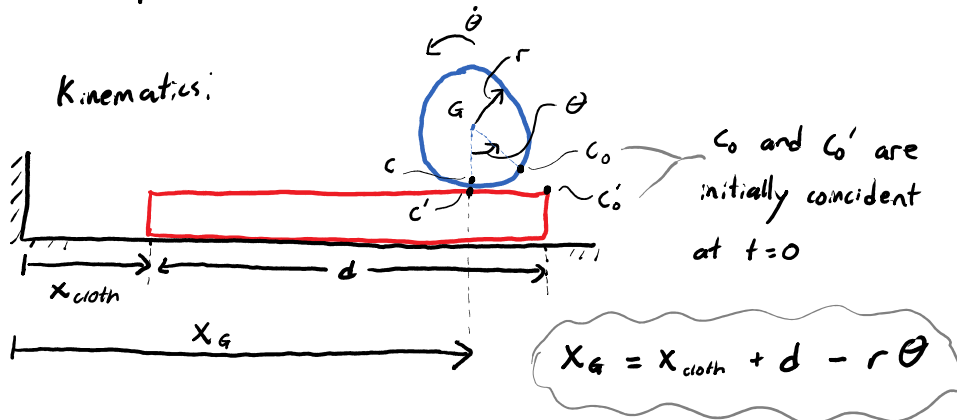
Wednesday, April 21, 2021 2:09 PM

17.4.9 A napkin ring lies on a thick velvet tablecloth. The thin ring (of mass m , radius r) rolls without slip as a mischievous child pulls the tablecloth (mass M) out with acceleration a_A . The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

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- Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?
- Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?



Let x_G track the position of G



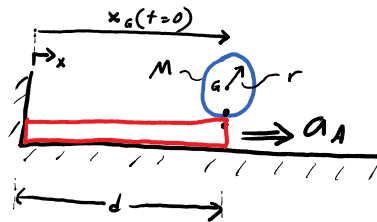
Solution: Michael Zakworotny

Given: Ring has m, r

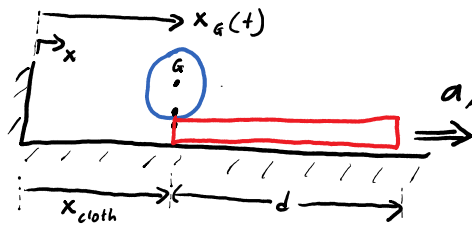
Tablecloth has M , pulled at acceleration a_A

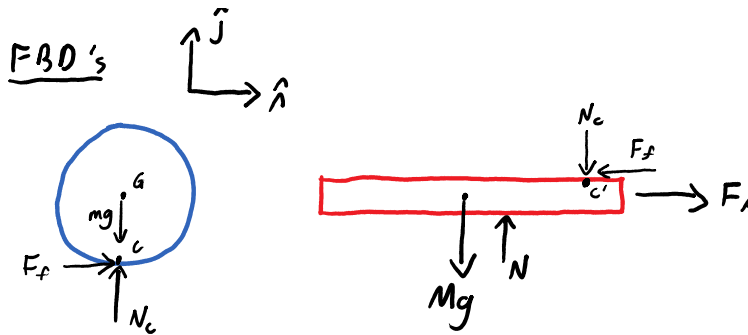
Ring starts at $x=d$

At $t=0$



At a later time





a) Find ring's acceleration when being pulled

Consider AMB of ring about point C

$$\sum \vec{M}_{i/c} = \dot{\vec{H}}_{i/c}$$

$$\{ \vec{0} = \vec{r}_{G/c} \times m \vec{a}_{G/F} + I^G \ddot{\theta} \hat{k} \}$$

$$\vec{r}_{G/c} = r \hat{j} \quad \vec{a}_{G/F} = a_G \hat{i}$$

$$\{ \} \cdot \hat{k} : -r m a_G + I^G \ddot{\theta} = 0$$

$$a_G = \frac{I^G}{mr} \ddot{\theta} \quad (1)$$

From kinematics of the system

$$x_G = x_{cloth} + d - r\theta$$

$$\dot{x}_G = \dot{x}_{cloth} - r\dot{\theta}$$

$$\ddot{x}_G = a_G = \dot{x}_{cloth} - r \ddot{\theta}$$

$$v = a_A$$

$$\ddot{\theta} = \frac{a_A - a_G}{r} \quad (2)$$

Plug (2) into (1):

$$a_G = \frac{I_G}{mr} \cdot \frac{a_A - a_G}{r}$$

$$a_G \left(1 + \frac{I_G}{mr^2} \right) = \frac{I_G}{mr^2} \cdot a_A$$

$$a_G = \frac{I_G}{mr^2 + I_G} \cdot a_A$$

Hold even if $a_A = a_A(t)$

We know $I_G = mr^2$ for a thin ring

$$a_G = \frac{mr^2}{mr^2 + mr^2} \cdot a_A \Rightarrow a_G = \frac{a_A}{2}$$

b) Find distance moved by ring by the time at which it rolls off left end of cloth

This occurs when $x_G = x_{cloth}$

From kinematics: $x_G = x_{cloth} + d - r\theta$

From ABB: $a_G = \frac{I_G}{mr} \ddot{\theta} \Rightarrow \{ \ddot{x}_G = r \ddot{\theta} \}$

$$\int \{ \} dt \Rightarrow \{ \dot{x}_G = r \dot{\theta} \} + c_1 \quad \text{since } \dot{x}_G(0) = 0 \text{ when } \dot{\theta}(0) = 0$$

$$\int \{ \} dt \Rightarrow x_G = r\theta + c_2 \quad \text{when } \theta = 0, x_G = d$$

$$x_G = r\theta + d \quad \text{so, } c_2 = d$$

$$r\theta = x_G - d$$

Combine kinematics and AMB:

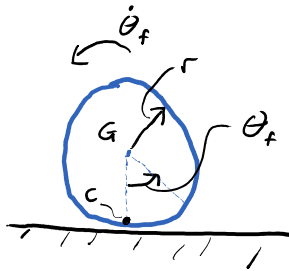
$$x_G = x_{\text{cloth}} + d - (x_G - d)$$

$$2x_G - x_{\text{cloth}} = 2d$$

Apply $x_G = x_{\text{cloth}}$

$$x_{\text{cloth}} = 2d$$

c) Describe ring's motion after cloth has been pulled



From kinematics, we now have

$$v_{G,f} = -r\dot{\theta}_f \quad (3) \quad \text{for no-slip}$$

rolling on the table

(t = final state)

Again, from AMB:

$$\sum \vec{M}_{I,C} = \dot{H}_{I,C}$$

We can use the same AMB formula as a) and b) because we only looked at FBD of ring, and it does not depend on whether ring is actually rolling, or whether the ring is on a moving cloth or stationary table. We therefore obtain $\ddot{x}_{G,f} = r \ddot{\theta}_f$, which can be integrated to obtain

$\dot{x}_{G,f} = r \dot{\theta}_f$. However, now kinematics tells us that $\dot{x}_{G,f} = -r \dot{\theta}_f$ (since ring is rolling on a stationary surface). Combining:

$$\dot{x}_{G,f} = r \dot{\theta}_f = -r \dot{\theta}_f \Rightarrow \dot{\theta}_f = 0, \quad v_{G,f} = 0$$

Initially, there must be some slip between point C on the ring and the surface, because C moved at same velocity as point C' on the cloth. Eventually, AMB necessitates that the ring must come to rest.

d) Would c) be different if ring slipped on cloth?

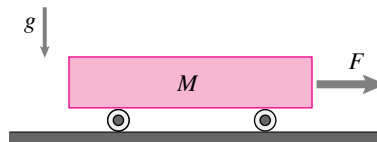
No, the ring would still come to rest

because this reasoning does not depend on the kinematics of the ring when it is on the cloth.

17.4.10 A block of mass M is supported by two rollers (uniform cylinders) each of mass m and radius r . They roll without slip on the block and the ground. A force F is applied in the horizontal direction to the right, as shown in the figure. Given F , m , r , and M , find:

- the acceleration of the block,
- the acceleration of the center of mass of this block/roller system,
- the reaction at the wheel bases,

- the force of the right wheel on the block,
- the acceleration of the wheel centers, and
- the angular acceleration of the wheels.



Problem 17.10

16.4.10

$$\sum \vec{F} = m \vec{a}_b$$

$$F \hat{i} - Mg \hat{j} + N_{b1} \hat{j} + N_{b2} \hat{j} = M \vec{a}_b$$

$$\vec{a}_b = a_x \hat{i} + a_y \hat{j}$$

$$a) F = Ma_x \therefore a_x = \frac{F}{M}$$

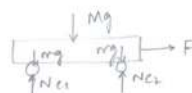
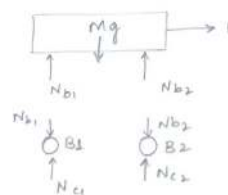
$$\therefore \vec{a}_b = \frac{F}{M} \hat{i}$$

$$b) \sum \vec{F} = m \vec{a}_{cm}$$

$$F \hat{i} + N_{c1} \hat{j} + N_{c2} \hat{j} - Mg \hat{j} - 2mg \hat{j} = (M+2m) \vec{a}_{cm}$$

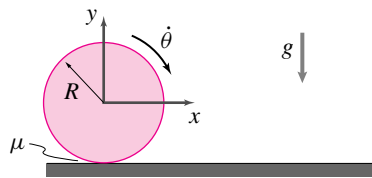
$$N_{c1} + N_{c2} = Mg + 2mg$$

$$\vec{a}_{cm} = \frac{F}{M+2m} \hat{i}$$



17.4.11 Dropped spinning disk. 2-D

. A uniform disk of radius R and mass m is gently dropped onto a surface and doesn't bounce. When it is released it is spinning clockwise at the rate $\dot{\theta}_0$. The disk skids for a while and then is eventually rolling.



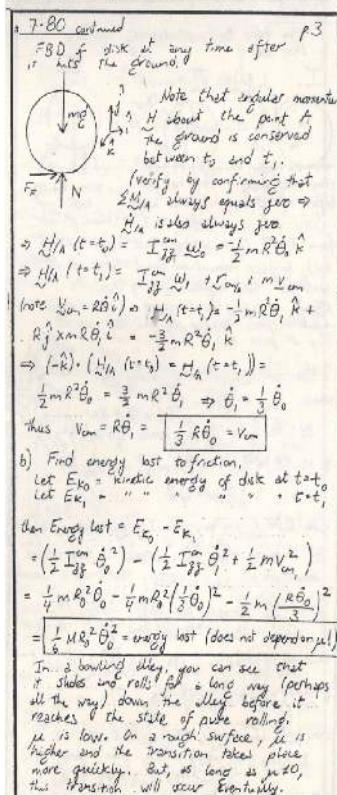
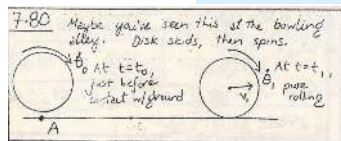
- What is the speed of the center of the disk when the disk is eventually rolling (answer in terms of g , μ , R , $\dot{\theta}_0$, and m)?
- In the transition from slipping to rolling, energy is lost to friction. How does the amount lost depend on the coefficient of friction (and other parameters)? How does this loss make or not make sense in the limit as $\mu \rightarrow 0$ and the dissipation rate $\rightarrow$ zero?

Problem 17.11

Answer:

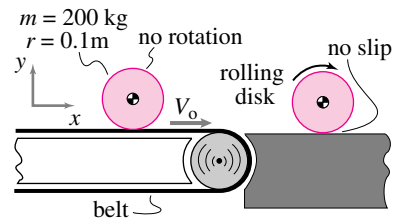
(a) speed $v = \frac{R\dot{\theta}}{3}$.

(b) The energy lost to friction is $E_{fric} = \frac{mR^2\dot{\theta}_0^2}{6}$. The energy lost to friction is independent of μ for $\mu > 0$. Thus, the energy lost to friction is constant for given m , R , and $\dot{\theta}_0$. As $\mu \rightarrow 0$, the transition time to rolling $\rightarrow \infty$. It is not true, however, that the energy lost to friction $\rightarrow 0$ as $\mu \rightarrow 0$. Since the energy lost is constant for any $\mu > 0$, the disk will slip for longer and longer times so that the distance of slip goes to infinity. The dissipation rate $\rightarrow 0$ since the constant energy is divided by increasing transition time. The energy lost is zero only for $\mu = 0$.



Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

17.4.12 Disk on a conveyor belt. A uniform metal cylinder with mass of 200 kg is carried on a conveyor belt which moves at $V_0 = 3$ m/s. The disk is not rotating when on the belt. The disk is delivered to a flat hard platform where it slides for a while and ends up rolling. How fast is it moving (i.e. what is the speed of the center of mass) when it eventually rolls?

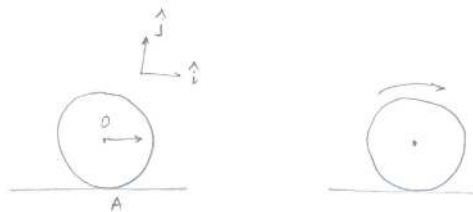


Problem 17.12

Answer:

$$V = 2 \text{ m/s.}$$

16.4.12



Just before rolling

$$\vec{H}_{/A} = \vec{r}_{O/A} \times m \vec{v}_{cm} \longrightarrow (1)$$

During rolling

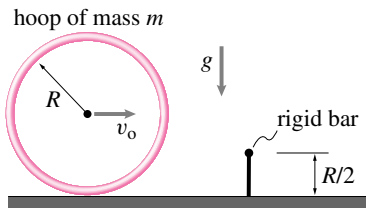
$$\vec{H}_{/A} = \vec{r}_{O/A} \times m \vec{v}_f + \vec{I}^{cm} \cdot \vec{\omega}_f \longrightarrow (2)$$

Equating (1) and (2)

$$-mR V_0 = -Rm v_f - \frac{m R^2}{2} \times \frac{v_f}{R}$$

$$-V_0 = -v_f - \frac{v_f}{2} \Rightarrow V_0 = \frac{3v_f}{2} \Rightarrow v_f = 2 \text{ m/s}$$

17.4.13 A rigid hoop with radius R and mass m is rolling without slip so that its center has translational speed v_0 . It then hits a narrow bar with height $R/2$. When the hoop hits the bar suddenly it sticks and doesn't slide. It does hinge freely about the bar, however. The gravitational constant is g . How big is v_0 if the hoop just barely rolls over the bar?



Problem 17.13

Answer:

$$v_0 = 2\sqrt{\frac{2gR}{3}}.$$

Page 13.

(7.73) mass m . $\downarrow g$ P $\uparrow R/2$

The hoop rolls w/o slipping. How big should v_0 be such that the hoop barely climbs above bar. Assume it sticks at P and hinges freely at that point.

Solution:

Since the force of friction does no work during rolling w/o slip, we use conservation of energy.

Initial energy: $E_P = 0$ (assumed reference position)

$$E_K = \frac{1}{2} m v_0^2 + \frac{1}{2} I_{cm} \omega_0^2$$

Since it rolls w/o slip $\omega_0 = \frac{v_0}{R}$

$$E_K = \frac{1}{2} m v_0^2 + \frac{1}{2} (m R^2) \cdot \frac{v_0^2}{R^2}$$

I_{cm} for a hoop

$$E_K = m v_0^2$$

Final energy: $E_K = 0$

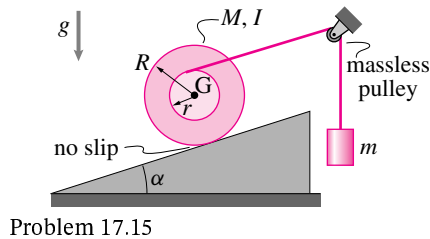
$$E_P = mg \frac{R}{2}$$

Equating $E_{initial}$ and E_{final} .

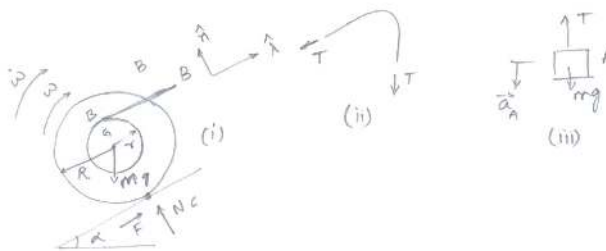
$$v_0 = \sqrt{\frac{gR}{2}}$$

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17.4.15 Spool and mass. A reel of mass M and moment of inertia $I_{zz}^{cm} = I$ rolls without slipping upwards on an incline with slope-angle α . It is pulled up by a string attached to mass m as shown. Find the acceleration of point G in terms of some or all of M, m, I, R, r, α, g and any base vectors you clearly define.



16.4.15



$$\sum \underline{M}_{/c} = \dot{\underline{H}}_{/c}$$

$$R \hat{n} \times mg(-\hat{j}) + (R+r) \hat{n} \times T \hat{\lambda} = R \times m \underline{a}_G + I \dot{\omega} \hat{k}$$

$$mgR \sin \alpha \hat{k} - (R+r) T \hat{k} = R \hat{n} \times m(-\dot{\omega} R) \hat{\lambda} + I \dot{\omega} \hat{k}$$

$$(mgR \sin \alpha - (R+r)T) \hat{k} = (mR^2 \dot{\omega} + I \dot{\omega}) \hat{k} \rightarrow \textcircled{1}$$

Now, From the FBD of the mass 'm'.

$$T \hat{j} - mg \hat{j} = m \underline{a}_A$$

$$|a_A| = |a_B|$$

$$\therefore (T - mg) = m(\underline{a}_G + \underline{a}_{B/G})$$

$$(T - mg) = m(\dot{\omega} R + \dot{\omega} r)$$

$$T = mg - m \dot{\omega} (R+r)$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

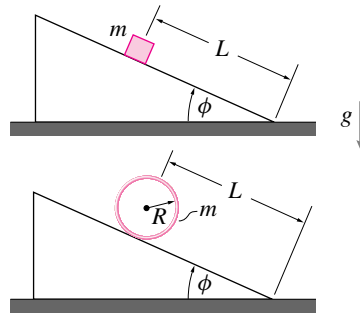
on substituting the value in (1).

$$mgR \sin \alpha - mg(R+r) + m\dot{\omega}(R+r)^2 = (mR^2 + I)\dot{\omega}$$

$$\dot{\omega} = \frac{mg(R \sin \alpha - (R+r))}{(mR^2 + I - m(R+r)^2)}$$

$$\underline{a}_m = \frac{mgR [R \sin \alpha - (R+r)]}{(mR^2 + I - m(R+r)^2)} \hat{\lambda}$$

17.4.16 Two objects are released on two identical ramps. One is a sliding block (no friction), the other a rolling hoop (no slip). Both have the same mass, m , are in the same gravity field and have the same distance to travel. It takes the sliding mass 1 s to reach the bottom of the ramp. How long does it take the hoop? [Useful formula: " $s = \frac{1}{2}at^2$ "]



Problem 17.16

16.4.16

$\Sigma \vec{F} = m\vec{a}$
 $N\hat{n} - mg\hat{j} = m\vec{a}$
 $N\hat{n} + mg \sin\phi \hat{\lambda} - mg \cos\phi \hat{n} = m\vec{a}$
 $\hat{j} = -\sin\phi \hat{\lambda} + \cos\phi \hat{n}$
 $\{ \} \cdot \hat{\lambda} \Rightarrow a = g \sin\phi$
 $\{ \} \cdot \hat{n} \Rightarrow N = mg \cos\phi$
 $s = \frac{1}{2}at^2 \Rightarrow L = \frac{1}{2}g \sin\phi t^2$

$\vec{a}_G = \vec{\omega} \times R\hat{n}$
 $= -\vec{\omega} \hat{k} \times R\hat{n}$
 $\vec{a}_G = a_n \hat{\lambda}$
 $\Sigma M_{/C} = \frac{I}{C}$
 $R\hat{n} \times mg(-\hat{j}) = R\hat{n} \times m\vec{a}_G + I_{cm}(-\vec{\omega})\hat{k}$
 $-mgR\hat{k} \sin\phi = -ma_n R\hat{k} - mR\vec{\omega}\hat{k}$
 $a_n = \frac{g}{2} \sin\phi$

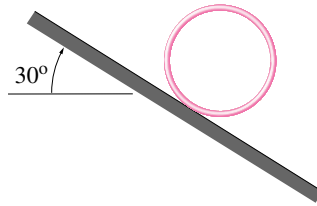
$t^2 \propto \frac{1}{a}$
 $\frac{t_h^2}{t_b^2} = \frac{a_b}{a_h} \times t_b^2 = \frac{a_b}{a_h} \cdot t_b^2$
 $= \frac{g \sin\phi}{\frac{g \sin\phi}{2}}$
 $t_h = \sqrt{2} t_b$

The hoop takes $\sqrt{2}$ times longer than the mass

17.4.17 The hoop is rolled down an incline that is 30° from horizontal. It does not slip. It does not fall over sideways. It is let go from rest at $t = 0$.

- At $t = 0^+$ what is the acceleration of the hoop center of mass?
- At $t = 0^+$ what is the acceleration of the point on the hoop that is on the incline?
- At $t = 0^+$ what is the acceleration of the point on the hoop that is furthest from the incline?
- After the hoop has descended 2 vertical meters (and traveled an

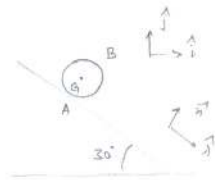
appropriate distance down the incline) what is the acceleration of the point on the hoop that is (at that instant) furthest from the incline?



Problem 17.17

9

16.4.17



a) $a_G = ?$ (from previous problem)

$$a_G = \frac{g}{2} \sin \theta = \frac{g}{4} \hat{\lambda}$$

or

$$a_G = \vec{\omega} \times R \hat{\lambda}$$

b) $a_A = ?$

$$\vec{a}_A = \vec{a}_G + \vec{a}_{A/G}$$

$$\vec{a}_A = \vec{\omega} \times R \hat{\lambda}$$

$$\vec{a}_A = \vec{a}_G + \vec{a}_{A/G} = \omega^2 R \hat{n}$$

$$\begin{aligned} \vec{a}_{A/G} &= \vec{\omega} \times R \hat{n} + \vec{\omega} \times (\vec{\omega} \times R (-\hat{n})) \\ &= -\vec{\omega} \hat{k} \times R \hat{n} + (-\omega \hat{k}) \times (-\omega R \hat{n}) \\ &= \vec{\omega} R (\hat{k} \times \hat{n}) - \omega \hat{k} \times \omega R \hat{n} \\ \vec{a}_{A/G} &= -\omega R \hat{\lambda} + \omega^2 R \hat{n} \end{aligned}$$

c) $\vec{a}_B = ?$

$$\vec{a}_B = \vec{a}_G + \vec{a}_{B/G}$$

$$\vec{a}_{B/G} = -\vec{\omega} \hat{k} \times R \hat{n} + [-\omega \hat{k} \times (-\omega \hat{k} \times R \hat{n})]$$

$$= \omega R \hat{\lambda} - \omega^2 R \hat{n}$$

Now,

$$\vec{a}_B = 2\omega R \hat{\lambda} - \omega^2 R \hat{n}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

(10)

d)

$$\text{As } a_G = \dot{\omega} R = \frac{g}{2} \sin 30^\circ = \frac{g}{4}$$

$$\dot{\omega} = \frac{g}{4R} \Rightarrow \omega = \frac{gt}{4R} + C_1 \quad \text{at } t=0, \omega = 0$$

$$C_1 = 0$$

$$\Rightarrow \omega = \frac{gt}{4R} \Rightarrow \theta = \frac{1}{2} \frac{gt^2}{4R}$$

No slip $\Rightarrow$

$$x = R\theta \Rightarrow L = R \cdot \frac{1}{2} \frac{gt^2}{4R}$$

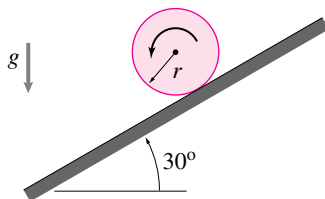
$$t = \sqrt{\frac{8L}{g}} = 4\sqrt{\frac{y}{g}}$$

$$\omega = \frac{g \cdot 4\sqrt{\frac{y}{g}}}{4R} = \sqrt{\frac{yg}{R^2}} = \frac{\sqrt{2g}}{R}$$

$$\therefore \vec{a}_B = 2 \cdot \frac{g}{4} \hat{\lambda} - \frac{2g}{R^2} \cdot R \cdot \hat{n}$$

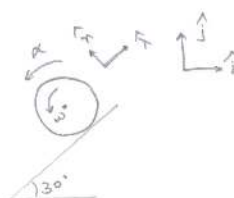
$$\vec{a}_B = \frac{g}{2} \hat{\lambda} - \frac{2g}{R} \hat{n}$$

17.4.18 A uniform cylinder of mass m and radius r rolls down an incline *without slip*, as shown below. Determine: (a) the angular acceleration α of the disk; (b) the minimum value of the coefficient of friction μ that will insure no slip.



Problem 17.18

16.4.18



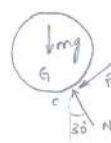
(11)

$$a) \sum M/c = \dot{H}/c$$

$$r \hat{n} \times mg(-\hat{j}) = r \hat{n} \times m \vec{a}_G + I^G \dot{\omega} \hat{k}$$

$$\sin \lambda \, mg r \hat{k} = m r^2 \dot{\omega} \hat{k} + \frac{m r^2}{2} \dot{\omega} \hat{k}$$

$$\dot{\omega} = \frac{2g \sin \lambda}{3r} \Rightarrow \dot{\omega} = \frac{g}{3r}$$



b) Minimum value of co-efficient of friction to ensure no slip

$$\sum \vec{F} = m \vec{a}_G$$

$$N \hat{n} - F \hat{\lambda} - mg \hat{j} = m a_G (-\hat{\lambda})$$

$$\Rightarrow \left\{ N \hat{n} - F \hat{\lambda} - mg \sin \hat{\lambda} - mg \cos \lambda \hat{n} = -m \dot{\omega} r \hat{\lambda} \right\}$$

$$\left\{ \right\} \cdot \hat{n} \Rightarrow N = mg \cos \lambda, \quad \left\{ \right\} \cdot \hat{\lambda} \Rightarrow -F - mg \sin \lambda = -m \dot{\omega} r = -\frac{mg}{3}$$

$$F = \frac{mg}{3} - \frac{mg}{2} \Rightarrow F = -\frac{mg}{6} \Rightarrow F = \mu N, \quad \mu = \frac{F}{N}$$

$$\mu = \frac{-\frac{mg}{6}}{mg \cos \lambda} = \frac{1}{3\sqrt{3}}$$

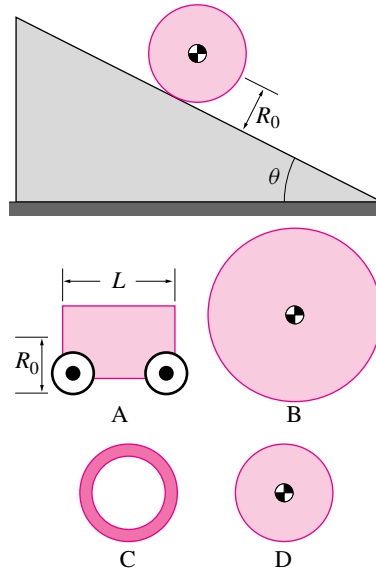
Any $\mu < \frac{1}{3\sqrt{3}}$, at $\theta = 30^\circ$ results in cylinder slip

17.4.19 Race of rollers. A uniform disk with mass M_0 and radius R_0 is allowed to roll down the quite slip-resistant ($\mu = 1$) 30° ramp shown. It is raced against four other objects (A, B, C and D), one at a time. Who wins the races, or are there ties? First try to construct any plausible reasoning. Good answers will be based, at least in part, on careful use of equations of mechanics.

- Block A has the same mass and has center of mass a distance R_0 from the ramp. It rolls on massless wheels with frictionless bearings.
- Uniform disk B has the same mass ($M_B = M_0$) but twice the radius ($R_B = 2R_0$).
- Hollow pipe C has the same mass ($M_C = M_0$) and the same radius ($R_C = R_0$).
- Uniform disk D has the same radius ($R_D = R_0$) but twice the mass ($M_D = 2M_0$).

Can you find a round object which will roll as fast as the block slides? How about a massless cylinder with a point mass in its center? Can you find an object which will go slower than the slowest or faster than the fastest of these ob-

jects? What would they be and why? (This problem is harder.)



Problem 17.19

Answer:

Accelerations of the center of mass, where $\mathbf{i}$ is parallel to the slope and pointing down: (a) $\mathbf{a}_{cm} = g \sin \theta \mathbf{i}$, (b) $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta \mathbf{i}$, (c) $\mathbf{a}_{cm} = \frac{1}{2} g \sin \theta \mathbf{i}$, (d) $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta \mathbf{i}$. So, the block is fastest, all uniform disks are second, and the hollow pipe is third.

Solutions to Problem Set 10: FAM 203 S.96
 TA: Erich Schneider • July 26, 1996

7.75 Race of rollers.

Diagram showing objects: Disk "O", Block "A", Disk "B", Pipe "C", Disk "D".

Rolling down a 30° ramp. Which one wins the race?

Generic FBD: $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm}$ about contact point O: $\mathbf{a}_{cm} = R\ddot{\theta}$.

For things that roll: (Disks "O", "B", "D", pipe "C")

Some reasoning/guesses first:
 Note that each term in the AMB is proportional to the mass of the object. This is true, as well for the block.

Block "A": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = g \sin \theta$.

Disk "O": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta$.

Disk "B": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta$.

Pipe "C": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{1}{2} g \sin \theta$.

Disk "D": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta$.

Note: see addendum on last pg for more of prob. 7.75

7.75 continued p 2

term in (1), as well as the LHS of the AMB, is the same for each of our 5 objects. The answer depends only on the relative moment of inertia of each of our rolling bodies. Since the moment of inertia of hoop "C" ($= MR^2$) is bigger than the moment of inertia of a disk ($= \frac{1}{2} MR^2$), I would expect that the disks will all accelerate at the same rate (a_{cm} does not depend on R or M), which is faster than the rate at which the hoop will accelerate. The block will accelerate fastest of all, since it is undergoing straight line acceleration ($\ddot{w} \neq 0$).

Thus: Block A (Disks O, B, D) Hoop C
 Fastest same accel. slowest

Calculations:

Disk "O": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta$.

Block "A": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = g \sin \theta$.

Disk "B": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta$.

Pipe "C": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{1}{2} g \sin \theta$.

Disk "D": $\mathbf{F}_{net} = M\mathbf{a}_{cm}$. $\mathbf{a}_{cm} = \frac{2}{3} g \sin \theta$.

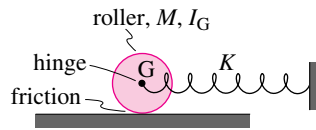
Note from (1) above that $\ddot{\theta}$ or $\frac{1}{R} \ddot{a}_{cm}$, but $\ddot{a}_{cm} = R\ddot{\theta}$, so the acceleration of the center of mass will not depend on the radius of the body. In fact, the first

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17.4.20 A roller of mass M and polar moment of inertia about the center of mass I_G is connected to a spring of stiffness K by a frictionless hinge as shown in the figure. Consider two kinds of friction between the roller and the surface it moves on:

1. Perfect slipping (no friction), and
2. Perfect rolling (infinite friction).

- a) What is the period of oscillation in the first case?
- b) What is the period of oscillation in the second case?



Problem 17.20

(12)

16.4.20

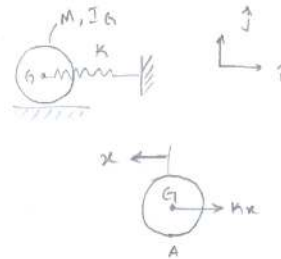
a) No friction

$$\sum \underline{M}_{/A} = \underline{\dot{H}}_{/A}$$

$$\sigma \hat{j} \times Kx \hat{i} = \sigma \hat{j} \times m \vec{a}_G$$

$$Kx \sigma (\hat{k}) = -\sigma \hat{j} \times m \ddot{x} \hat{i} = m \ddot{x} \sigma \hat{k}$$

$$m \ddot{x} + Kx = 0 \Rightarrow \omega = \sqrt{\frac{K}{m}}, \quad T = 2\pi \sqrt{\frac{m}{K}}$$



b) Perfect rolling

$$\sum \underline{M}_{/A} = \underline{\dot{H}}_{/A}$$

$$\sigma \hat{j} \times Kx \hat{i} = \sigma \hat{j} \times m \ddot{x} (-\hat{i}) + \underline{I} \dot{\omega} \hat{k}$$

$$Kx \sigma (\hat{k}) = m \ddot{x} \sigma \hat{k} + \underline{I} \frac{\ddot{x}}{r} \hat{k}$$

$$\left(m + \frac{I}{r^2}\right) \ddot{x} + Kx = 0$$

$$\omega = \sqrt{\frac{K}{m + \frac{I}{r^2}}}$$

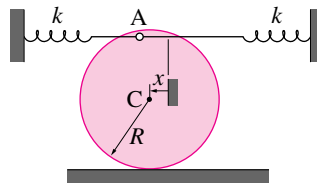
$$T = 2\pi \sqrt{\frac{m + \frac{I}{r^2}}{K}}, \quad I = \frac{mr^2}{2}$$

$$T = \left(\sqrt{\frac{3}{2}}\right) 2\pi \sqrt{\frac{m}{K}}$$

17.4.21 A uniform cylinder of mass m and radius R rolls back and forth without slipping through small amplitudes (i.e., the springs attached at point A on the rim act linearly and the vertical change in the height of point A is negligible). The springs, which act both in compression and tension, are unextended when A is directly over C.

center.

- b) Calculate the natural frequency of the system for small oscillations.



Problem 17.21

16.4.21

$$\sum \mathbf{M}_{/C} = \dot{\mathbf{H}}_{/C}$$

$$a) \cos \lambda \, 2R \hat{j} \times 2kx \hat{i} = R \hat{j} \times m \underbrace{a_{cm}}_{-\omega^2 R \hat{i}} + I_{cm} \dot{\omega} \hat{k} - m \dot{\omega} r \hat{i}$$

$$\cos \lambda (-4kxR) \hat{k} = m \dot{\omega} R^2 \hat{k} + \frac{mR^2}{2} \dot{\omega} \hat{k} \\ = mR \ddot{x} \hat{k} + mR \frac{\ddot{x}}{2} \hat{k}$$

$$(\cos \lambda)(-4kxR \hat{k}) = \frac{3}{2} mR \ddot{x} \hat{k}$$

$$\frac{3}{2} m \ddot{x} + 4kx \cos \lambda = 0$$

$$m \ddot{x} + \frac{8}{3} kx \cos \lambda = 0$$

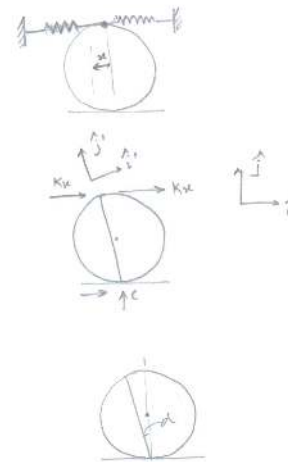
$$\omega = \sqrt{\frac{8K \cos \lambda}{3m}}$$

Differential Eqn

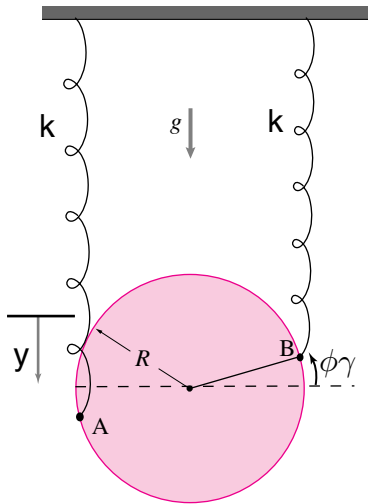
$$3m \ddot{x} + 8K \cos \lambda = 0$$

- b) For small oscillations $\cos \lambda = 1$

$$f = \frac{1}{2\pi} \sqrt{\frac{8K}{3m}}$$



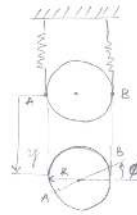
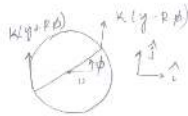
17.4.22 Hanging disk, 2-D. A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k . In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating their stretch ($\sin \phi \approx \phi$ and $\cos \phi \approx 1$).



Problem 17.22

- Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of ϕ , $\dot{\phi}$, y , $\dot{y}$, k , m , R , and g .
- Find the natural frequencies of vibration in terms of some or all of k , m , R , and g .

16.4.22



$$\sum \underline{M}_{I_0} = \dot{\underline{H}}_{I_0}$$

Let $\phi = \text{small}$

$$\underline{I} \ddot{\phi} = KR(y - R\phi) - KR(y + R\phi) \Rightarrow \underline{I} \ddot{\phi} = -KR^2\phi - KR^2\phi$$

$$m \ddot{\phi} + 4KR\phi = 0$$

$$\omega_1 = \sqrt{\frac{4K}{m}}$$

Now,

$$\sum F = m \ddot{a}$$

$$[K(y - R\phi) + K(y + R\phi)] \hat{j} = m \ddot{y} (-\hat{j})$$

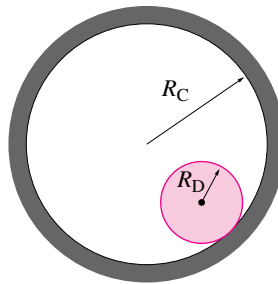
$$m \ddot{y} + 2Ky = 0$$

$$\omega_2 = \sqrt{\frac{2K}{m}}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

17.4.23 A disk rolls in a cylinder. For all of the problems below, the disk rolls without slip and rocks back and forth due to gravity.

- Sketch.** Draw a neat sketch of the disk in the cylinder. The sketch should show all variables, coordinates and dimension used in the problem.
- FBD.** Draw a free-body diagram of the disk.
- Momentum balance.** Write the equations of linear and angular momentum balance for the disk. Use the point on the cylinder which touches the disk for the angular momentum balance equation. Leave as unknown in these equations variables which you do not know.
- Kinematics.** The disk rolling in the cylinder is a *one*-degree-of-freedom system. That is, the values of only *one* coordinate and its derivatives are enough to determine the positions, velocities and accelerations of all points. The angle that the line from the center of the cylinder to the center of the disk makes from the vertical can be used as such a variable. Find all of the velocities and accelerations needed in the momentum balance equation in terms of this variable and its derivative. [Hint: you'll need to think about the rolling contact in order to do this part.]
- Equation of motion.** Write the angular momentum balance equation as a single second order differential equation.
- Simple pendulum?** Does this equation reduce to the equation for a pendulum with a point mass and length equal to the radius of the cylinder, when the disk radius gets arbitrarily small? Why, or why not?

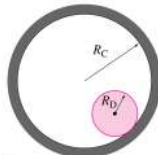


Problem 17.23: A disk rolls without slip inside a bigger cylinder.

16.4.23

Saturday, April 20, 2019 10:38 PM

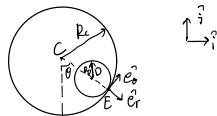
A disk rolls in a cylinder



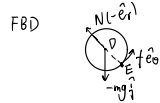
Problem 16.4.23: A disk rolls without slip inside a bigger cylinder.

No slip disk rolls back and forth due to gravity

- Sketch the disk in the cylinder. Show all variables, coordinates and dimension



- Draw FBD of the disk



$$\begin{aligned} e_r &= \sin\theta \hat{i} + \cos\theta \hat{j} \\ e_\theta &= \cos\theta \hat{i} - \sin\theta \hat{j} \end{aligned}$$

- Find LMB & AMB/E for the disk

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

$$\text{LMB: } \Sigma \vec{F} = m \vec{a}_c$$

$$-mg \hat{j} + f \hat{e}_\theta - N \hat{e}_r = m \vec{a}_c$$

$$\text{AMB/E: } \Sigma \vec{M}_E = \dot{\vec{H}}_E$$

$$\text{where } \Sigma \vec{M}_E = R_D \hat{e}_r \times mg \hat{j} = R_D (\sin \theta \hat{i} + \cos \theta \hat{j}) \times mg \hat{j} \\ = R_D mg \sin \theta \hat{k}$$

$$\text{and } \dot{\vec{H}}_E = \vec{r}_{D/E} \times m \vec{a}_D + I \dot{\omega} \hat{k} = -R_D \hat{e}_r \times m \vec{a}_D + \frac{1}{2} m R_D^2 \dot{\omega} \hat{k}$$

(d) Kinematics

Find all the velocities and accelerations needed in momentum balance equation in terms of θ .

$$\vec{v}_D = \vec{v}_E + \dot{\theta} \hat{k} \times (R_C - R_D) \hat{e}_r \\ = \dot{\theta} (R_C - R_D) \hat{e}_\theta \quad (1)$$

Take derivative

$$\vec{a}_D = \ddot{\theta} (R_C - R_D) \hat{e}_\theta - \dot{\theta}^2 (R_C - R_D) \hat{e}_r \quad (2)$$

Note: $\dot{\theta} \neq \omega$, angular velocity of disk, find ω as follows

Rolling w/o slip:

$$\vec{v}_E = \vec{0}$$

$$\vec{v}_D = \vec{v}_E + \vec{\omega} \times (-R_D \hat{e}_r) = -\omega R_D \hat{e}_\theta \quad (3)$$

$$(1) (3) \Rightarrow \omega = -\frac{R_C - R_D}{R_D} \dot{\theta}$$

$$\text{Take derivative } \dot{\omega} = -\frac{R_C - R_D}{R_D} \ddot{\theta}$$

(e) Equation of motion

Find AMB/E as a 2nd order ODE.

Substitute $\vec{a}_D, \dot{\omega}$ into AMB/E.

$$mg R_D \sin \theta \hat{k} = -m R_D \ddot{\theta} (R_C - R_D) \hat{k} + \frac{1}{2} m R_D^2 \left(-\frac{R_C - R_D}{R_D} \right) \ddot{\theta}$$

$$\Rightarrow \ddot{\theta} + \frac{2g}{3(R_C - R_D)} \sin \theta = 0 \quad (*)$$

(f) Simple pendulum?

Does (*) reduce to the equation for a pendulum with a point mass and length equal to the cylinder, when $R_D \rightarrow 0$?

$$\text{If } R_D \rightarrow 0, (*) \text{ becomes } \ddot{\theta} = -\frac{2}{3} \frac{g}{R_C} \sin \theta$$

which is different from simple pendulum equation

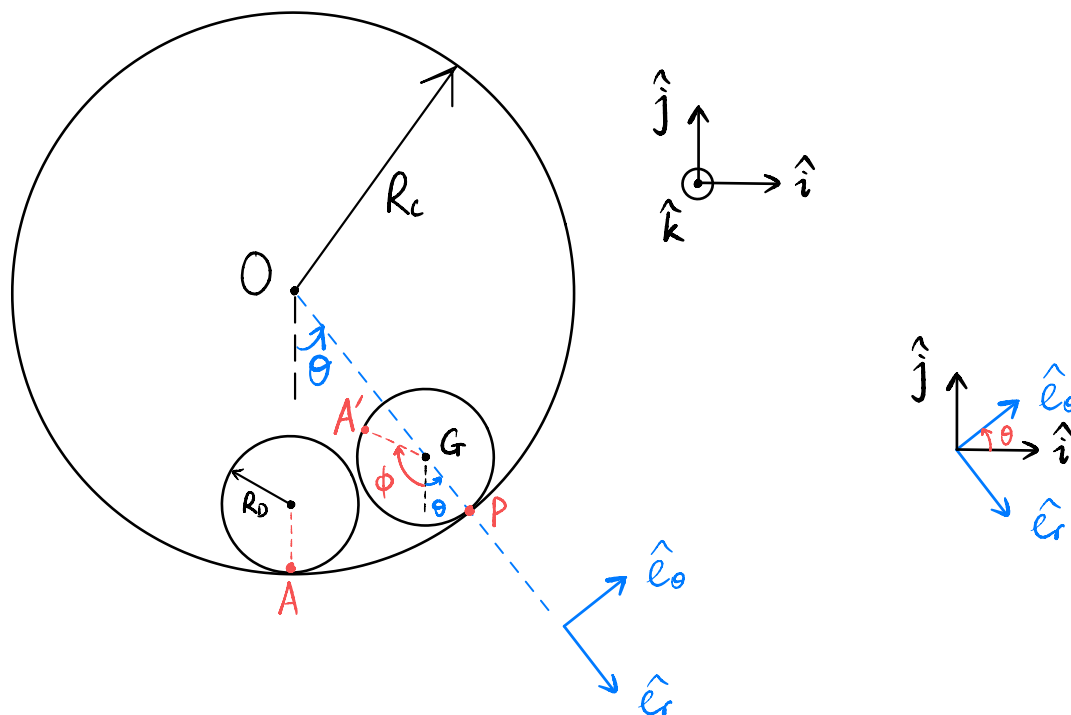
$$\ddot{\theta} = -\frac{g}{R_C} \sin \theta$$

MAE 2030 HW P60 (17.4.23) Solution

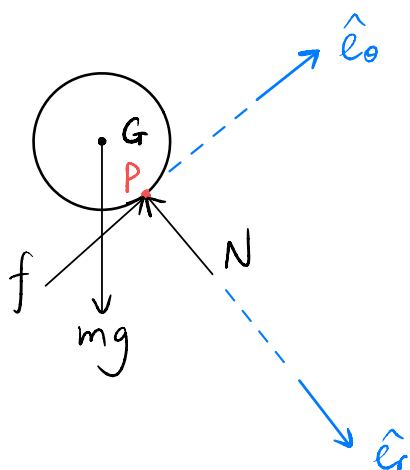
Duan Li

Given : R_D , R_c , no slipping, gravity

a) Sketch



b) FBD



Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

c) Find LMB and AMB/p

LMB

$$m \vec{a}_G = \sum \vec{F}$$

$$m \vec{a}_G = f \hat{e}_\theta - mg \hat{j} - N \hat{e}_r$$

AMB/p

$$\dot{\vec{H}}_{/p} = \sum \vec{M}_{/p}$$

$$m \vec{r}_{G/p} \times \vec{a}_G + I^G (-\ddot{\phi}) \hat{k} = \vec{r}_{G/p} \times (-mg \hat{j})$$

where

$$\vec{r}_{G/p} = -R_D \hat{e}_r$$

$$\vec{a}_G = (R_c - R_D) \ddot{\theta} \hat{e}_\theta - (R_c - R_D) \dot{\theta}^2 \hat{e}_r$$

$$I^G = \frac{1}{2} m R_D^2 \text{ for a disk}$$

Simplify

$$-m R_D (R_c - R_D) \ddot{\theta} \hat{k} - \frac{1}{2} m R_D^2 \ddot{\phi} \hat{k} = mg R_D \sin \theta \hat{k} \quad (1)$$

d) Find $\ddot{\phi}$ in terms of θ

Method 1

Rolling w/o slipping $\rightarrow \vec{v}_P = \vec{0}$

$$\begin{aligned}\vec{v}_P &= \vec{v}_G + \vec{v}_{P/G} \\ &= \dot{\theta} \hat{k} \times (R_c - R_D) \hat{e}_r + (-\dot{\phi}) \hat{k} \times R_D \hat{e}_r \\ &= \vec{0}\end{aligned}$$

$$\dot{\phi} = \frac{R_c - R_D}{R_D} \dot{\theta} \rightarrow \boxed{\ddot{\phi} = \frac{R_c - R_D}{R_D} \ddot{\theta}} \quad (2)$$

Method 2

Rolling w/o slipping $\rightarrow$ Curve length $|\widehat{AP}| = |\widehat{A'P}|$

$$R_c \theta = R_D (\phi + \theta)$$

$$\phi = \frac{R_c - R_D}{R_D} \theta \rightarrow \boxed{\ddot{\phi} = \frac{R_c - R_D}{R_D} \ddot{\theta}}$$

e) Find EoM (AMB as a 2nd-order ODE)

Combine Eqn. (1) and (2)

$$-m R_D (R_C - R_D) \ddot{\theta} \hat{k} - \frac{1}{2} m R_D \frac{R_C - R_D}{R_D} \ddot{\theta} \hat{k} = mg R_D \sin \theta \hat{k}$$

$$-\frac{3}{2} m R_D (R_C - R_D) \ddot{\theta} \hat{k} = mg R_D \sin \theta \hat{k}$$

$$\ddot{\theta} = -\frac{2}{3} \frac{g}{R_C - R_D} \sin \theta$$

f) Compare w. a simple pendulum

$$\text{As } R_D \rightarrow 0, \quad \ddot{\theta} = -\frac{2}{3} \frac{g}{R_C} \sin \theta$$

$$\text{Simple pendulum w. length } R_C: \quad \ddot{\theta} = -\frac{g}{R_C} \sin \theta$$

The EoM does **NOT** reduce to that for a simple pendulum w. length R_C .

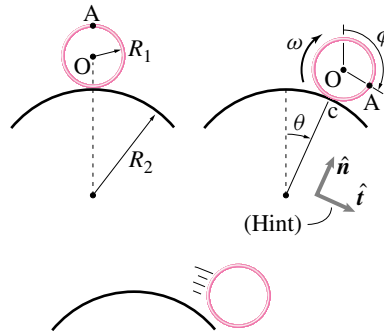
Comments:

- ① As $R_D \rightarrow 0$, $\dot{\phi} \rightarrow \infty$, i.e. the rotation of the disk doesn't vanish.
- ② If the disk has mass distribution of just a point mass at its center, then $I^G = 0$, and the EoM would be the same as that for a simple pendulum w. length $(R_C - R_D)$.

17.4.24 A uniform hoop of radius R_1 and mass m rolls from rest down a semi-circular track of radius R_2 as shown. Assume that no slipping occurs. At what angle θ does the hoop leave the track and what is its angular velocity ω and the linear velocity $\bar{v}$ of its center of mass at that instant? If the hoop slides down the track without friction, so that it does not rotate, will it leave at a smaller or larger angle θ than if it rolls without slip (as above)? Give a qualitative argument to justify your answer.

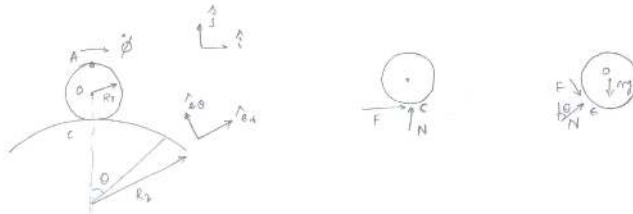
HINT: Here is a geometric relationship between angle ϕ hoop turns through and angle θ subtended by its center when

no slipping occurs: $\phi = [(R_1 + R_2)/R_2]\theta$. (You may or may not need to use this hint.)



Problem 17.24

16.4.24



'O' - goes in circles with radius $(R_1 + R_2)$

$$\begin{aligned}\underline{V}_O &= \underline{\omega} \times (R_1 + R_2) \hat{e}_r \\ &= \dot{\theta} (-\hat{k}) \times (R_1 + R_2) \hat{e}_r\end{aligned}$$

$$\underline{V}_O = \dot{\theta} (R_1 + R_2) (-\hat{e}_\theta)$$

$$\begin{aligned}\underline{a}_O &= \underline{\omega} \times \underline{\omega} \times (R_1 + R_2) \hat{e}_r + \dot{\underline{\omega}} \times (R_1 + R_2) \hat{e}_r \\ &= -\dot{\theta} \hat{k} \times (-\dot{\theta} \hat{k} \times (R_1 + R_2) \hat{e}_r) - \ddot{\theta} \hat{k} \times (R_1 + R_2) \hat{e}_r \\ &= \dot{\theta} \hat{k} \times (R_1 + R_2) \dot{\theta} \hat{e}_\theta - (R_1 + R_2) \ddot{\theta} \hat{e}_\theta \\ &= -(R_1 + R_2) \dot{\theta}^2 \hat{e}_r - (R_1 + R_2) \ddot{\theta} \hat{e}_\theta\end{aligned}$$

$$\hat{j} = \cos \theta \hat{e}_r + \sin \theta \hat{e}_\theta$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

$$\sum \underline{F} = m \underline{a}$$

$$\left\{ -F \hat{e}_\theta + N \hat{e}_r - mg \hat{j} = m \left[-(R_1 + R_2) (\ddot{\theta} \hat{e}_r + \dot{\theta}^2 \hat{e}_\theta) \right] \right\}$$

$$\left\{ \right\} \cdot \hat{e}_r \Rightarrow N - mg \cos \theta = -m(R_1 + R_2) \dot{\theta}^2$$

Hoop loses contact $\Rightarrow N = 0$

$$\Rightarrow \dot{\theta}^2 = \frac{g \cos \theta}{(R_1 + R_2)}$$

Energy Balance

$$E_1 = E_2$$

$$E_1 = P.E = mg(R_1 + R_2)$$

$$E_2 = K.E + P.E = \frac{1}{2} m v_0^2 + mg(R_1 + R_2) \cos \theta + \frac{1}{2} I \omega^2$$

No slip

$$v_0 = \underline{\omega} \times R_1 \hat{j}$$

$$= -\dot{\phi} \hat{k} \times R_1 \hat{e}_r$$

$$v_0 = \dot{\phi} R_1 (-\hat{e}_\theta)$$

$$\Rightarrow \dot{\phi} = \left(1 + \frac{R_2}{R_1} \right) \dot{\theta}$$

$$\Rightarrow mg(R_1 + R_2) = \frac{1}{2} m (\dot{\phi} R_1)^2 + mg(R_1 + R_2) \cos \theta + \frac{1}{2} m R_1^2 \dot{\phi}^2$$

$$= \frac{2}{2} m \left(1 + \frac{R_2}{R_1} \right)^2 \dot{\theta}^2 R_1^2 + mg(R_1 + R_2) \cos \theta$$

$$\Rightarrow mg(R_1 + R_2)(1 - \cos \theta) = \frac{2}{2} m (R_1 + R_2)^2 \dot{\theta}^2$$

$$g(1 - \cos \theta) = \frac{(R_1 + R_2)}{(R_1 + R_2)} g \cos \theta$$

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$$\cos \theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ$$

Angular velocity at that instant,

$$\dot{\phi} = \frac{R_1 + R_2}{R_1} \sqrt{\frac{g \cos \theta}{R_1 + R_2}}$$

$$\dot{\phi} = \underline{\omega} = \frac{1}{R_1} \sqrt{\frac{g(R_1 + R_2)}{2}} (-\hat{k})$$

$$\underline{v} = \sqrt{\frac{g(R_1 + R_2)}{2}} (-\hat{e}_\theta)$$